

Counting mental strategies as new mathematical operations DAVID WOMACK

Abstract: Drawing on my previous published work with both children and teacher trainees, I will present a variety of arguments for allowing children to develop a view of addition which sees numbers primarily as *positions*. This is generally known as a ‘transformational’ understanding of number operations; a view which considers addition as transforming a number from one position to a ‘higher’ position. In such a view there is a difference between the two ‘aspects’ of subtraction known to teachers as ‘take-away’ and ‘comparison’. If these ‘subtractive operations’ are given their own signs, algebraic equations can be formed in which terms can be substituted for and transposed like orthodox operations. Also, if the system is extended to negative positions, then algebraic findings such as ‘two minus’s make a plus’ are seen to be clearly demonstrable.

Participants are invited to give their own opinions as to why this formalisation has not been attempted before and also give their views on the pedagogical and theoretical significance of the idea.

Background – the Investigation

This paper builds on several previously published papers describing an Investigation conducted with a small group of 5- and 6-year-old children in a 35-pupil rural school in the Langdale Valley (Lake District, U.K.) over a period of 14 weeks. The findings were presented first to a meeting of the British Society for Research into Learning Mathematics at Oxford University (Womack, 1997).

The Investigation began from the premise that the intuitive number framework which young children adopt prior to schooling regards numbers primarily as *objects with positions* rather than symbols for collection size (see Rationale 1). The distinctive characteristic of this mental model, is that there are two ‘subtractive’ strategies, referred to in the literature as *counting-back* and *counting-up* (Fuson, 1988), both of which yield the same numerical answer. For example, $11 - 6$ can be mentally calculated as ‘count back 6 from 11’ (to reach position number 5) or ‘count-up from 6 to 11’ (in which case 5 numbers have to be counted – 7, 8, 9, 10, 11).

Aims

One of the aims of the Investigation was to discover whether these mental strategies of *counting-back* and *counting-up* could be taught in a similar pedagogical manner to the teaching of the conventional operations of addition and subtraction in school? However, to do this, these strategies would need to be given signs to instruct children which operation to carry out, just as conventionally, ‘ $8+3$ ’ instructs children to ‘add 3’, whilst ‘ $8 - 3$ ’ instructs children to ‘subtract 3’? Hughes’ work (1986) suggested that children have a very shaky understanding of the conventional operation signs but in the Investigation it was found that children could handle new signs – provided they were signs which they themselves had invented (see Rationale 2).

Rationale

1: The ‘stepping stone’ model of numbers

Dufour-Janvier et al (1987) claimed that many children mentally envisage numbers as a series of stepping stones in which children do not see the necessity for placing the ‘stones’ at equal distances. Gallistel and Gelman (1992) appear to give support to this model when they suggested that children seem to possess the ability to “directly enter the *positional representation* for a number upon hearing its name”. More recently, Butterworth (1999) has claimed that many children and adults visualise numbers as a sequence of bubbles stretching away into the distance. Based on such findings, it was argued (previously), that children’s understanding of numbers is primarily one in which the number symbols represent *positions*

in a hierarchical sequence, rather than *sizes* of collections. In the Investigation, children did not identify the cardinal aspect with an unordered collection of objects in a set, but with the ordered sequence of counting actions between two positions (Womack, 1995) [See Note 1]. It is claimed that this intuitive theory of numbers is frequently not replaced when conventional definitions are introduced but remains a covert model through which mental calculations are made throughout adult life (Womack, 1998b). In the case of unschooled adults, living in non-technological societies, it remains their *only* model and is analogous to the body-counting systems used by various orate African societies (e.g. Saxe, 1982; Petito & Ginsberg, 1982; Womack, 2000c) [See Note 2].

I have shown in previous papers that in this mental model, the strategies of *counting-on*, *counting-back* and *counting-up* can be replicated by walking on, back or between the stepping stones (Womack and Williams, 1998). In the 'stepping stone' setting of the Investigation, two basic types of question could be asked (and answered): *Where will you reach?* and *How many steps did you take?* *Where will you reach?* questions require children to *count-on* (or *back*), whereas to answer the question, *How many steps did you take?* requires children to *count-up* (Womack, 2000a, 2000b) [See Note 3].

