New Research Findings, Ideas, and Techniques in Teaching and Learning Mathematics (Plenary Speech)

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Introduction For this session, I have been thinking of which discipline I should focus my speech. Is it Algebra? Calculus? Or Geometry? Eventually, I decided to talk about the status of geometry today. Specifically, and for many reasons, geometry is alive, well and sound, it has not died. For, it is essential to many other human activities and is so deeply embodied in how people think. I am inspired by my colleague, Walter Whiteley of York University, Toronto, Canada, who gave an independent, but related, description of the decline and rise of geometry through the 20th century [1]. The availability of computers with dynamic graphic capacities and the realization of a variety of ways of learning, our current state of affairs in the field of geometry offers an unprecedented opportunity for geometers and others whose interest is visualization and informal reasoning.

I. The Fall of Geometry: Why and How As the mathematics community knows that a field of mathematics 'dies' when it is no longer considered as an 'important' area of mathematical research. Geometry 'died" in this sense by the mid 20th century in North America and the rest of the world. Over the last few decades, the path of this fall proceeded from the graduate schools to the elementary classrooms passing through the high and middle schools [1 p. 7]. In sum, the decline proceeded from left to right passing through the center! Knowing this path may help us plan strategies for speeding up and accelerating the rise of geometry. "We do not have a half-century to spare for a comparable, gradual 'rise' of geometry".

In a mini history over the 19th and 20th centuries, Philip Davis gave an account on the fall and rise of geometry focusing on a specific field of discrete geometry known as 'triangle geometry' [2]. In this regard, Eric T. Bell states that "The geometers of the 20th century have long since piously removed all these treasures to museum of geometry where the dust of history quickly dimmed their luster" [3, p. 323]. Walter Whiteley states that:

"Discrete geometry virtually died as an 'important' field of mathematical research through the twenties and the thirties and the forties, at least in North America and parts of Europe. It survived in pockets (Hungary, Germany, Switzerland, Austria, Russia ...) and through a few key people in other places (H.S.M. Coxester, D. Pedoe, B. Grunbaum). In the Canadian context, this death was confirmed as Professor Coxeter retired at the University of Toronto several decades ago. The department followed a policy of not hiring in discrete geometry and shifted to the 'hotter' areas such as algebraic geometry" [1, p. 8].

This state of affairs in discrete geometry shows how the situation has been deteriorated. It marked the turning point toward a chain of events that explains how the decline was transmitted down from 'research activities' to the rest aspects of mathematics education. It reminisces the 'domino effect' sequence of events. In particular, as research in geometry declined, the importance of teaching geometry in graduate programs declined, and so the number of faculty proposing courses in geometry declined. As a result, teaching geometry in pre-service teacher education programs declined. After a few decades, there were new graduates moving out to teach undergraduate mathematics who possessed no experience in discrete geometry as an important and vital field of mathematics and who may not studied any geoemetry course. Over a span of a few more decades, the decline of teaching geometry at undergraduate level resulted in having a generation of school teachers who hardly have any geometry within their undergraduate program. Thus, we have teachers in the classrooms who, more likely, will implement their curriculum leaving geometry to the end of the year, a situation often ended into no time left for geometry. Such understanding that geometry is something extra or optional continues to spread wider and wider among curriculum designers, writers, teachers, and the parents.

The message that geometry is not relevant is embedded in the dominant culture in undergraduate mathematics departments and in high school curricula at least in North America today [4, p. 184]. The 'new math' movement started early of the second half of the 20th century, with its emphasis on algebra and set theory, has further deepened this decline in teaching geometry. This classification of mathematics with language and formulas is an attribute of professionals interested in the foundations of mathematics, and of professionals interested in algebra and analysis - this is strictly evident in the Bourbaki school of thoughts for example. The famous urge of Dieudonne for a "strict adherence to the axiomatic methods, with no appeal to the 'geometric intuition', at least in the formal proofs: a necessity which we have emphasized by deliberately abstaining from introducing any diagram in the book" [5 pp. 173-174], [1]. It is no wonder to witness this cultural misconception that geometry is irrelevant in the recent literature of educational psychology and cognitive science. Educational psychologists and cognitive scientists associate 'mathematical intelligence' with numbers and their applications using formulas (e.g. algebra) while 'visual intelligence' with 'art and architect' [6, 7].

II. The Rise of Geometry Nowadays, geometry is not only alive, rather it is rising as an area of research not only in mathematics but in other areas outside mathematics. Geometry is now very active as a field of research. It is rising in several research arenas: (1) Research on Cognitive Mental Processes: Young Children's Spatial Structuring, Concepts of Shape and Area, Visualization and informal Reasoning (2) Research on Applications of Geometry (3) Research on Dynamic Geometry Programs

(1) Research on Cognitive Mental Processes: Young Children's Spatial Structuring, Concepts of Shape and Area, Visualization and informal Reasoning

Out of many research reports in geometry , I chose three more recent research reports:

(a) Michael Battista et al. Michael Battista et al [8] studied the mental processes by which students structure space. The study implicates that studying the processes by which students structure space offers us a *new and powerful perspective on investigating children's construction of geometric and spatial ideas.* The study calls for a reexamining of the traditional treatments of multiplication and area concept. In their word:

In the traditional view of learning, it is assumed that row-by-column structuring resides in 2D rectangular arrays of squares and can be automatically apprehended by all. However, as we have seen in the present study, and consistent with a constructivist view of the operation of the mind, such structuring is not "in" the arrays - it must be personally constructed by each individual. Consequently, traditional instructional treatments of multiplication and area need to be rethought. If students do not see a row-by-column structure in these arrays, how can using multiplication to enumerate the objects in the array, much less using area formulas, make sense to them?

