Some Valuable and Applicable Mathematics.

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Abstract: Mathematics that can be presented with some historical introduction, and that may be useful in the real world, often helps the enquiring mind want to delve deeper into the finer points of the workings of the queen of the sciences. With this in mind we present some examples of mathematics that may help to spark enthusiasm for the enquiring mind

Numbers In this vignette we look at some ideas dealing with numbers. The early Greeks were fascinated with numbers and discovered many of their properties. For Pythagoras, numerical perfection depended on the numbers divisors and he defined a *perfect* number, as a number n whose divisors, less than n add up to the number n itself. For example:

$$6 = 1+2+3, \quad \text{and } 28 = 1+2+4+7+14.$$

The perfection of the numbers 6 and 28 were also acknowledged in other cultures; 6 as the number of days taken by God to create the earth and 28 as the moons orbit around the earth. In *The City of God*, St. Augustine argues that although God could have created the world in an instant he decided to take 6 days in order to reflect the universe's perfection. Other perfect numbers are:

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248,$$

$$8128 = 1 + 2 + 4 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064,$$

33,550,336 and 8,859,869,056. Other perfect numbers become harder to find as the counting numbers get bigger.

Prime numbers are defined as those, which may be divided by itself and one; for example 2,3,5,7,11,13,17,19,23,….. Marin Mersenne (1588-1648), a French monk announced in 1644 that if p is a prime number, then $M_p = 2^p - 1$ is also prime. M_2, M_3, M_5 and M_7 are all primes, however M_{11} $=2047=23\times 89$, which is not prime. The 38th and largest known prime is $2^{6972593}$-1 which is a number with 2,098,960 digits. Hajratwala, Woltman and Kurowski using GIMPS, Greatest Internet Mersenne Prime Search, obtained the 38th prime in June 1999. This is a search that is available on the Internet and we can, if we wish all become Internet prime number detectives. Hajratwala et. al., will pocket a US$50,000 prize for the discovery of the 38th prime. The prize is being offered by the EFF, Electronic Frontier Foundation and larger primes will earn an award of up to US$250,000. For more details on prime numbers you may visit the site

http://www.utm/research/primes/mersenne.

Cristian Golbach (1690-1764) an amateur Prussian mathematician living in St. Petersburg claimed that 'Every even number greater than two can be expressed as the sum of two primes'. For example

4=2+2,…,42=5+37, 11+31, 13+29, 19+23,…

Goldbach stated his conjecture in a letter to L. Euler, in 1742 who at first regarded the conjecture as trivial. Goldbach's conjecture remains unsolved to this day. For more details, see the site http://www.plus.maths.org/issue2/xfile

In conjunction with the publication of the book 'Uncle Petros and Goldbach's Conjecture' by Apostoles Doxiadis, Faber and Faber Publishing and the Bloomsbury Publishing Company have offered US $1,000,000 for a proof of the Goldbach conjecture. Good Luck. You may see the announcement of March 26, 2000 at

http://www.seattletimes.nwsource.com/news/nation-world/htm198/math26-20000326.html

Another interesting problem dealing with numbers has been Fermat's theorem. Fermat (1601-1665), a brilliant French amateur mathematician was well aware of a result given by Pythagoras some 2200 years earlier, viz: that there exist integers x, y and z such that $x^2 + y^2 = z^2$, some numbers are $3^2 + 4^2 = 5^2, \ \dots\ , 99^2 + 4900^2 = 4901^2, \dots$

If the power of 2 was changed to 3, then it seemed that $x^3 + y^3 = z^3$ did not hold for integers, Pythagorean triples {*x*, *y*, *z*}, indeed generations of mathematicians failed to find such triples. Fermat claimed $x^n + y^n = z^n$ has no whole number solutions for *n* integer and greater than two, in fact he had a proof but never recorded it on paper. Fermat's 'marginal note' appeared in his edited works, published by his son, Simon, in 1670. "*It is impossible to divide a cube into two cubes, a fourth power into two fourth powers, and in general any power except the squares into two powers with the same exponent,…. I have discovered a truly wonderful proof of this, but the margin is too narrow to hold it.*" Fermat's claim was eventually proved by Wiles and published in 1995.

