A Picture is Worth a Thousand Words

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Abstract: The well-known author of the Mathematical Recreations column in *Scientific American*, Martin Gardner, wrote that 'a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance' (Gardner 1973). However visual methods of problem solving or of illuminating mathematical results are all too rare occurrences in school mathematics. This paper argues that visual thinking should be an integral part of students' mathematical experiences, and discusses its importance in developing algebraic understanding, in providing a powerful problem-solving tool, and in valuing a variety of learning styles. It includes examples of visual thinking from across the secondary school mathematics curriculum, and discusses some ways in which teachers can develop students' capacity to think visually.

Why visual thinking?

Visual thinking has always been an important part of the thinking of mathematicians (Hadamard 1945), but perhaps less so an integral part of school students' mathematical experiences. It was the subject of some discussion in the mid 1980s, and again in the early 1990s as neuro-psychologists looked at the functioning of the brain. In the current educational climate there are at least three reasons to re-evaluate the role of visual thinking in school mathematics. The first is that the current trend that identifies mathematics with the study of patterns, together with the ready availability of hand-held technology that will easily develop a general rule for a given pattern, has the potential to devalue algebraic thinking. The second is that visualisation can often provide simple, elegant and powerful approaches to developing mathematical results and solving problems, in the process making connections between different areas of mathematics. The third is the importance of recognising and valuing different learning styles, and of helping students to develop a repertoire of techniques for looking at mathematical situations. This may well be a significant challenge to teachers of mathematics who, as successful students of mathematics at school and tertiary level, almost instinctively opt for a verbal-logical style of thinking, which may not always be the most effective in solving some mathematical problems.

Visual algebra: a study of patterns

Mathematics has been described as the study of patterns (Steen 1990). Nowhere is this emphasis on patterns more evident than in recent approaches to the learning of algebra. Texts and curriculum documents abound with examples of patterns involving matchsticks and various arrangements of dots or squares. The clear aim of these pattern-generalisation examples is to develop students' algebraic thinking. Yet there is a real danger that students may miss the point and fail to develop the generalised thinking these exercises attempt to develop.

Geometric patterns

I well remember a session on matchstick patterns at a previous conference. The presenter showed how he helped his students to observe and generalise a pattern of squares made from matchsticks (Fig 1).

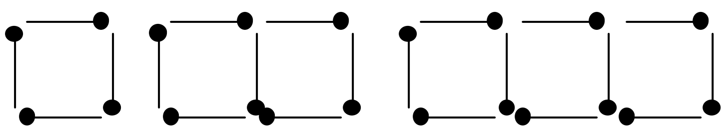


Figure 1: A matchstick pattern

As in most of these questions the students began by constructing a table of values (Table 1) showing the number of matchsticks required for varying numbers of squares.

Number of squares	1	2	3	
Number of matchsticks	4	7	10	

Table 1

They then adopted the problem-solving strategy of 'look for a pattern'. Typically the students observed that the number of matchsticks required increased by 3 each time. As a well-meaning and helpful teacher, the presenter went on to explain how this observation could help students to induce that the relationship between the number of matches required and the number of squares to be constructed was linear, and of the form $3s + k$, where k was some number that made it work. The

students then used substitution to find that $k = 1$ gave correct values for the number of matchsticks.
The advent of graphics calculators makes such a process even easier. It is now a very simple matter to generate a list showing the data collected, then plot a graph, and perform a regression to obtain the equation expressing the relationship.

In these inductive approaches the endpoint seems to be the development of an algebraic relationship, rather than the development of a sense of generality. Stacey (1989) identifies two problems with such an approach. The first is that of false proportionality where students see the construction as a whole and assume that, for example, the number of matchsticks required for the tenth pattern must be five times the number required for the second. The second, and perhaps more insidious, is a focus on the recurrence relationship rather than the functional relationship. In such a pattern-spotting activity, algebra neither illuminates nor provides a means for validating the functional relationship generated (Noss, Healy and Hoyles 1997). Furthermore there is a real danger that the very nature of the mathematical process itself may be misunderstood when numerical properties alone are used to construct general results.

"… attention tends to become focused on the numeric attributes of the output. Worse still, school mathematics becomes constructed – by students and teachers alike – as a stereotypical data-driven 'pattern-spotting' activity in which it is acceptable to search for relationships by constructing tables of numeric data without appreciating any need to understand the structures underpinning them." (Noss, Healy and Hoyles 1997). Rather than fixating on one variable and using some form of algorithm to generate a functional relationship, powerful algebraic thinking arises when students attach meaning to variables, and visualise the relationship in a number of different ways (Thornton 2000). For example the relationship can be visualised as one matchstick to commence the pattern and three for each square [$m = 1 + 3s$], four matchsticks for the first square and three for each subsequent square [$m = 4 + 3(s - 1)$], a row of matchsticks across the top, a row along the bottom and a set of vertical matchsticks with one additional matchstick to complete the pattern [$m = s + s + (s + 1)$], or as four matchsticks per square with all but one of the vertical matchsticks removed to eliminate overlaps [$m = 4s - (s - 1)$]. The equivalence of these different algebraic expressions is an obvious outcome of such a visualisation activity.