Taking a broader view, we suggest that structuring 2D and 3D space is the foundation for geometric and visual thinking. All of geometry is, in essence, a way of structuring space and studying the consequences of that structuring. We structure space when we organize it by arrays or coordinate systems. We structure space when we conceptualize it in terms of specific shapes (such as lines, angles, polygons, polyhedra) or in terms of geometric transformations. Studying the processes by which students structure space offers us a new and powerful perspective on investigating children's construction of geometric and spatial ideas. [8, p. 531].

(b) Douglas H. Clements et al.

Douglas H. Clements et al. [9] investigated criteria preschool children use to distinguish members of a class of shapes from other figures. They conducted individual clinical interviews of 97 children ages 3 to 6 emphasizing identification and descriptions of shapes and reasons for their identifications. Clements and his associates found that young children initially form schemas on the basis of feature analysis of visual forms. They further indicated that while these schemas are developing, children continue to relay on visual matching to distinguish shapes. They are, however, also capable of recognizing components and simple properties of familiar shapes. This ***new finding*** is an interesting result to geometers and other individuals interested in visualization and informal reasoning. This finding suggests that the van Hiele [10] Level 1 ("visual level") would seem not to be the bottom level (the 'floor' level) within the van Hiele

5 levels of thoughts development in geometry, rather a prerecognitive level exists before van Hiele's level 1 (visualization level) [9, p. 206].

Clements and his associate [9] urge geometers, mathematics educators and specialists in cognitive mental processes to give a close attention to the children early conceptions of geometric shapes. They state:

Descriptions of children's early conceptions of geometric shapes are important not only for theory but also for teacher education (e.g., for cognitively guided instruction models) and for developers of constructivist-oriented curricula. Too often teachers and curriculum writers assume that students in early childhood classrooms have little or no knowledge of even simple shape identification (Thomas, 1982 [11, Author]). Obviously, this belief is incorrect ; preschool children exhibit working knowledge of simple geometric forms (even in the paper-based situations in the present study). Instruction should build on this knowledge and move beyond it. Students fail to reach the descriptive level of geometry in part because they are not offered geometric problems in their early years (van Hiele, 1987 [12, Author]). The "prolonged period of geometric inactivity" (Wirszup, 1976, p. 85 [13, Author]) of the early grades leads to "geometrically deprived" children (Fuys, Geddes, & Tischler, 1988 [14, Author]), p. 208.

Below is a brief reflection on van Hiele's levels of geometric thoughts development:

Pierre Marie van Hiele[10] identified five levels of reasoning which students go through in dealing with geometric concepts and figures. The van Hiele research indicates that these levels (identified as Levels of Thought Development in Geometry) are not biologically achieved during the person's maturation; they can only be achieved by instruction and should be learned in their proper order: Levels 1, 2, 3, 4, and 5. The progress through these levels seems different from that of Piaget levels [15]. As we know from psychology, the Piaget Levels of Cognitive Development occur naturally and progressively during biological maturation. A description of van Hiele five levels can be found in several NCTM publications such the *Mathematics Teacher* (see Shaughnessy & Burger [16] and the *NCTM 1987 Year Book* (see, M. Crowley [17]). A description of these levels may be helpful. Level 1: students identify and operate on shapes in their global appearances (holistic); Level 2: students recognize shapes by their properties (part-whole); Level 3: students recognize relationships among properties and shapes (part-part; and, whole-whole); Level 4: students understand the deductive reasoning process; Level 5: students can work in different axiomatic systems.

(c) Medhat H. Rahim et al. Medhat Rahim and his associate Alton Olson [18] conducted an investigation (at grade 8 level) for the purpose of identifying and explain students' geometric thinking processes in judging and verifying, visually and through hands-on manipulation, whether or not area equivalent polygonal regions of different shapes are 'congruent by pieces'. The process of cutting and covering of cardboard models of geometric shapes were used by the students.