2: Children's use of symbols

Gifford (1990) notes that children tend to see the equals sign as a prompt for an adding procedure and suggests that children tend to read signs as *actions*, rather than relationships between numbers. Gifford argues that children have difficulty relating *plus*, *minus* and *equals* signs to all the different aspects of the concepts involved – in this case, difficulty relating minus signs *only* to the 'take-away' and not to the 'difference' situations. For example one child's idiosyncratic notation for finding the largest difference between pairs of numbers is a sign consisting of a sort of skipping rope linking the two numbers - in children's minds, clearly a different mental process from that of 'take-away'. Another example is the child who, when faced with a *How many more?* type problem said, 'I don't know the sign for adding on'. However, it seems children do not consider this 'operation' as substantially different from the conventional operation of subtraction (signified by the 'hyphen' sign). Atkinson, (1992) provides many more examples.

In summary, although children's understanding of *conventional* mathematical symbols is greatly overestimated, it was found that they can invent and use their *own* symbols with great facility (Gardner and Wolf, 1983; Hughes, 1986; Resnick et al, 1990; Atkinson, 1992; Neuman, 1987, 1993).

Issues

The 5 year old children in my Investigation clearly understood the difference between these two 'processes' on the numbered 'stepping stones' and I now believe that these strategies are analogous to (and perhaps preparatory to) a *transformational* understanding of addition. [See Note 3].

In transformational addition, children regard the addition sign as an instruction to *do* something (Hughes, 1986). Transformational addition implies two subtraction 'operations' - *take-away* and *find the difference* which are clearly differentiated in school teaching contexts. [See Note 4].

Pedagogically, *take-away* and *find the difference* have always been considered as alternative mental strategies which achieve the same end – the subtraction of one number from another. However, within a transformational model, where the addition operation is essentially non-commutative, these are two different processes. This can be demonstrated by considering the

higher but equally non-commutative operation of *exponentiation* in the following equation: 3 powered by $4 = 81$, in which the operator (4) transforms the operand (3). To find the operand, requires the operation of taking the 4th root of 81 but to find the operator, requires the operation of finding the logarithm of 81 (to the base 3). Therefore, if *taking the root* and *finding the logarithm* are considered different mathematical operations associated with the transformational operation of exponentiation, should not the operations of *take-away* and *find the difference* be considered in the same way - as different mathematical operations associated with the transformational operation of addition?

Proposed participant discussion

I will refer to these mental operations as ‘operations’ and put forward for consideration the possibility that with suitably devised signs, these could be used initially with children. The transition to conventional operations can then be made at a later time when commutativity is confidently espoused and operation signs can be understood as standing for both count-back and count-up.

However, irrespective of the practical and pedagogical merits or demerits of this approach, this paper invites

discussion of the *mathematical* implications of regarding the non-commutative form of addition as a valid (and alternative) mathematical operation.

NOTES

1. Note that a cardinal model of numbers may be adequate for dealing with collections up to about 6 (the subitizable range), but beyond this, numerical *size* cannot be adequately envisaged mentally.
2. *Where will you reach?* questions were asked using vertical arrows, written on cards. *How many steps did you take?* questions were asked much later in the investigation sessions, using another sign invented by the children - a horizontal arrow which instructed children to find the number of steps linking two numbers. For example, $9 \rightarrow 6$ meant *Walk to 9 from 6*. Effectively, this was a finding-the-counted-on-steps question. The relation between count-on signs and count-up signs is more fully discussed in Womack (1998a).
3. Vergnaud (1982) has given a comprehensive account of different types of transformational addition and subtraction problems.
4. The confusion sometimes caused by using a single ‘subtraction’ sign to represent distinct number-related situations can be clearly seen when negative integers are introduced. Some researchers have made this distinction very clear (e.g. Rowland, 1982; Haylock, 1995, p.95). One significant difference between the two mental processes subsumed under ‘subtraction’ is that ‘take-away’ is an instruction for action without requesting an answer, whereas ‘find the difference’ asks for an answer without giving any instructions as to how this might be achieved. For the further implications of the fundamental difference between ‘take-away’ and ‘difference’ methods for subtraction, (see Womack, 1998a, 1998b).

References

- Atkinson, S. (Ed) (1992) *Mathematics with Reason*. London. Hodder & Stoughton. Contains various accounts by Sue Atkinson, Shirley Clarke and Sue Gifford.
- Butterworth, B. (1999) *The Mathematical Brain*. London. Macmillan.
- Dickson, L., Brown, M. and Gibson, O. (1984) *Children learning mathematics: A teacher's guide to research*. Holt, Rhinehart and Winston.

