The behaviour of future teachers dealing with proof problems in arithmetic

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Abstract: We present some results of a didactical experiment with future mathematics teachers concerning their ability in solving proof-questions in the field of aritmetics. For this purpose a coordination between various representation of natural numbers and especially polynomial one is requested. The students showed various types of behaviour when faced with these questions, and they had experienced many difficulties. The most problematic result concerns the meaning of proof they came up with, more closely linked to the validation of a wide plurality of numerical cases than the development of reasoning through the algebraic language.

Introduction

Proofs in the field of arithmetic is hardly used in our schools, partly for reasons connected to the history of the teaching of arithmetic and algebra in our country (Vita 1986). Mathematics teaching in Italy has always adopted an approach to arithmetic based on the unquestioning learning of algorithms and processes of calculus which lead to a distorted approach to algebra. This approach, for the most part, is based on the syntactical aspects of the algebraic language, instead of those for codifying relationships between and the production of thought (Chevallard 1989, Kieran 1992, Arzarello et al 1994, Malara 1994).

Arithmetic, especially the ambit of natural numbers, constitutes ideal ground for exploratory activities, the formulation of conjectures, and the development of proofs. In particular, these activities offer an important basis for the move from argumentation, (based on the use of verbal language) to proof (based on the use of algebraic language). Furthermore, simple proof problems in arithmetic (and not only) if well introduced in class, may be used to form the basis and motivation for self-study of algebraic transformations. This is also implicitly proposed in middle school curriculum which strongly recommends activities on the individualisation of patterns as an introduction to algebra seen as a language with which to codify them. But in fact, apart from a few exceptional cases, this is not put into practice.

It must be noted that, unlike with geometry, it is possible to do this without having to deal with the question of constructing the formal system and the axioms of arithmetic. Multiplication and addition operations as well as their relative properties, apart from the usual order, are founded during early experiences of mathematics in class – usually in the first years of primary school – often with the support of the naïve set theory. The proof of arithmetic properties, when carried out, are obviously "local" results. However, this is not exclude that the same results may not be reconsidered at higher scholastic levels when studying theoretical systems.

If we take into consideration problems regarding the proof of arithmetic properties, we see that not all researchers agree on the use of the algebraic language in cases in which proof given orally is more intuitive and straight-forward than the formal one. For example, let us consider the task "prove that the result of the multiplication of three consecutive numbers is necessarily divisible by six" (which can be simply resolved considering that if there are three consecutive numbers, at least one of the three must be even and that at least one of the three divided by three has a remainder of zero), or the task "show that the square of an odd number is odd", (which can be solved considering that an odd number is characterised by the fact that - in the canonical 10-base representation - it ends with an odd digit and that, since the square of the odd digits is an odd number, then the square of the number given is odd because its digit of the units is odd). We do not wish to deny the aspects of intuition or simplicity in these deductions, but it is impossible not to reflect on their expository complexity. It is therefore necessary to lead students to appreciate algebraic language because it allows us to codify and resolve situations which are difficult to handle with natural language. In fact, it is interesting to carry out a comparison with the students between intuitive verbal strategies and algebraic strategies. For example, in cases like this, the teacher might propose the task of the translation into algebraic language of the reasoning process and, when comparing resolution strategies with the whole class, point out that first of all, verbal proof depends on the representation of numbers, while algebraic proof does not. The teacher may extend this generalisation on the second point saying that it can be understood by people who do not understand Italian, underlining the nature of "mathematic Esperanto" inherent in the algebraic language.

The experimental studies carried out with students from 12 to 16 years of age on proof in arithmetic at various levels of difficulty (Bell 1976, Frielander et al. 1989, Garuti *et al.* 1996, Malara & Gherpelli 1997, Savadosky 1999, Healy & Hoyles 2000), allow us to state that the students, when appropriately taught, can come up with good proof productions right from middle school, and, as time goes on, go on to appreciate the correct significance and role of examples (inductive-explanatory role of conjectures and preparatory to proof) and counter examples (resolutory compared to the non-validity of a conjecture) and to understand the role of proof, necessary to sustain the validity of a proposition in general terms.

From the linguistic point of view, the development of a proof in this field requires on the students' part:

- the knowledge of the meaning of specific terms in natural language which characterises predicates in association with the verb to be (double, consecutive, even, greater than, lesser than, divisible by, multiple of, etc. and their combinations;

as well as the ability to:

- reformulated propositions adequately to the aim (for example, express "greater" in terms of "equal"),
- translate from natural language into the algebraic one,
- interpret algebraic expressions transformed from others within the terms of the situation in question;
- check the outcome of what is assumed (if n represents an odd number, recognise n + 1 as an even number).

In this process, and the role of the teacher's fundamental. S/He will have to act as a model showing students, through various carefully chosen situations, how to:

a) translate the hypotheses into algebraic language,
b) transform writing in various ways in order to open up the field to his/her own different interpretations;
c) Interpret the formulae obtained for syntactic elaboration and select those which are useful to these ends.

