ON STRUCTURAL FEATURES OF MODELLING

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Summary: The process of mathematical modelling proceeds certain stages to reach the final result. Due to its diversity a number of paths (trajectories) passing through the stages may be distinguished. Accordingly, structural viewpoints of modelling and features of particular stages will be studied. Examples taken from realistic situations are presented and relevant didactic aspects are pointed out.

Introduction

The role of modelling in mathematics instructing is very diverse (see Blum, Niss [1], Galbraith and others [5], Saaty,Alexander [10] among others). Modelling plays a key role in primary mathematics didactic employing "separated models" in the mechanism of understanding (see e.g. Sierpinska [11]). On higher educational levels, modelling is associated often with applications and essentially contributes to the role of mathematics as an integrating subject in the system of education (see ICMI [6]).

As usual, a ***model*** is to be understood as a simplified representation of a real situation (analogous to Dilwyn, Hamson [2]). A *mathematical model* (briefly a *model*) is a model set up by means of mathematical tools. Accordingly, by *mathematical modelling* (briefly *modelling*) we understand a procedure of creating models. For all practical purposes of mathematics didactic a certain classification of types of modelling techniques has been suggested, namely *quantitative, qualitative, visual, analytic and abstract* (for the details see Meznik [9]). The mentioned notions point at the way the model was set up not at the character of resulting structure. For instance, the result of quantitative or analytic or abstract modelling may be a polynomial. All the modelling types contribute to modelling process in a fruitful symbiosis reflecting the variety of the entire process. With a view to our further mathematics instruction considerations, we will focus mainly on visual, analytic and qualitative modelling, although quantitative and abstract modelling will be taken into account when investigating modelling trajectories. It should stressed, that mainly he use of visual tools in mathematics instructing is presently spreading and is connected with the issue of *visualization* in mathematics (see Fomenko, Kunii [4]).

From the viewpoint of teaching aims we will concentrate to modelling of dependencies formulated verbally by means of declarations that are of particular didactic value. We will consider situations, that can be expressed as functional correspondence between two sets of numbers (usually reals), which is the basic necessary simplification The resulting models will be functions of one real variable. In particular, visual models will be sets of points in the plane forming continuous curves, analytic models will be elementary functions. So, continuity will be typical property of such models. It tackles one of crucial points of modelling, namely the problem of discreteness and continuity. More precisely, continuous models of real situations with discrete character. Handling (mainly calculating) nearly exclusively elementary functions at secondary schools, students brought in mind the perception of the function concept as continuous. On the other hand, they hold continuity for ideal, i.e. practically non existing. In engineering courses, they encounter real actions that are evidently more continuous rather than discrete (e.g. electricity), but also those that are very far from being continuous (e.g. economics). The problems may effectively be overcome by doing modelling. When employing modelling, mathematics instructors should also realize that (see also [1]) students must face a new phenomenon - *heuristics*. Up to now, students were led to find a unique solution (or for example in geometry small finite number of solutions) of a problem using step by step defined process (algorithm) of deterministic nature. Therefore it is desirable to lay emphasis on different models of a given problem (real situation). There is another important principle to strenghten motivation for mathematics and to avoid annoying of students , namely that the given problem should be theoretically simple and easily comprehensible. From this viewpoint the problems, for instance, taken from everyday "economics" seem to be useful. At the end we will present an example using slightly advanced mathematics tools , but it is clear how to simplify it to lower educational levels.

Trajectories of modelling process

When doing modelling in the process of instructing, the initial real (life) situation is usually described by means of a rough declaration (verbal or written) or some raw data collection, which frequently contains vague, inconsistent and sparse data and its form is mostly far from mathematical language. In the next step the declaration must be specified (or data organized) to get the form required for further mathematical treatment – modelling, which ends with the "solution" as a mathematical model. The result is usually demanded in a prescribed mathematical form (e.g. graph, equation, function, formula, etc.) in terms of tools that are at the disposal on the particular educational level. The prescribed resulting form determines to a large extent the modelling process. On the other hand, the process may be influenced by didactic aspects. So, the modelling process may pass different stages forming various trajectories. The following Diagram 1 depicts basic trajectories of modelling process.

