Autoreport of the dissertation
of Martha Isabel Fandiño Pinilla

Introduction

In this paper we dissertation the findings of a principally bibliographical long-term research project, concerning “fractions”. This is one of the most studied questions in Mathematics Education, since the learning of fractions is one of the major areas of failure. Here we present a way of understanding lack of success based on Mathematics Education studies, rather than on mathematical motivation.

The current situation concerning the problem studied in the dissertation

Introducing the concept of fractions has a common basis the world over. A given concrete unit is divided into *equal* parts and some of these parts are then taken. This intuitive idea of fraction of the unit is clear and easily grasped, as well as being simple to modelize in everyday life. It is, however, theoretically inadequate for subsequent explanation of the different and multiform interpretations given to the idea of fraction. As we shall see, one single “definition” is not sufficient.

When a child of between 8 and 11 years of age has understood that $\frac{3}{4}$ represents the concrete operation of dividing a certain unit in 4 *equal* parts, of which 3 are then taken, it would seem that everything is proceeding smoothly. Unfortunately, almost immediately it is clear that the simple construction of that knowledge is blocking the way, it is an obstacle to subsequent real learning. This is knowledge, but inadequate to continue in the construction of further correct knowledge. If, for example, we have a unit divided in 4 *equal* parts, what does it mean, from this point of view, taking $\frac{5}{4}$ of it?

At times it seems that many teachers are unaware of the conceptual and cognitive complexity involved. I believe that it is necessary to dedicate a whole section to different ways of intending the concept of fraction, that we would like the pupil to acquire.

To give reliability to my work I am obliged to propose an overview of international research in this delicate field, certainly one of the most cultivated the world over. It is impossible to quote the whole of these researches, since its vastness goes beyond our imagination. I will quote only the works which have been directly influential for my subsequent choices, dropping the others. They will be rather a lot anyway. My hope is that this painstaking bibliographical research (I shall propose mainly quotations with regard to the period 1970-1990, and a more detailed bibliography with reference to the period 1990-2000) may be of use also to others who wish to pursue research in the same field. It has not been trivial constructing it.

The objectives of the dissertation

The research conducted for this dissertation aims to show that:

- the mathematical concept of fraction, generally considered as having one unique definition, in fact assumes various meanings in different classroom situations, not only from a didactic point of view (concerning the interpretations on the part of students) but also from a scientific point of view; this fact has already been documented in the international research literature and will be investigated in action in concrete teaching situations;
- in the face of widespread failure in the teaching-learning of fractions, attempts to find solutions are all based on conceptual reformulations; the dissertation will rather analyse the teaching approaches highlighted in the research of the last 20 years.

Research methods used for the dissertation

The research has been conducted with numerous class a different school levels with learners at:

- Primary school
- Secondary school (the majority of the students involved)
- University degree course in Education for Primary School teachers

as well as teachers attending in-service training courses
Given the variety of subjects and situations, the research has been conducted through:

- classroom observation
- activities proposed to participants
- interviews and discussions;

as relevent facts emerged they were immediately used for discussion and clinical interviews.
A considerable part of this research is also of a theoretical nature, in that it contains a detailed analysis of the vast literature concerning this field, a study concerning over 200 authors conducted over 6 years. This analysis has allowed me to classify the enormous quantity of research on the basis of criteria of analogy.

Theoretical points of departure

The idea of fractions is formally introduced at primary school level, in Italy usually in the third year, even though it is already present in the most immediate sense of "half" an apple or "a third" of a bar of chocolate, or divide a handful of chocolates in 4 equal parts, at a much earlier age.
What schooling does is formalise the written form and institutionalize its meaning.

Roughly speaking, we can say that the universal first approach is that of taking a "concrete object of reference", considered as unit, which should have the following requirements:

- be perceived as pleasant and thus fun,
- clearly unitary and
- already familiar, thereby not requiring further learning.

Normally a round cake or a pizza is chosen in almost all countries the world over; both these objects have the above requirements

Situations are then imagined in which this given *unit* (a cake, a pizza or similar) must be shared between a number of pupils or people in general. In this way the pupils arrive at the idea of a half (dividing by 2), a third (dividing by 3), and so on: the "Egyptian fractions", which are our first historical example.

