STUDENTS' PERCEPTIONS OF THE IMPORTANCE OF CLOSURE IN ARITHMETIC: IMPLICATIONS FOR ALGEBRA

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Abstract: Traditionally, many students have found the study of algebra difficult. This paper examines arithmetic prerequisites for algebra study, particularly those associated with the concept of equals and the operational laws. A sample of secondary students who were about to begin a unit of algebra was tested for their ability to do the prerequisite arithmetic. Analysis of their responses revealed that most of them were poorly equipped for algebra study. In particular, students performed poorly on problems where the equals sign did not simply designate where the answer is to be placed and where operations cannot be easily closed.

Algebra is an abstract system in which interactions reflect the structure of arithmetic (Cooper, Williams & Baturo, 1999). Its processes are abstract schemas (Ohlsson, 1993) or structural conceptions (Sfard, 1991) of the arithmetic *operations, equals*, and *operational laws*, combined with the algebraic notion of *variable* (Cooper, Boulton_Lewis, Atweh, Willss & Mutch, 1997). Arithmetic does not operate at the same level of abstraction as algebra for, although they both involve written symbols and an understanding of operations (e.g., order of operations, inverse operations – Herscovics & Linchevski, 1994), arithmetic is limited to numbers and numerical computations (Sfard & Linchevski, 1994). Arithmetic and algebra differ fundamentally in that in arithmetic computational procedures are separated from the object obtained (Linchevski & Herscovics, 1996). That is, students in arithmetic are not expected to conceive of groups of numbers and symbols as objects, where as in algebra this is necessary.

A fundamental requirement of algebra is an understanding that the equal sign indicates equivalence and that information can be processed in either direction (Kieran 1981; Linchevski, 1995). It has been noted previously that many students' understanding of equals is action indication (e.g., "makes or gives" – Stacey & MacGregor, 1997, p. 113) or syntactic (showing the place where the answer should be written – Filloy & Rojano, 1989). Misconceptions relating to the equal concept make it very difficult for students to transform and solve equations (Kieran, 1992; Linchevski & Herscovics, 1996).

Understanding the arithmetic operational laws (commutative, associative and distributive) and the order convention for the operations are also important in algebra (Bell, 1995; Boulton-Lewis et al., 1998; Herscovics & Linchevski, 1994). For example, Kieran (1992) reported that some students read algebraic expressions from left to right and ignored bracketing, a behaviour that indicates inability to apply the order convention.

The difficulties students who study algebra face without adequate arithmetic prerequisite knowledge can be easily seen in the following Year 9 task: "Solve for x: 2(x-1)+2 = - (4-3x)". Completing this algebra task requires understanding of the equal concept, order convention, operational laws and directed numbers. In this study, Year 9 students who were about to begin a unit on algebra were the subject of a probe into these prerequisites.

Method

Subjects: The subjects of the study were forty-one Year 9 and nine Year 10 students who were about to begin a 35-hour unit on algebra. All these students had passed Year 8 prerequisite subjects that incorporated the arithmetic operations included in this study. The Year 10 students had previously completed the algebra unit in Year 9 but, since their marks were unsatisfactory, they were repeating the unit. Observations of 20 lesson within 3 classes showed that the students were generally cooperative and well behaved, although some students were reluctant to complete homework tasks. The students were enrolled in a middle sized coeducational school located in a middle class suburb in Queensland, Australia.

Instrument: The instrument in the study was a pencil-and-paper test of eight questions designed by the authors to test students' abilities to do some of the arithmetic necessary for beginning symbolic algebra study. That is, the test items (see Table 1) were arithmetic tasks that reflected the prerequisites for the types described above and consisted of items testing student understanding of the equal concept; order conventions and operational laws and directed numbers. The students' ability to work with directed number was assessed with a simple algebra task that was also used to check understanding of variable as unknown (for a broader study). The test was designed so that descriptive statistics and qualitative analysis of its items would provide insight into the students' understanding of the mathematics topics. Students were encouraged to show the procedures they used in completing the tasks. However, it should be noted that pencil and paper tests are limited in their ability to probe students' thought.

Procedure: The test was administered at the beginning of the unit prior to the commencement of algebra study and arithmetic revision. There was no time limit, students were encouraged to show all working, calculator use was optional, no written or support material was allowed and students worked individually.

Analysis: The test papers were collected and marked. Students who did not attempt to answer a question were given no marks. Students' responses for each of the items were analysed in terms of correctness and, where common mistake patterns were observed, they were described and analysed in terms of the probable thinking that underlaid the error. The overall results were then presented in terms of the number of students who correctly answered each question.

Results

The correctness of students' results are described and summarised in Table 1. The table provides the item, the major concept it is meant to probe, and the number of students who answered the item correctly.

Table 1.
Summary of the Results (n=50).