Many famous and some not so famous mathematicians before Wiles had tried, but failed, to prove Fermat's conjecture, however, at least in one case Fermat's problem saved the life of one amateur mathematician. Paul Wolfskehl, a wealthy German industrialist spurned by a beautiful woman, decided to do away with himself. Wolfskehl, a passionate but not impetuous man planned his death with specific detail, at the stroke of midnight the desperate measure was to be enacted. German efficiency shone through and Wolfskehl completed his gruesome preparations well ahead of the midnight deadline. To while away the hours before midnight he wandered to his library and began examining Kummer's paper explaining the shortfall of Cauchy and Lamè's arguments on the proof of Fermat's theorem. Wolfskehl was surprised when he found what he thought was a gap in the logic, he became engrossed in the problem and by dawn had remedied the gap. Despair and sorrow dissipated and Wolfskehl, thanks to mathematics, found a new desire for life. In 1908 Wolfskehl died and he bequeathed a reward of 100,000 marks to whomsoever could prove Fermat's last theorem. Giles collected the Wolfskehl prize in 1997 worth $50, 000. A fuller account of the Fermat saga may be read in Simon Singh's brilliant book 'Fermat's Enigma', published in 1998.

In this talk, time permitting, I will give some more intriguing properties of numbers.

Modelling In many cases it may be possible to introduce a mathematical idea based on a real physical situation already appreciated by the student. A mathematical model may be built up which describes a situation and then extended and tried in different situations.

Mathematical ecology, for example, is the study of various species in relation to their environment, of the competition for resources within and between species, of the poisoning or polluting effects on the environment due to metabolic action of the species and so on. Since rates of change of population sizes are involved, the mathematical models generally consist of differential equations or systems of dfferential equations. Differential equations are also used extensively in biology, economics, chemistry and the social sciences. Population dynamics, for example, can be fruitfully used to motivate many types of differential equations and has advantages in that population models are relatively easy to formulate, and the topic itself is current. I would now like to discuss some single species and two species interactions in population models.

Single Species Interaction

The growth of human populations, as well as that of crystals, plants, animals and lower organisms presents a characteristic pattern which suggests that they may be described by a single differential equation. The British economist Thomas Malthus (1766-1834) observed that biological populations, including human ones, tended to increase at a rate proportional to the population size.

Consider a small time interval, δt, in which births and deaths are assumed to be proportional to the total population size and to the time interval. Hence, if $N = N(t)$ is the population size, in a time interval δt there are $a.N.\delta t$ births and $b.N.\delta t$ deaths, where a and b are assumed to be positive constants, and the increase in population size in time δt is

$$ä\, N = a \cdot N \cdot ä t \quad - \; b \cdot N \cdot ä t$$

Dividing both sides by δt and taking the limit as δt 0 we have that

$$\frac{dN}{dt} = cN, \quad \text{where } c = (a - b).$$

This is a first order separable differential equation, and the solution is

$$N = N_0 \exp(ct) \tag{1}$$

where N_0 is the population size at $t = 0$.
Depending on the sign of c there is exponential growth, steady state or exponential decay.

It was estimated that the earth's human population in 1962 was 3060×10^6 and that during that decade the population was increasing at a rate of 2% per year. Based on these statistics, equation (1) predicts that the earth's population will be 2×10^5 billion in the year 2670. The total surface area of this planet is approximately 1.67×10^5 billion square metres, and eighty percent of this surface is covered by water. Assuming that we are willing to live on boats as well as land, we can see that by the year 2510 there will be 0.86 square metres per person, by 2635 each person will have only 0.093 square metres on which to stand, and by 2670 we will be standing two deep on each others shoulders. Malthus reasoned that such an exponential growth of the world's population could not go on indefinitely and therefore some sort of catastrophe (such as wars) must intervene from time to time.