Number patterns

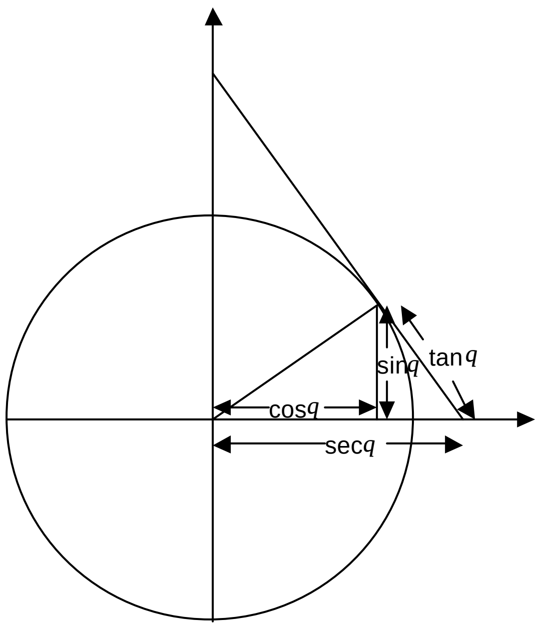
My other vivid conference recollection is of a session at a recent National Council of Teachers of Mathematics Conference in Chicago, advertised as being about graphics calculators and number theory. The presenter had developed some ideas involving graphics calculators to help prospective secondary teachers to look at some ideas in number theory. He started with some number patterns involving square numbers and asked us to generalise the result:

$1.3 + 1 = 4 \quad 2.4 + 1 = 9 \quad 3.5 + 1 = 16$

He then asked us to prove the generalisation. As successful verbal-logical thinkers, the people in the group were quick to formulate an algebraic proof of the generalisation $n(n + 2) + 1 = (n + 1)^2$. However, to try to make the problem a little more mathematically interesting, I drew a dot picture (Thornton 2000) representing a rectangle n by $(n + 2)$ with an additional square [$n(n + 2) + 1$], and showed how it could be rearranged, by changing the bottom row into a column and adding the single dot, into a square pattern representing $(n + 1)^2$. The session leader asked whether participants felt that the dot picture was really a proof. Regrettably, many felt it did not, and believed that, for a proof to be valid, it had to conform to conventions of layout and notation. Eisenberg and Dreyfus (1991) report similar criticisms levelled at visual proofs during a session at the Sixth International Congress on Mathematics Education (ICME-6) in Budapest in 1988.

The mathematical power of visual thinking If one accepts that the purpose of proof is to illuminate a mathematical result, then the dot picture certainly shows the result in a different light. It develops the connection between the process of multiplication and the number of elements in a rectangular array, and shows that the array can be transformed by manipulating parts of it. To generate and understand such proofs the student needs to be able to see parts of the figure as entities in themselves that can be transformed and manipulated as necessary. In the dot picture proof, for example, students need to be able to take one row of dots and transform it into a column, in the process transforming a rectangle into a square with missing corner. The capacity to operate on mathematical entities as objects in their own right is the essence of algebraic thinking. In his delightful book *'Proof Without Words'*, Nelsen (1990) provides some 100 visual proofs of results from number, algebra and geometry. It is interesting that many of these proofs were originally published as space-fillers in the journal *Mathematical Magazine* and the *College Mathematics Journal*, perhaps disregarding their mathematical significance and aesthetic value. Shear (1985) stresses the importance of studying trigonometric functions using visual methods on a unit circle, and of using visual as well as analytical methods to find elegant methods of solving problems. He uses as one example the proof of the identity $(\sec q - \cos q)^2 = \tan^2 q - \sin^2 q$. While this is not a difficult identity to verify analytically, it lends itself to a visual proof using a unit circle (Figure 2).

The importance of visual thinking in mathematical discovery is graphically illustrated in the work of Hadamard (1945), himself a mathematician of some note, who surveyed many other mathematicians and scientists, asking them about their thought processes as they solved problems or investigated new ideas. He identified a remarkable consistency in the way in which leading mathematicians used images to develop their thoughts, only resorting to more formal algebraic conventions when they wished to communicate their results with others. Among those interviewed by Hadamard were Henri Poincaré and Albert Einstein, who later wrote of their thought processes when engaged in mathematical and scientific discovery.