For each of these fractions specific written forms are established that for the above cases are $\frac{1}{2}$ and $\frac{1}{3}$ and reading these forms as "a half" and "a third" poses few problems. Nor does generalizing from these examples the written form $\frac{1}{n}$, which assumes the meaning of an initial unitary object divided into *n equal* parts. With young pupils various examples are considered, assigning different appropriate values to *n*.
If then the guests, for different reasons, have the right to different amounts of the equal parts into which the unitary object had been divided, this gives rise to different written forms such as $\frac{2}{5}$ (two fifths) meaning that two of the five *equal* parts into which the unitary object had been divided are taken. Several new ideas thus arise and a number of characteristics of these written forms are then established:

- the number beneath the little horizontal line is called the *denominator* and this indicates the number of *equal* parts into which the unit has been *divided*;
- the number above the little horizontal line is called the *numerator* and this always indicates the number of parts *taken* (in this way, the numerator expresses the number of times the fractional part must be taken, and thus a *multiplication*);
- to give sense to this, the fractional parts of the unit must be *equal*, a point much stressed and to which we will return later in a critical way.

We shall see how the understanding of these elements, and in particular those marked by italics, end up being an obstacle to the construction of the concept of fraction.

The dissertation analyses over 200 research publications subdivided on the basis of criteria of chronology and analogy in terms of the objectives of the research.
1. *From the 1960s to the 1980s*
2. *From the 1990s to the present day*

Preparation and conduct of the experiment

The preparation of the dissertation consisted principally of an analysis of the vast literature related to the field.
Moreover, in order to guarantee that all interpretations of the idea of fraction, contrasting and real, were considered, I also conducted:

- classroom observations of teaching practices
- an analysis of the differences between what teachers taught and required of students and the contents of teaching manuals
- clinical interviews with teachers after lessons;

in certain cases I also used in-service teacher training courses to propose:

- reflective dialogues

- personal reflective writing.

This approach has given rise to a new area of research and the already-published article:

Campolucci L., Fandiño Pinilla M.I., Maori D., Sbaragli S. (2006). Cambi di convinzione sulla pratica didattica concernente le frazioni. Una learning story basata su una ricerca-azione di gruppo e sua influenza sulle decisioni relative alla trasposizione didattica delle frazioni. *La matematica e la sua didattica*. 3, 353-400.

During the experiment it became immediately clear that the teachers themselves needed to reflect on:

- preceding mathematical aspects
- mathematical aspects after the didactic trasposition
- the effects of their teaching action

in order to subsequently consider

- possible solutions offered by current research literature.

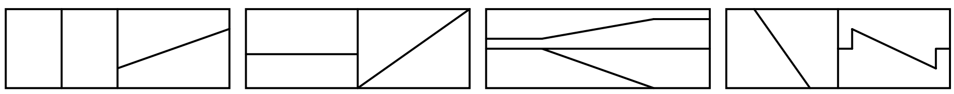
Conclusion of the results of the experiments (a posteriori analysis)

The experiment based on the classroom observation of teachers has led to a number of various considerations briefly illustrated in the following points:

Different ways of understanding the concept of fraction

Something which often strikes teachers on training courses is how an apparently intuitive definition of fraction can give rise to at least a dozen different interpretations of the term.

1) A fraction as part of a one-whole, at times continuous (cake, pizza, the surface of a figure) and at times discrete (a set of balls or people). This unit is divided into "equal" parts, an adjective often not well defined in school, with often embarrassing results such as the following, concerning continuous situations:



or discrete ones: how to calculate $\frac{3}{5}$ of 12 people.

Providing students with concrete models and then requesting abstract reasoning, independently of the proposed model, is a clear indicator of a lack of didactic awareness on the part of the teacher and a sure recipe for failure.

2) At times a fraction is a quotient, a division not carried out, such as $\frac{a}{b}$, which should be interpreted as a:b; in this case the most intuitive interpretation is not that of part/whole, but that we have a objects and we divide them in b parts.

3) At times a fraction indicates a ratio, an interpretation which corresponds neither to part/whole nor to division, but is rather a relationship between sizes.

4) At times a fraction is an operator.