On a lighter note we address ourselves to the question, "What was the world's population when Cain killed Abel?". In *Gen.* 4:4-15 Cain expresses his fear that everyone he meets will want to slay him. Now Cain and Abel were the first born of Adam and Eve and therefore the world's population must have consisted of no more than Cain, his parents and perhaps a few brothers and sisters. Who then would have tried to kill him? The answer is to be found in Saint Augustine's book 15 *The City of God*. We know from chronology supplied in *Genesis 4* and *5* that Abel was murdered about the year 129, and we know from 5:4 that Adam begat sons and daughters. If we assume that Adam and Eve had a child each year, the sexes of which were equally divided, by the year 129 there would have been 129 offspring. If there were no deaths before Abel was murdered and brothers and sisters married and had one offspring per year from the age of say 18, then we can calculate that at the time Cain killed his brother, Adam and Eve had more than 3,000 grandchildren and more than 90,000 great-grand-children. Add to this number the great-, great-, and great –grandchildren, we have that the world's population at the time of the murder was approximately 500,000.

It appears then that population growth is satisfactory as long as the population is not too large, when the population gets very large, then the Malthusian model cannot be accurate since it does not reflect the fact that individual members are now competing with each other for the limited living space, natural resources and food availability. The Belgian mathematical biologist, P.F. Vehulst (1804-1849), modified the Malthusian model by incorporating the "Competition" term $-bN^2$, where b is a positive constant, and hence the growth model is now

$$\frac{dN}{dt} = cN - bN^2 . \tag{2}$$

This differential equation is known as the logistic law of population growth and the constants b and c are called the vital coefficients of the population. Equation (2) is separable which can be integrated by resorting to partial fraction decomposition, and the solution becomes

$$N = \frac{(c/b)N_0}{N_0 + (c/b - N_0)\exp(-ct)}$$

The nature of the solution curves depends on the sign of $c/b - N_0$.

We may observe that from (2), if $0 < N < c/b$ then $\dot{N} > 0$, so that the population is increasing, and if $N > c/b$ then $\dot{N} < 0$, so the population is decreasing. This is an example of a *stable equilibrium*. If the population is not at equilibrium, it is moving towards it. Equation (2) may be differentiated implicitly, in which case

$$\frac{d^2N}{dt^2} = \dot{N}(c - 2bN)$$

from which we see that there is an inflection point at $N = c/2b$, and the growth rate is maximal when the population reaches half its maximum supportable level. It is interesting to note that equation (2) raises a possibility that does not occur with the Malthusian model and, that is, that for $\dot{N} = 0$ where $N = c/b$ we have a non-zero equilibrium population. Technological developments, energy limitation, pollution considerations and sociological trends have significant influence on the vital coefficients b and c, therefore they must be re-evaluated every few years. However, we know that the logistic population model does not give a good description of the world's population. In case of pollution, the population model is modified to

$$\frac{dN}{dt} = cN - bN - aN\int f(t-T)N(T)dT,$$

where $f(t)$ is a certain function.

Equation (2) is also a simple mathematical model to describe how: A rumour spreads through a community; An innovation or technological change spreads in an industry; and An infectious disease spreads through a community.

For this case the model unfortunately predicts that all susceptible people in the community will eventually catch the disease. This is not a realistic result for many diseases, since it does not include the possibility of only a small number of susceptibles catching the disease. Consequently, more sophisticated models of the spread of infectious disease involving systems of non-linear differential equations have been developed. These models allow for the removal of infectives from the population by death, subsequent immunity, and so forth. A controversial model for world population growth was put forward by Foerster, Mora and Amiot [1] in an article entitled *Doomsday: Friday, 13 November A.D. 2026*, in which they consider the differential equation

$$\frac{dN}{dt} = aN^{1+1/k}$$

with a & k constants. This model also has deficiencies, some of which are noted in an article by J. Serrin [2].

Two Species Interactions

When a number of living species co-exist in a restricted environment, a variety of mutual relations are possible. Two species for example, may form a mutually beneficial partnership with each other. We call such an association *symbiosis*, an example of which is the union between the hermit crab and sea anemones. On other cases one species may live at the expense of the other, we call this *parasitism* or *antagonistic symbiosis*, an example of which is the cat-mouse relationship.

The incessant competition between species in the struggle for existence has been the subject of extensive study by both mathematicians and biologists.