A simple application of Pythagoras' Theorem reveals visually that $(\sec q - \cos q)^2 = \tan^2 q - \sin^2 q$.

Figure 2: A visual approach to a trigonometric identity

'(Poincaré perceived) mathematical entities …whose elements are harmoniously disposed so that the mind without effort can embrace the totality while realising their details.' (Poincaré 1968)

'(For Einstein) words or…language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements of thought are certain signs and more or less clear images…The above-mentioned elements are, in my case, of visual and some of muscular type.' (Einstein 1979)

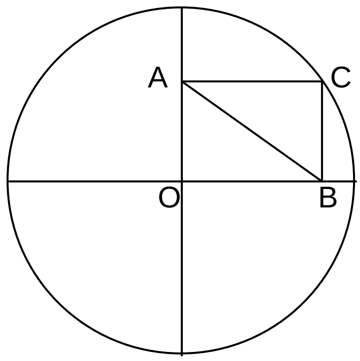
Valuing different learning styles

Krutetskii (1976) surveyed a large number of students aged 10 to 14, and identified two distinct types of problem solvers: verbal-logical and visual-pictorial. Verbal-logical thinkers had no need of diagrams, and attempted to solve all problems algebraically, even if a visual representation was available. On the other hand visual-pictorial thinkers tried to form a picture, even when it was unnecessary. Krutetskii further categorised some students as being harmonic thinkers, able to think both ways.

Moses (1982) found that the problem-solving performance of fifth grade students improved significantly following a course in visualisation. She asked them to try to feel a part of the situation being considered, and to identify with the people or elements involved in the situation. She identified seeing, imagining and designing as three overlapping strategies that helped students to obtain a gestalt picture of the entire situation. The solutions developed through visualisation then made sense in the context of the original problem, rather than being the result of a formal, and potentially meaningless, mathematical operation. Campbell, Collis and Watson (1995) affirmed the role of concrete-pictorial imagery in motivating students, in helping them to clarify the structure of the problem, and in assessing the reasonableness of their results. This was particularly the case for students who struggled to understand mathematical concepts.

Presmeg (1992) identified five categories of imagery: concrete-pictorial imagery, pattern imagery, memory images of formulae, kinaesthetic imagery and dynamic imagery. She described the importance of pattern imagery to expert chess players. When presented with an arrangement of chess pieces arising from an actual game, these experts were able to reproduce the situation from memory after a short exposure to the arrangement. Their performance was significantly better than non-chess players. However, when presented with a random arrangement of chess pieces, experts were unable to reproduce the situation any better than non-chess players. The development of a similar mathematical imagery, in which students are able to focus on relationships and patterns, is surely one of the principal goals of mathematics education.

The larger the repertoire of strategies available to students, the more likely they are to be successful problem-solvers and to develop a deep understanding of mathematical concepts. Consider, for example, the problem illustrated in Figure 3, and used by Krutetskii (1976) as an evaluation item in his research.



AOBC is a rectangle, inscribed in one quadrant of the circle, centre O. The circle has diameter 2cm. Calculate the length of OC.

Figure 3: A problem from Krutetskii

Ira, a verbal-logical thinker, tried for a long time to solve the problem. She tried many different positions of AB and used analytical geometry to try to find a generalisation. It was only when the experimenter advised her to construct the radius OC that she found a simple visual solution.

In recent times a significant amount of research has been devoted to a study of how the brain functions. While historically such research was used to justify theories of innate ability differences between races and genders, it has great potential for informing the learning of mathematics. Sword (2000) describes the problems experienced by highly capable visual thinkers who may be 'at risk' in the school system because their learning style is not recognised. She maintains that traditional teaching techniques are designed for auditory-sequential learners, and hence disadvantage visual-spatial learners. Material introduced in a step by step manner, carefully graded from easy to difficult, with repetition to consolidate ideas, is not only unnecessary for the visual-spatial learner, but, by failing to create links in a holistic picture, actively works against such students progressing to their potential. As a result gifted visual-spatial learners often exhibit characteristics such as lack of motivation, inattentiveness, weaknesses in basic calculations, and disorganisation.

Visual-spatial and verbal-logical learning styles are associated with 'right brain' versus 'left brain' thinking. The characteristics of these two learning styles are summarised in table 2.