5) A fraction is an important part of work on probability, but it no longer corresponds to its original definition, at least in its ingenuous form.
6) In scores fractions have a quite different explanation and seem to follow a different arithmetic.
7) Sooner or later a fraction must be transformed into a rational number, a passage which is by no means without problems.
8) Later on a fraction must be positioned on a directed straight line, leading to a complete loss of its original sense
9) A fraction is often used as a measure, especially in its expression as a decimal number.
10) At times a fraction expresses a quantity of choice in a set, thereby acquiring a different meaning as an indicator of approximation.
11) It is often forgotten that a percentage is a fraction, again with particular characteristics.
12) In everyday language there are many uses of fractions, not necessarily made explicit, e.g. for telling the time ("A quarter to ten") or describing a slope (a 10% rise"), often far from a scholastic idea of fractions.

In this respect the studies of Vergnaud are illuminating. I am personally convinced that conceptual learning is the first stage of mathematical learning. So many different meanings for the concept of "fraction" require an attempt to find some unifying principle. Following Vergnaud, we can consider a *concept* C as three sets C = (S, I, *S*) such that:

- S is the set of situations that give sense to the concept (the referent);
- I is the set of the invariants on which is based the operativity of the schemata (the signified);
- *S* is the set of linguistic and non-linguistic forms that permit symbolic representation of the concept, its procedures, the situations and treatment procedures (the signifier).

Thus it is evident that the choice of a single meaning of fraction cannot conceptualize the fraction in its multiple features.
As we have seen:

- Behind the same term "fraction" are hidden may different situations which give sense to the concept
- Each of these situations contains invariants on which are based the operativity of the schemata,
- Various linguistic forma can be used to represent the concept.

Thus it is necessary to conceptualize the fraction via all of these meanings and not just through one or two of them, a scholastic choice that would lead to failure.

Vergnaud proposes also a theory of conceptual fields: "a set of situations, concepts and symbolic representations (signifiers) closely interdependent which cannot be analyzed separately" … "a set of problems and situations the handling of which requires different concepts, procedures and representations which are strictly interconnected".

On the one hand, it is impossible to imagine an approach to teaching fractions in isolation from the mathematical context which gives them sense: fractions, ratios, proportions, multiplications, rational numbers, are but a few of the emerging features from all of that which gives sense to fractions. These concepts must not be separated, but rather should flow together in one sole learning process. On the other, all the different acceptations of fractions must be explored and put in relationship between one another, since there are considerable differences between some of them.

Gérard Vergnaud's ternary schema is important and useful, as we have seen, but other approaches have been proposed for the conceptualization. More recently, Raymond Duval has replaced the ternary schema with a binary schema containing the pair "meaning-object" or "sign-object", thereby expressing the idea that conceptualization passes through the sign which expresses its own object. The occurrences of the mathematical object "fraction" are multiple and refer back to a variety of signs each one belonging to an appropriate system of signs.

The noetics and semiotics of fractions

The term "noetics" refers to conceptual acquisition and thus within the school environment to conceptual learning.
The term "semiotics" refers to the representation of concepts through systems of signs.
Both are of extraordinary importance in Mathematics.

On the one hand any form of mathematical activity requires the learning of its concepts. On the other it is impossible to study the learning in Mathematics without referring to semiotic systems.

It is important to bear in mind that the concepts of Mathematics do not exist in concrete reality. The point P, the number 3, addition, parallelism between straight lines, are not concrete objects which exist in empirical reality. They are pure concepts, ideal and abstract, and therefore , if we want to refer to them, they cannot be "empirically displayed" as in other sciences. In Mathematics concepts can only be *represented* by a chosen semiotic register.
As a matter of fact, in Mathematics we do not work directly with objects (i.e. with concepts), but with their *semiotic representations*. So semiotics, both in Mathematics and in Mathematics Education, is fundamental.

To represent a given concept there are many possible registers. Passing from one representation to another within the same register is called "transformation by treatment", while a change of semiotic representation into another register is called "transformation by conversion". In 1993 Duval called attention to a cognitive paradox hidden within these issues. We shall see that as regards the didactics of fractions this is an extraordinary important issue.

The objective is conceptual learning. The teacher (who knows the concept) proposes some of its semiotic representations to the student (who does not yet know the concept), in the hope, with the desire and will, that, via the semiotic representations, the student will be able to construct the desired conceptual learning (noetics) of the concept. But the student possesses only semiotic representations, objects (words, formulae, drawings, diagrams, etc.), but not the concept itself. If the student already knew the concept, he could *recognize it in those* semiotic representations, but since he does not know it, *he sees only* semiotic representations, i.e. concrete objects, ink marks on sheets of paper, chalk marks on a blackboard, etc.