However, if our aim is to give space to the concept of proof in arithmetic in secondary education, we must invest heavily in the training of future teachers. As we shall see, even those with a mathematical background have a certain amount of difficulty in solving proof problems.

An experiment with future teachers

Let us now look at some of the results from the treatment of several proof problems in the ambit of natural numbers (see table 4) carried out with 15 students in the last year of the mathematics degree and 13 future teachers[1] taking part in a teacher training course.

Our initial aims were manifold. Partly, we wanted to verify their proving abilities and pick out any blocks or difficulties of the future teachers, but above all, we wanted to put them into a situation in which they had to concentrate their attention on the issue of proof and discuss the role of this kind of activity in mathematics education at secondary school level. Our hypothesis was that the teachers, given the tradition of teaching and the age-old habits of blindly trusting in whatever the textbook says, were not aware of the importance nor of the feasibility of this kinds of problem from the students' point of view. Furthermore, given their unfamiliarity with proof and the difficulties they find in solving the problems, they would be far from even thinking about making space for it in the classroom.

The problems posed, taken from "Giochi d'aritmetica e problemi interessanti" by G. Peano

1. Write down a three digit number, invert the order of the digits and write down the difference between the two numbers, the greater minus the lesser. Give me the last (first) digit of the difference and I will tell you the difference.

[1] The course was devoted to the training of future teachers of Mathematics and Science in middle school (6th to 8th grade). It was attended by 11 non-mathematics graduates.

2. Write down a number with several digits, multiply it by 10 and take it away from the first one, cross out one of the (not zero) digits from the difference and give me the sum of those left. I will guess the one that you crossed out. Explain to me how this is possible.
3. Write down a three-digit number with the digits in descending order, invert the order of the digits and write down the difference between the two numbers. Add the same number to this difference with the digits inverted. Whatever the number is, you always get 1089. Why?
4. A two-digit number has this characteristic: its square minus the square of the previous number is the same as the first number with the digits inverted. What number is it?
5. Write down a two-digit number. Write down what you get from this when you invert the digits. Prove that the sum is divisible by 11. Investigate what happens when you use a three or four-digit number.

The problems

The problems considered were taken from the text by G. Peano entitled "Giochi d'aritmetica e problemi interessanti" (Arithmetic games and interesting questions) written back in 1924 for training primary school teachers. These are based on the use of the canonical polynomial representation of the number, and require the ability to:

- formerly express the verbal conditions given in the text of a problem,
- distinguish the canonical polynomial representation of a number (under the positional 10 base representation) from other (though polynomial) representations,
- represent a natural number in polynomial form, about which certain indications or conditions on the digits have been given,
- carry out reasoned transformations about which the results must be proved and interpreted; in particular, functionally apply the principal properties of an arithmetical operation to the solution of the problem,
- make generalisations.

From a general point of view, the main difficulties in solving the task, quite apart from the awareness of what proof means and consists of, regard the ability to:

- interpret a verbal text (the language is discursive and of a 19th-century style),
- formalise properties and relations between the data considering implicit conditions and elaborate them so as to give the necessary proof,
- correctly interpret representations deriving from the execution of given algorithms on numbers expressed through their canonical polynomial representation,
- come up with functional transformations to that which is necessary in order to prove and interpret the results,
- grasp the general patterns of the situation in question (for example, moving from the case of a number in the hundreds to one in 10^n).

More precisely, the first task requires the individualisation of the canonical polynomial representation of the difference between two related numbers and then the highlighting of a property - these are the resolutory key to this task. The second concerns difficulties owing to the reciprocal influence between polynomial and algebraic representation; furthermore, if polynomial representation is used, the direct proof of the problem is more intricate. Moreover, if it is not independently proved that a multiple of 9 has the sum of its digits which is a multiple of 9, it is in fact impracticable. The third task is comparable to the first, and requires co-ordination between the algorithmic aspects of addition and subtraction operations and the algebraic aspects resulting from the formulation of the problem. The fourth presents a variant on the third; the principle of polynomial identity must be applied and information must be suitably co-ordinated according to the different cases. The fifth task is more simple compared to the previous ones; an element of difficulty which distinguishes it lies in the request to investigate the extension of the pattern proved for two-digit numbers in the case of higher numbers.

Behaviour encountered

The students had been asked to solve the problems, analyse their own thinking processes while doing so, and say what their difficulties were. As regards their initial approach to the problems, all except one mathematics undergraduate, started off making numerical trials to verify the results of the theses. Their behaviour can be classified in the following categories:

Category 1. *Naive trialing subjects*. This includes those who stopped once they had carried out some numerical checks without facing the question of looking for/trying out a proof or without declaring that the did not know how to proceed in the proof process.