Prey-Predator Model

The Italian biologist U. D'Ancona, during the mid-1920's, was studying the population variations of various species of fish that interact with each other. In the course of the research, he came across some very odd data, for the years 1914-1923, on percentages of total catch of selachians during World War I. The increase in the percentage of selachians he thought, was due to the greatly reduced level of fishing, but how does the intensity of fishing affect the fishing populations? The answer to this question was of great concern to D'Ancona and to the fishing industry. Now, what distinguishes the selachians from the food fish, is that the selachians are predators, while the food fish are prey; the selachians depend on the food fish for survival. After exhausting all possible biological explanations of this vexing question, D'Ancona consulted his colleague Vito Volterra who formulated a mathematical model of interaction and completely solved the question of D'Ancona. Volterra concluded that a moderate amount of fishing actually increases the number of food fish, on the average, and decreases the numbers of selachians. Conversely, a reduced level of fishing increases the number of selachians, in the average, and decreases the number of food fish.

Volterra began his analysis of this problem by separating all the fish into the prey population $x(t)$ and the predator population $y(t)$. He reasoned that food fish do not compete very intensively among themselves for their food supply since this is very abundant, and the fish population is not very dense. In the absence of selachians, the food fish grow according to the Malthusian law of population growth $\dot{x} = ax$, where a is a positive constant. The number of contacts per unit time between predators and prey is bxy, where b is a positive constant. Hence

$$\frac{dx}{dy} = ax - bxy.$$

Similarly, the predators have a natural rate of decrease $-cy$ proportional to their present number, and that they also increase at a rate dxy proportional to the present number y and their food supply x, where c and d are positive constants. Therefore

$$\frac{dy}{dt} = -cy + dxy.$$

Hence the autonomous system of differential equations

$$\begin{aligned} \frac{dx}{dy} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \tag{3}$$

governs the interaction of the selachians for food fish in the absence of fishing. To include the effect of fishing on our system (3) would complicate the equations and be too difficult for us to analyse, nevertheless, the same sort of conclusion follows from system (3) and has the benefit of being easier to analyse.

We may see that this Volterra model leads to a prediction of population cycles about the equilibrium point (c/d , a/b).

It is instructive to set up difference equations for system (3) and solve them on a computer and plot the trajectories. An example of this for a fox-rabbit ecological model can be seen in an article by Sofo [3].

System (3) is a simple model of two species interactions and would therefore need modification to take into account internal competition of the prey and/or predators and any other peculiar behaviour of the particular populations.

Model for Competition

Two species of animals have populations $x(t)$ and $y(t)$ on a particular island. These two species are in competition with one another because they consume the same food supply. For actual colonies of animals, the quantities $x(t)$ and $y(t)$ would be limited to positive integers, but, if they are taken as percentages of an initial large population, then $x(t)$ and $y(t)$ can be regarded as continuous variables. This struggle between two species with similar habits is often observed in nature and is known as the principle of competitive exclusion. The simple model which describes this phenomenon is

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= cy - dxy \end{aligned} \tag{6}$$

where a, b, c, d, are positive constants, $x(t)$ is the prey population and $y(t)$ is the predator population. In this model, the two species expand in a Malthusian manner in the absence of the other, but due to competition for resources, contact between the two species is detrimental.

This competition model leads, by analysing (6), to the conclusion of the possible extinction of one or the other of the populations, whereas in the Volterra model we have the phenomenon of the population cycles.

Many population models of this nature exist whether they be for insect pests or grazing systems of human host-parasite systems. Differential equations are not exclusive only in the field of population dynamics, but also in a diversity of other areas, for example, in fluid flow, astrophysics, heart beat and nerve impulse, relativity theory, national economics, sociology and a mathematical theory of war.
Some internet sites on applicable mathematics are:

http://www.cs.uidoho.edu; http://www.library.thinkquest.org/

http://www.sigmatc.org/amsci/; http://encarta.msn.com/

http://archives.math.utk.edu/; http://www.mathpages.com/

http://www.seanet.com/wksbrown/; http://www.univi.ac.at/

The consortium for mathematics and its applications is founded on the principle that mathematics is a necessary tool for understanding the physical and social worlds in which we live. It presents mathematics in the way we actually use mathematics in real life

http://www.comap.com/

REFERENCES

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