Item	Concept	Number correct
1) If I add 56 to the right hand side, how do I keep the equation $139 \times 43 = 5977$ equal?	Equal concept	5
2) Calculate: $6 + 4 \div 2 =$	Order conventions	15
3) Calculate: $7 \times (2 + 1) =$	Order conventions	48
4) Calculate: $20 \div (5 - 1) =$	Order conventions	42
5) Can you work out the answer to $\frac{(36+24)}{6}$ if you are not allowed to add 36 + 24? Explain.	Operational laws (Distributive law)	12
6) Could you work out the answer to the problem $5 \times (6 + 7)$ if you were not allowed to add 6 and 7? Explain.	Operational laws (Distributive law)	4
7) The triangle σ and the square ν represent unknown numbers. Can you calculate the answer to σ + (ν + 7) if you know σ + ν = 11?	Operational laws (Associative law)	13
8) Solve for x: $-4x = 20$.	Directed number (Variable as unknown)	3

The equal concept (*Item 1*): Only five of the 50 students could correctly suggest that 56 be added to the left-hand side if it had been added to the right-hand side. Thirty-five students either did not attempt to answer or reported, "I do not know what to do". The remaining ten students suggested various numerical procedures such as "divide 56 by 43 and add the result to 13", two students suggested "subtract 56 from the other side". The responses support the conclusion that many of the students had a action or syntactic (Filloy, & Rojano, 1989; Stacey, & MacGregor, 1997) rather than a sense making approach to the equal concept.

Order conventions (*Items 2 to 4*): Only fifteen students correctly answered 8 for *Item 2*, "$6 + 4 \div 2$". Thirty-three reported that 5 was the answer; clearly these students had done the operations in order left to right. Other incorrect responses included 20 and the decimal number 4.1. The student who answered 20 may have added 6 to 4 then multiplied 10 by 2, indicating confusion with both order convention and division and multiplication. The response of 4.1 is difficult to explain. Only two students did not succeed in *Item 3*, "$7 \times (2 + 1) =$ ". The answers of these two students were "9" and "12". Most students (42 out of 50) correctly computed the answer to *Item 4*, "$20 \div (5 - 1)$". One student responded with an incorrect answer of 4. This may have indicated that this student had made a computational error. One student responded with 80 suggesting that she multiplied instead of divided.

Other answers included 3½ which is difficult to fathom, and ¼ that may have indicated that the student divided 5 by 20 and ignored the subtract one. Clearly the data supports the finding that almost all the students appreciated that the brackets part of the order convention (in Queensland this is based on the learning hinge or memory prompt BOMDAS - Brackets, Of, Multiplication, Division, Addition, and Subtraction) must be done first.

Operational laws (*Items 5-7*): Only twelve students correctly answered *Item 5*, "Can you work out the answer to (36 + 24) / 6 if you are not allowed to add 36 and 24? Explain". Most of the students who gained no marks simply responded with, "no" or "can't be done," but ten students made comments similar to, "no because you have to do the brackets first". Similarly most of the students who gave reasons for not being able to complete *Item 6*, "5 × (6 + 7)" reported that the brackets needed to be done first. One has to wonder if the use of the order convention as a learning hinge or memory prompt has not clouded students' memory of the distributive law. One conclusion from these observations is that order convention needs to be taught along side the distributive law.

The second law checked was the associative law, and it yielded similar results. Only thirteen of the students correctly answered *Item 7*, "The triangle σ and the square ν represent unknown numbers. Can you calculate the answer to σ + (ν + 7) if you know σ + ν = 11?". As in previous responses, many students did not elaborate on why the problem could not be solved. However, six students responded with numbers which added up to 11 such as "1 + (3 + 7) = 11". Several others said no "because σ + ν could be any number" and went on to suggest "6 + 5 or 8 + 3 or 10 + 1". Clearly, many students did not appreciate that the brackets did not have meaning in the case where all the operations were addition. It is possible that the use of brackets and symbols confused students, however the use of such symbols is common in the primary curricula in Queensland. These results provide further evidence that using BOMDAS without understanding has hindered the student performance on these tasks.

Directed numbers (*Item 8*): Directed numbers were not directly included in the test; however, in *Item 8*, the students were asked to solve for x in the following equation - 4x = 20. This type of problem has been termed operational algebra in that it can be solved through arithmetic operations. Twenty-one students reported that they could not do the problem, eight responded with numbers like 16, 24 or 6. These responses indicated that arithmetic as well as the directed number and equal concepts was problematic for them. Eighteen reported 5 indicating that they were able to solve the problem except for the directed number component part of the operation. Only three students of the 50 had the correct answer of –5. The observation that eighteen reported 5 and eight others had other positive numbers is clear evidence that many of the students could not work with directed numbers.

Discussion

In summary, students showed poor understanding of the concept of equal, order conventions where brackets are not central, operation laws and directed numbers operations. In contrast, students showed good understandings of the order convention where brackets were present. Interestingly, many of the deficiencies are such that they would cause difficulties in arithmetic as well as algebra. However, others (concept of equals, application of distributive and associative laws and directed number concept) are such that many arithmetic procedures may not be affected; but, as argued by Kieran (1992), they may cause difficulties in the transition to algebra. However, it should be noted that weaknesses such as those with respect to the concept of equals would only affect algebraic manipulations of equations. It is possible for students with poor understanding of equals to solve algebraic equations by backtracking (working backwards) or trial and error (Boulton-Lewis et al., 1997).