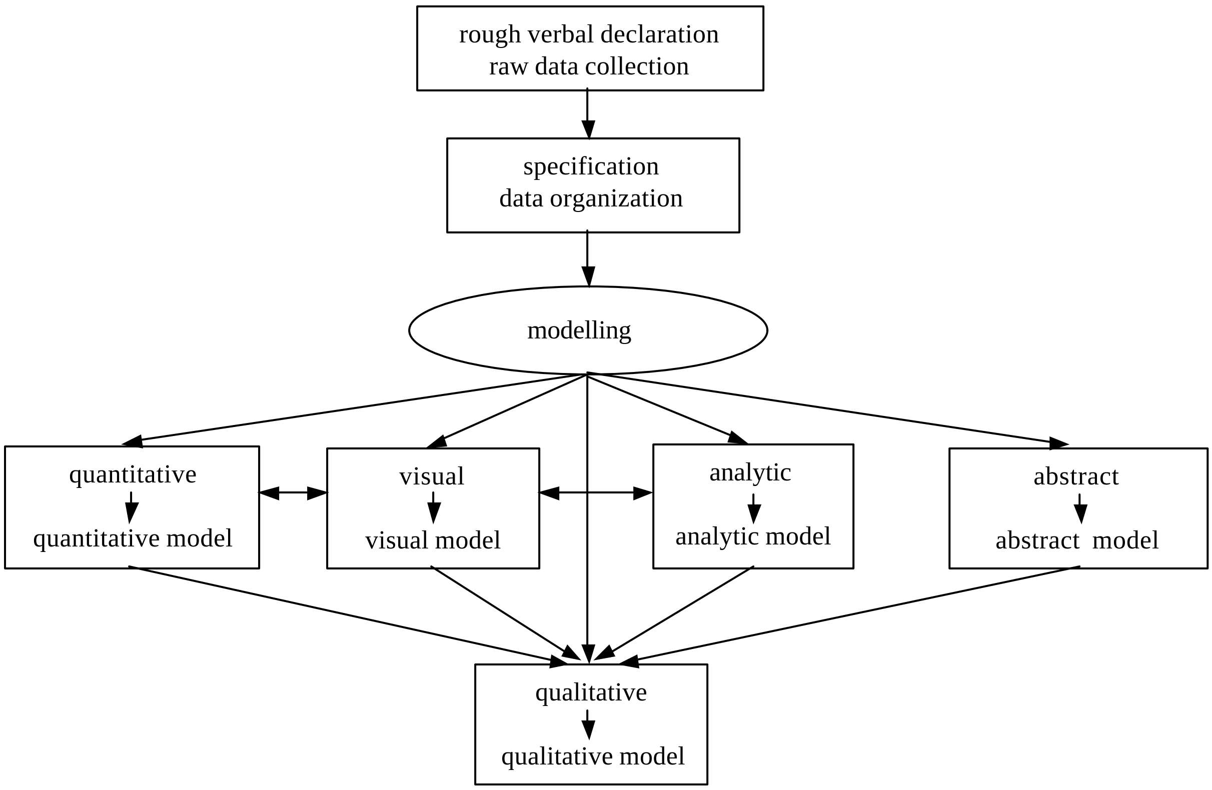


Diagram 1

As to the use of modelling as a support in instructing of mathematics, the trajectories passing visul, analytic and qualitative modelling stages are of great importance. It is apparent, that the order of employing various types of modelling plays an important role and their influence on the final result is different. Above all, visual modelling plays a key role. The experience of mathematics educators justifies the hypothesis, that the process of finding the model and accordingly the process of understanding in mathematics is significantly sensitive to visual modelling. In more general form, the understanding is strongly elastic to visualization. Hence, emphasis should be layed on tools of visual modelling in didactics. So, to reach the final stage claims some preceding stages, particularly those based on visual modelling, resulting in a visual model. The following Diagram 2 is to depict mostly employed trajectories regarding the mentioned aspects..