The 'congruence by pieces' or 'piece-wise congruence' concept was adopted by the investigators and used as an extension of the rigid shape congruence concept. The reason for using congruence by pieces lies in the relationship between congruence and area equivalence of two geometric figures. The relationship between congruence and equivalence is somehow restricted; it is a 'one-way road' relationship. For, a congruence of two identical polygonal regions implies equivalence in the area they occupy, but the reverse is not necessarily true. That is, two area equivalent polygonal regions are not necessarily congruent unless they are of identical shapes. On the hand, two polygonal regions of different shapes such that one of them can be cut into a few pieces to completely cover the other region, are necessarily area equivalent. Conversely, two area equivalent polygonal regions of different shapes are necessarily congruent by pieces or piece-wise congruent. In other words, there exits a way of ***dissecting*** one of them into few pieces so that, through ***motions,*** they can be fit together to completely cover the other region . The operations involved are (a) dissecting a polygonal region into pieces and (b) moving the pieces around. It is then a combination of more than one operation referred to as ***dissection-motion operations*** or DMO and that constitutes the extension. The relationships among these three ideas are shown in Figure 1 below.

Congruence -------------> Equivalence
Piece-to-piece Congruence <-----> Equivalence

Figure 1

Each student was given 8 tasks, one at a time, in a one-on-one interview . The 8 tasks labeled as "equivalence/piece-to-piece congruence"; consist of 8 pairs of different polygonal regions with equal area. The two regions in each pair were made of cardboard of different colors. A pair of scissors, a pencil and a ruler were available for the students. One of the regions in each pair was a rectangular region equal in area to the other region. The second region was one of the following: right triangle, acute triangle, parallelogram, right trapezoid, non-isosceles trapezoid, quadrilateral with no specific property, rhombus, and a regular pentagon.

The students responses on these tasks suggest that the tasks were processed through a particular sequence of strategies that constitutes a general emerging pattern. The emerging pattern has the following sequence of events:

superimposing one region on the other for a partial cover ---> attempting to find a linear congruence among the edges and/or angular congruence among the angles ---> dissecting one region along an edge of the other ---> holding on the initial partial cover and moving the other piece around to cover the remaining part of the other region ---> and the above process starts again till the completion of the task [18, p. 386].

Through dissecting area equivalent shapes for regional congruence, the data suggest that polygonal regions were directly compared through matching sides, angles, and sub regions. Whenever students performed a task successfully they responded that the two regions were: "equal", "congruent", "the same." And when they were asked "why they are equal? Congruent? The same?" Their answer was "Because they fit." These responses suggest that a regional sameness or congruence of polygonal regions seemed to have been visualized as a process of fitting a region on the other through cutting. Further, the students responses revealed that they were led by principles that reflect knowledge of component parts of polygonal regions. The dominant regional congruence justification of grade 8 students for two equiareal polygonal regions with different shapes was superposition with linear and/or angular congruence of one region on the other. This would resemble an extended analogy to Beilin's [19] conclusion on the congruence-justification strategy of 7- and 8-years-olds where superposition of one region on the other was the dominating pattern of the children's responses. [18, p. 387].

Elsewhere, Medhat Rahim, Daiyo Sawada, and Joanne Strasser [20] highlighted that understanding geometric shapes and the many relationships connecting shapes with one another is a major component of spatial awareness: to know how a particular shape can arise or be created from others provides a way for learners to understand the visual aspects of their everyday experience. The NCTM Standards [21] put it this way: In grades k-4, the mathematics curriculum should include two- and three-dimensional geometry so that students can (a) describe, model, draw, and classify shapes; (b) investigate and predict the results of combining, subdividing, and changing shapes. With these goals in mind we created a set of simple geometric materials designed to facilitate student exploration of shapes undergoing transformation from one configuration into another. During the transformation process, a major attribute of the shape would remain invariant. If time allows, I would like to share some classroom research episodes in which children explore what we eventually called "shape transforms". As a story line within the lesson we focus on "the boy with the ruler". In doing so we highlight how an overemphasis on quantitative aspects can sometimes divert from the development of qualitative geometric understandings. [20, p. 23].

(2) Research on Applications of Geometry

Many current applications in industries and computer science are deeply rooted in geometry; they have solid geometric components. Very often, the problem under consideration involves getting geometric information into a computer and the output, a solution, will be in a visual form, as a figure to entertain, design to build, or action to execute. Each of these possibilities requires substantial knowledge and the computers users would benefit understanding the outcomes and the geometry behind them. There are many application areas where geometry is extremely involved as a backbone:

(a) Computer Aided Design and Geometric Modeling

For example, the most resent Boeing plane was entirely designed inside a computer with no physical models involved [1].

(b) Robotics

The problems of operating and control the robot have generated a major area of research known as 'computational geometry' with many books and new results, [22].

(c) Medical Imaging

This field of computer applications (e.g. ultra-sound and MIR devices) is dense with geometric problems, research, and new results in fields such as geometric tomography [23].

(d) Computer Animation

(e) Computer Visual Presentations

(f) Linear Programming.

All these fields of research activities do need geometry; geometry is out there to serve and is essential for many current applications in science and in technology.

(3) Research on Dynamic Geometry Programs

There are several products of software known as tools for dynamic geometry such as Geometer's Sketchpad, Cabri Geometry, and Cinderella. The development of dynamic geometry programs such as these is decisively changing what teachers and hence students would do when solving geometric problems. This has opened a wide field for learning, teaching, and research in dynamic geometry [24, 25, 26].

Epilogue

"... **the universe stands continually to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wondering about in a dark labyrinth." -- Galileo.**

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