- Dufour-Janvier, B. Bednarz, N & Belanger, M. (1987) Pedagogical considerations concerning the problem of representation. In: C. Janvier (ed) *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, N.J., Lawrence Erlbaum. pp. 109-122.
- Garner and Wolf, (1983)
- Fuson, K., (1988) *Children's Counting and Concepts of Number* N.Y. Springer-Verlag.
- Gallistel, C.R. & Gelman, R.: 1992, Preverbal and verbal counting and computation. *Cognition*, 44. pp43-74.
- Gardner, H. & Wolf, D. (1983) Waves and streams of symbolization: Notes on the development of symbolic capacities in young children. In: Rogers, D. & Sloboda, J.A. (Eds.) *The Acquisition of Symbolic Skills* (Proceedings of a NATO Conf., 1982, Keele). Plenum Press, NY. pp. 19-42.
- Gifford, S. (1990) Young children's representations of number operations. *Maths Teaching*. Journal of Association of Teachers of Mathematics. Also in Sue Atkinson (ed). (1992) *Mathematics with Reason*. Hodder & Stoughton.
- Haylock, D. (1995) *Mathematics explained for primary teachers*. London. Paul Chapman Publ. Ltd.
- Hughes, M. (1986) *Children and Number*. Oxford Blackwell Publishers.
- Neuman, D. (1987) *The origin of arithmetic skills: A phenomenographic approach*. Acta Universitatis Gothoburgensis.
- Neuman, D. (1993) Exploring numbers through subtraction in primary mathematics. In: Breiteig, T., Huntley, I., Kaiser-Messmer, G., *Teaching and Learning Mathematics in Context*. Chichester. Ellis Harwood Ltd. pp.37-50.
- Petito, A.L. & Ginsberg, H.P. (1982) Mental arithmetic in Africa and America: Strategies, principles and explanations. *International Journal of Psychology* 17. 81-102.
- Resnick, L.B. & Lesgold, S. (1990) From Protoquantities to Number Sense. *Psychology of Mathematics Education Proceedings* 14. III.
- Vergnaud,
- Rowland, T.: 1982, 'Teaching directed numbers: an experiment.' *Mathematics in School*, Vol 11 No.1, pp. 24-27.
- Saxe, G.B. (1982) Culture and the development of numerical cognition: Studies among the Oksapmin of Papua New Guinea. In: C.J. Brainerd (ed). *Children's Logical and Mathematical Cognition*. NY. Springer-Verlag.
- Vergnaud, G. (1982) A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In: Carpenter, T.P., Moser, J.M, Romberg, T.A. *Addition and Subtraction: A Cognitive Perspective*. Lawrence Erlbaum Associates, Inc., Hillsdale, N.J. pp. 39-59.
- Womack, D. 1995, Arithmetic Images of Reality. In *Proceedings of the Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM conference). Berlin. Freie Universität. pp.121-126.
- Womack, D. 1997, Investigating Children's Intuitive Understanding of Number Operations by Formalising their Mental Strategies. In: *Proceedings of the British Society for Research into Learning Mathematics*. Oxford University. pp.106-111.
- Womack, D. (1998a). Teacher trainee student's understanding of operation signs. *Proceedings of the British Society for Research into Learning Mathematics*. Birmingham University. pp.117-122. ('Paper 1')
- Womack, D. 1998b, Mental calculation strategies employed by teacher trainee students. In *Proceedings of the British Society for Research into Learning Mathematics*. Open University. pp.77-81. ('Paper 2')
- Womack, D. (2000a) Hopping onto a good idea. *Early Years Educator*, Vol. 2. No. 2. June pp19-21.
- Womack, D. (2000b) Children inventing signs for mental number strategies. In: T. Rowland & C. Morgan (eds) *Annual Proceedings of the British Society for Research into Learning Mathematics – Year 2000*. BSRLM Publications.
- Womack, D. (2000c) Children's intuitions and African insights: An alternative framework for number? *Proceedings of British Congress of Mathematics Educators*. Nene College, Northampton. UK.
- Womack, D. (2001) Children's intuitions of number. Pedagogical implications and mathematical insights. Ph.D thesis submitted to the University of Manchester.
- Womack, D. & Williams, J. (1998) Intuitive counting strategies of 5- to 6-year-old children within a transformational arithmetic framework. *Proceedings of the International Group for Psychology of Mathematics Education*. (Eds. Alwyn Oliver & Karen Newstead.) Stellenbosch, S.A., University of Stellenbosch. Vol.4. pp.185-192.