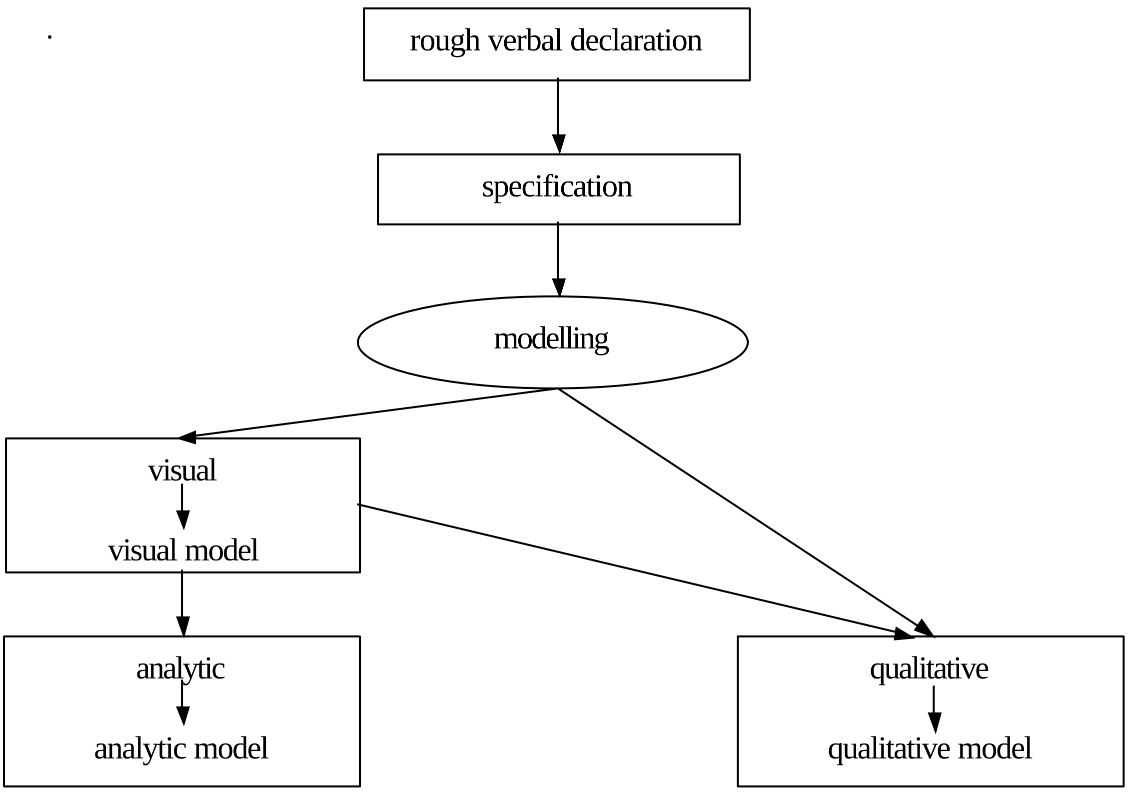


Diagram 2

In the sequel we will present an example to demonstrate the above formulated principles. Having in mind to grasp a real-life situation, such example can not be quite trivial. On the other hand the solution should claim more intuitive than sophisticated reasoning . For this reason the motivation will be taken from economics, we will set up a simple model of production. It is suitable for higher educational level, but it is obvious how to simplify it or to direct the creativity to find examples for lower educational levels in order to flourish didactics.