Category 2. *Aware experimental subjects*. This includes those who declared that they did not know how to investigate the reasons of what they tested in their numerical trials.

Category 3. *Theoretical subjects*. This includes those who went further than the numerical trials and attempted to find the reasons behind the patterns they tested, even if at times they did not manage to do it correctly.

Category 4. *Meta-cognitive theoretical subjects*. This includes those who did not only attempt to explain the proof in general terms, but who also analysed in their own streams of thought in certain detail, or who at least expressed their difficulties encountered.

Among those who attempted to explain the proof of these tasks (categories 3 or 4), two different approaches can be distinguished: there are those who favour verbal development of the proof, while others prefer an algebraic development, limiting the verbal operation to the interpretation of the concluding algebraic results. Obviously, those who chose verbal proof were generally wrong, though in a few cases (as in task 2) the aid of known theorems turned out to be fruitful.

As far as the aspects of meta-cognitive checking are concerned, only a few mathematics undergraduates (partly because they had been taught in previous activities) analysed their own reasoning processes, focusing carefully on the various frame changes (Arzarello et al. 1994), and explaining their difficulties. The others merely "solved" the tasks, simply noting when the one of them seemed particularly difficult.

The problems considered most difficult were the second and the fourth, the main reason lies in the conflict due to the contemporary handling of two different kinds of representation: the algebraic, used to translate and the relations expressed between numbers, and the polynomial (or general positional) to represent the numbers themselves. The formal tradition of the sum, product, or difference of generic numbers represented in polynomial form gave place blocks, difficulties, or errors owing to the inability to control the algorithms of the operations in general terms and to conveniently interpret or convert syntactic elaboration of such representations so as to recognise the
relative polynomial at the end of the operation.

An attitude common among the subjects, when passing from the particular to the general, was that of representing a generic natural number with a sequence of letters. Though inappropriate and in conflict with standard algebraic representation (where the juxtaposition of two or more letters represent the product), this "naive" code, a generalisation of the positional one, was handled well enough during the proof process; however, at a certain point (not the same one for all students) they realised the inherent ambiguity and felt the need to move over to polynomial representation. This leap was made by some of them during task 2 (which was not strictly necessary and was perhaps more of a hinder),while others changed systems to solve task 4. The use of this representation was not therefore immediate or spontaneous, but it marked the end of a "dirty" and conflicting system of representation.

A frequent error was that of interpreting the sum or the difference between two numbers, represented polynomially as the results of the mere addition (subtraction) of the terms related to the same power of ten, without checking the conditions of the coefficients in the numerical ambit or in the case of considering the amount to be carried over.

Problem two turned out to be the most difficult for all those who did not use the theorem which characterises the multiples of 9 as numbers whose digits are a multiple of nine. As mentioned above, the strategy of linking given information verbally to other knowledge proved highly effective.

Another characteristic attitude is to be found in the solution to the fourth question. As regards the two-digit number to be found represented as 10a+b, the result 19a-8b=1 can be reached on the basis of the conditions given. Only one student remembered the conditions regarding a and b following the relations imposed by the problem, and then developed considerations of a general nature: the majority proceeded by numerical checks. Two or three, at times wrongly, made use of a known theorem about the greatest common divider, without using it however. Another common attitude was considering the polynomials as implicitly equal, expressing the same number. Only one student explicitly recalled the principle of identity of polynomials.

Behavioural differences also showed up with regard to the second point of question 5, which required an investigation of the possible extension of the pattern proved in the case of two-digit numbers to numbers of more digits. The interesting thing is that this pattern carries on unevenly; that is, in numbers with an even number of digits, while in numbers with an odd number of digits there is no pattern. As expected, behaviour in regard to this point differed somewhat. Most of the students simply identified a numerical counter example in the case of three digit numbers for their investigations and stopped there, declaring the rule invalid in the case of higher numbers. And there were some students who, though having identified a numerical counter example, set themselves to task of studying the general terms which the numbers must meet in order to verify the rule, while others approaced to the problem in general terms correctly deducing what happens in the cases of even or odd exponent of the biggest power of ten.

The non-mathematics students had notable difficulties and their notes show various comments, for example:

I found myself completely unable to formulate any coherent reasoning. I have great difficulties in proving rules; for the training I have, mathematical exercises are an archaic experience, difficult to grasp at all quickly or easily.

It must however be pointed out that one mathematics undergraduate was not even unable to conduct experimental checks adequately or verify and identify the patterns in question.

It was very interesting for the students when they were presented with different "solution" prototypes and asked to compare and discuss them, especially for the students who had declared no knowledge in the subject. In fact, among some of them (among the most curious and lively of the students, who were not mathematics graduates) showed themselves right from the beginning to be anxious to understand how they might proceed.

This experience was very important for most of the participants, for their own awareness-building as well as for their shortcomings and difficulties in regard to the particular tasks studied, and for the more general problem of teaching-learning strategies for proving.

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