The findings with respect to student performance are generally similar to the findings of previous studies such as Booth (1988), Cooper, et. al (1997), Herscovics and Linchevski (1994), and Kieran (1992). However, some results are different to some previous studies, for example, Boulton-Lewis, Cooper, Atweh, Pillay & Willss, (1998) found that most students in Year 9 had a sufficient understanding of the commutative law, order of operations and understandings of arithmetic processes to apply them to algebra.

Separating procedures from objects and the importance of closure: Linchevski and Herscovics (1996) argued that arithmetic was unique in separating procedures from objects obtained from those procedures, a situation, which was not possible in algebra. This study indicates students' failure in the arithmetic tasks was similarly a result of being unable to separate the two. Further, their failures appeared to be a consequence of *closure* (Biggs, & Collis, 1982). In their responses to the test items, the students had great difficulty with the problems in which closure was not possible or not easily obtained, and where operations and operational laws had to be considered generally (and, in some cases, as objects). This appears to reinforce the argument that students who have little experience of arithmetic situations where closure is limited, or the pathway to it is obscure, had little knowledge of how to handle expressions and equations as generalities. Such abilities are necessary for success in algebra study.

In terms of closure, *Item 1* would have been a very difficult example for students. The operation 139×43 is too difficult to mentally close, further, it had already been closed, to 5977. The place where answers are normally written, just after the equals sign, was occupied. This would have been very confusing to students with an action or syntactic understanding of equals. This is possibly the reason why thirty-five of the 50 students who did not attempt *Item 1* reported, "I don't know what to do".

Similarly, *Items 5* and 6 would also have been difficult. These items were presented without an equals sign – (to students with a limited understanding) a necessary signal to begin solving the items. This may have been why some students responded with, "can't be done". Where students did try to solve the items, they were guided by BOMDAS, reporting that the brackets must be done first. However, in *Items 5 and 6*, directions prevented the brackets from being closed, which led most students not to be able to solve the items. In *Item 7* (associative law), the brackets contained an unknown ν. There was an equals sign in the second equation ($\sigma + \nu = 11$) but, on its own, this equation did not contain sufficient information for the students unless they made the link with the first equation ($\sigma + (\nu + 7)$. This interpretation of student thinking is supported by the some students' comments of, "no because $\sigma + \nu$ could be any number". In summary it appears that the students were looking for closure but the three items required the expressions to be manipulated before such closure was possible.

In contrast, *Items 3* and *4* were done much better. Here, the equals sign appeared to act both as an indicator to do something and an indicator of where the answer should be placed, and the numbers were small. The students were familiar with the role of brackets in guiding what operations should be closed first. *Item 2* was a mixture of the difficulties with *Items 1, 5, 6* and *7* and the ease of *Items 3* and *4*. The equals sign was in its normal place and the numbers were small, but the expression had to be considered generally to determine the order of closure. However, most students simply worked left to right, closing as they read the numbers and operations.

Order convention, negative sign as an operation, and closure: The students' responses with respect to *Items 1* to *7* illuminates the work of Boulton-Lewis et al., (1998) and Herscovics and Linchevski, (1994) who recognised that understanding of order of operations was necessary for algebra study. When there is a variable, the arithmetic style of closure is not possible. The expressions and equations have to be considered as they are and manipulated in terms of general laws. Brackets draw students' attention to what should be operated on (or closed) first but without generalised understanding of the remainder of the BOMDAS convention and the operational laws, a solution is not readily obtained.

Two factors appeared to limit closure in *Item 8* "(- 4x = 20)" which was also poorly done. Eighteen students made the error (reported previously) of detaching the number from the sign (Herscovics & Linchevski, 1994). As well, twenty-one students reported that they could not do the problem or that the problem could not be done. For these students, the number after the equals sign may have given the appearance that closure had already been accomplished (and the presence of the variable x may have confused the students and hampered closure).

Conclusion

The ability to handle a sequence of operations without closure and to study an operation as an object, the same as the result of the computation, are essential for algebra in Year 9. The students in this study were nowhere ready for this. In particular, the learnt response of students to close on operations means that the presence of variables and the layout of many algebra exercises will cause great difficulties. To become ready for algebra, this study suggests that students will have to:

- come to terms with the placement of an equals sign not designating where an answer has to be placed,
- come to terms with the absence of an equals sign not meaning nothing needs to be done, and

- come to understand that an expression involving sequence of operations can be manipulated without closure.

Further the tendency of the students to rush to close terms within brackets even when this was meaningless indicates that the application of BOMDAS without understanding poses problems for algebra study. Finally, it is apparent that many of these students ignored the significance of the negative sign as an operational instruction in the case of the directed number item. That is, they ignored its significance because they were unable to use it as an operational sign in this instance. Clearly, with the misconceptions that many of these students have, it is likely that some of them will struggle when the additional cognitive load of algebraic symbolism is placed upon them.

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