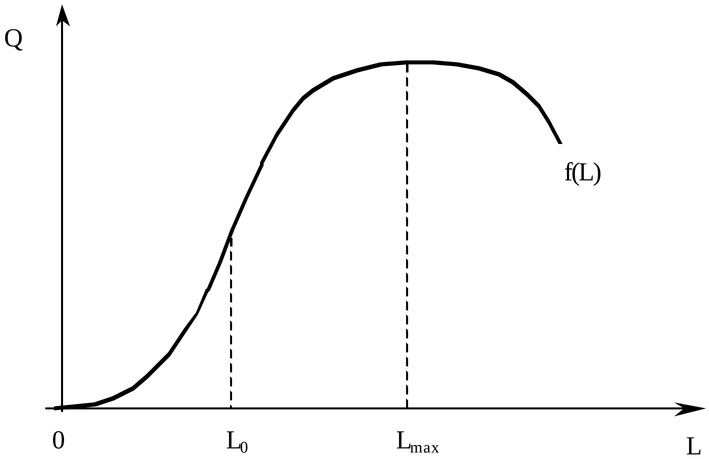
<u>**Example.**</u> *Production of a firm depends on a variety of inputs, called factors. The major factors are labour, capital and land. For simplicity we restrict our attention only to labour, considering the other factors as fixed. Supposing, that the dependence of production (output) Q on labour L is functional, then it may be expressed as a function Q = f(L), called a production function. Production Q is measured in physical units per unit time, labour L is measured in labour hours per unit time. Now, we will examine properties of production function to find a mathematical model (visual, analytic and qualitative) of a production process. For the purpose to grasp the behaviour of production function f, of particular interest is the effect on production Q when labour L is scaled in some way. In accordance with Diagram 2 we start with the initial stage.*

<u>*Rough verbal declaration.*</u> *Obviously, when L=0, then Q=0. Further, if labour increases then production increases, but only to some maximum value, then it decreases. There are basically two reasons for that – fixed values of the other factors and technical limitations (overcrowding in workplace, organizing the larger workforce,…). What else ? Still more – because of the maximum point, the increase in production due to a one unit increase in labour must eventually decline (note: this property is known as the law of diminishing returns). In other words, once the size of the workforce has reached a certain threshold level, the rate of change in production will go down. Up to this threshold level the rate of change in production may increase (it is not excluded, that in some cases the mentioned rate goes down from the beginning)*

<u>*Specification.*</u> *Production function f is defined for L 0 and passes [0, 0]. It increases to its maximum at some* L_{max} *in such a way, that up to some* L_0 L_{max} *the rate of increase in Q goes up and from* L_0 *the rate of increase*

in Q goes down. Conclusion : F passes [0, 0], f increases up to L_{max} , f decreases from L_{max} , f´ increases up to L_0 (it implies that f is convex up to L_0), f´decreases from L_0 (it implies that f is concave from L_0), L_0 is the point of inflection.

<u>Visual modelling</u>. *From the data collected in Specification, a curve may be sketched depicting a production function. From 0 up to L_0 the production curve steeply increases (it bend upwards, the slope function f´increases, f is convex), if L exceeds the threshold value L_0 then the production curve bends downwards (the slope function f´decreases, f is concave). If L exceeds L_{max} , then the production curve decreases preserving concavity. The result is a* ***visual model*** *of production function depicted in Picture 1.*



Picture 1.

<u>Analytic modelling.</u> *The shape of f (see Picture 1) reminds us of the graph of a polynomial. The degree of such polynomial is greater than 2 and it has root 0. Moreover f ´ ´ < 0 as from some L_0 and $f'(L_{max}) = 0$. It may be simply verified, that for* ***an analytic model*** *of production function may serve the polynomial*

$$Q = f(L) = aL^3 + bL^2, \text{ where } a < 0 \text{ and } b > 0.$$

Then $L = -(b/3a)$ and $L_{max} = -(2b/3a)$.

<u>Qualitative modelling .</u> *A production function f has the following characterization :*

$$[+ , + , +] \text{ on } (0, L_0) \text{ in case } L_0 > 0$$
$$[+ , + , -] \text{ on } (L_0 , L_{max})$$
$$[+ , - , -] \text{ for } L > L_{max}$$

This characterization gives ***a qualitative model*** *of production function .*

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