The common advice to students to read problems carefully, break them down into manageable steps, formulate algebraic expressions representing the situation, and then solve this formalised version of the original problem, could thus prove quite counter-productive for

Left brain thinking	Right brain thinking
Verbal	Visual-spatial
Analytical	Synthetic
Symbolic	Concrete-pictorial
Logical	Intuitive
Sequential	Multiple processing
Linear	Gestalt, holistic
Conceptual similarity	Structural similarity

Table 2. Left brain and right brain thinking (adapted from Tall 1991)

some students. Seldom do we ask our students to step back from the problem, to look at it holistically, and to try to visualise the situation in its entirety. As Leslie Hart vividly describes, the school curriculum is not designed for students who think in 'right brain' ways.

"We have all been brainwashed by the undeserved respect given to Greek-type sequential logic. Almost automatically curriculum builders and teachers try to devise methods of instruction, assuming logical planning, ordering and presentation of content matter…They may have trouble conceiving alternative approaches that do not go step by step down a linear progression…It can be stated flatly, however, that the human brain is not organised or designed for linear, one-path thought." (Hart 1974)

Promoting visual thinking in the classroom

Zimmerman and Cunningham (1991) note that our use of the term visualisation in mathematics is not the same as the everyday use of the term. It does not equate to just forming a mental image. Rather it is about visualising a concept or problem rather than an idea. So the visualisation can be on paper, or using computer graphics. Nemirovsky and Noble (1997) describe visualisation as the means of travelling between external representations and the learner's mind. Presmeg (1992) describes a continuum of visual imagery from concrete to abstract, and discusses the importance of students developing abstract pattern and dynamic imagery.

What is of crucial importance, then, is to promote flexibility of thinking, and to encourage students to look for the connections between alternative representations of mathematical entities. Noss (1997) described mathematical thought as being characterised by the capacity to move freely between the visual and the symbolic, the formal and the informal, the analytic and the perceptual and the rigorous and the intuitive. Brieske (1984) maintains that the transition from algebraic to geometric thinking and vice versa serves to significantly deepen students' understanding of underlying concepts.

The following suggestions are intended to be starting points for teachers to help students to become more effective visual thinkers.

Be sensitive to the possibility of finding visual solutions or representations of a given result.

Encourage concrete-pictorial imagery by asking students to picture themselves as part of the situation.

Encourage pattern imagery by connecting results in number and algebra with models such as area, length and arrays.

Encourage dynamic imagery by using software such as Cabri or Geometer's Sketchpad. Noss (1995) describes the development of a dynamic algebra program in which students were asked to construct geometric patterns by focusing on their algebraic properties. My Dynamic Visual Algebra Web site (Thornton 2000) provides a number of examples of dynamic imagery to illustrate results from number and algebra.

Promote discussion of alternative ways of thinking, and particularly of the transition from visual to symbolic.

Encourage students to look at problems holistically instead of breaking them into parts.

Draw three diagrams (a special case, a general case and a counter-example) when tackling geometric proofs. Ask why the result is true in the special case, whether it is still true in the more general case and why it is not true in the counter-example. Noss (1997) notes that diagrams have a tendency to take on a ritual character as mere appendages, particularly in geometric problems. The introduction of diagrams which illustrate a specific case and a counter-example can help focus attention on the key aspects of the geometric relationship, and make the diagram an integral part of the solution process. It may also help to avoid the pitfalls of metonymy (Presmeg 1992), in which students fail to recognise an object when it does not conform to their mental prototype, or in which students introduce extraneous properties by only considering a specific case.

Conclusion

Mathematical power involves the capacity to make connections, both between mathematical objects and concepts and between mathematics and the physical world. Visual thinking, whether in the form of concrete images, pattern images or dynamic images, has a key role to play in the development of students' mathematical power.

Visual thinking on the Internet

The outer angles of a polygon is a dynamic Java applet that beautifully illustrates the theorem that the exterior angles of a polygon sum to 360°. http://www.ies.co.jp/math/java/geo/gaikaku/gaikaku.html (accessed 24 July 2001)

Galton's Board illustrates a binomial distribution via a Quincunx, in which marbles are dropped through an array of pegs.
http://www.stattucino.com/berrie/dsl/Galton.html (accessed 24 July 2001)

Proofs Without Words provides examples of visual proofs of numerical relationships.
http://www.cut-the-knot.com/ctk/pww.html (accessed 24 July 2001)

Virtual reality polyhedra allows the user to explore polyhedra from a variety of angles and to visualise what they would look like from inside.
http://www.georgehart.com/virtual-polyhedra/vp.html (accessed 24 July 2001)

My Dynamic Visual Algebra site, still very much under construction, illustrates many of the examples discussed in this paper, and others, using animated graphics.
http://www.amt.canberra.edu.au/~sjt/dva.htm (accessed 24 July 2001)

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