The teacher who is unaware of noetics and semiotics may well cherish the illusion that, if the student manipulates the representations, then he is manipulating the concepts and thus the cognitive construction has taken place. In reality, it may well be that an incredible widespread ambiguity has arisen: the student has only learnt to manipulate the semiotic representations but has not at all constructed the concept and the teacher is suffering from an illusion.

In this respect, there are no miraculous recipes, there is only the need of awareness.
The teacher who is aware of this issue, cannot avoid focusing on the learning of his students, verifying if they really belong to the sphere of noetics and not only to the semiotic manipulation.

The fraction is a concept thus its learning is within noetics. As such, it cannot be concretely displayed. We can operate with a one-whole, an object, a cake, dividing it and obtaining a part. But the result is not the mathematical “fraction”, only the “fraction of that object”. Working with the semiotic register of concrete operations, we have shown a semiotic representation, not the concept.

We can use words to describe what we have done to the cake, thereby changing semiotic register and showing another semiotic representation, but not the concept.

We can pass to other examples, abstracting from the concrete object, the cake, going to another concrete object, for example a rectangle (better its area), but once again we have changed semiotic register thus providing another semiotic representation, not the concept.

At this point we normally go beyond the object (cake, rectangle, etc.) to its abstraction and the concept of fraction is supposed to have been constructed independently from the concrete model of departure considered as unit or whole. But often this is an illusion.

Up to this point the units are continuous objects: a cake or the surface of a rectangle. Passing to discrete units - e.g. 12 balls which must, however, be considered as one unit-whole - the register has been completely changed but it is taken for granted that the conversion spontaneously takes place. Indeed, at this point (maybe even before) an appropriate mathematical formalism is introduced, the written form of fractions, together with the terms “numerator” and “denominator”: as a matter of fact a new semiotic representation in a different register is supplied, and thus a new kind of conversion.
…
All this, and more, normally takes place within a lesson of 30-40-50 minutes.

The distinguishing features that characterize the different objects are chosen– the act of dividing, the cake (continuous), the surface (continuous) of a rectangle, the set of balls (discrete), the formal writing with its specific names – treatment transformations (few) and conversion transformations (many) are continuously carried out, taking for granted that, if the student is capable of reproducing them, the teaching has been successful, the learning achieved and the concept constructed.

If, however, we recognize that semiotics and noetics are not the same thing and that learning to manipulate semiotic representations is not noetics, we can understand how, usually after apparent initial success, within a few lessons or within the following weeks or months students may be in grave crisis, having learnt to manipulate a few passages and registers, nothing more, but he has not at all constructed the concept we wanted him to construct.

Conclusions of the dissertation (for teaching practice)

As the teachers themselves admitted, the points highlighted by the research had never previously been studied in such depth or so clearly illustrated. Even faced by so obvious a failure in class, the teachers however showed an inability to make use of the results of published research in Mathematics Education.
The question is essential for our research. I will now briefly present this aspect in the following points.

Research has illustrated some errors which are typical in students the world over. Research has thoroughly and precisely studied and listed them. Below we summarize the most important.

1) Difficulty in ordering fractions and numbers written in the decimal notation.
2) Difficulty with operations between fractions and between rational numbers.
3) Difficulty in recognizing even the most common schemata.
4) Difficulty in handling the adjective “equal”.
5) Difficulty in handling equivalences.
6) Difficulty in handling the reduction to minimum terms.
7) Difficulty in handling non standard figures.
8) Difficulty in passing from a fraction to the unit that has generated it.
9) Difficulty in handling autonomously diagrams, figures or models.

Research has highlighted these typical errors and has classified them, but without using modern Mathematics Education considered as Learning Epistemology in the specific case of fractions, thereby turning over its point of view, using the results of the copious research that has been conducted over the past 40 years.
This need has pushed me to consider the main research topics into Mathematics Education and come back to the previous classical research into fractions under this point of view, looking for didactic and not mathematical motivations of these “typical errors” The following list considers but a few of the issues involved.

Didactic contract

(a) The “sum” of fractions: $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$ is not something the student proposes to the teacher because he believes it to be true, but because he thinks it may be acceptable by the teacher in terms of its form …

(b) In the context of a problem involving fractions, it is illusory to imagine that the student reasons, while choosing the appropriate operation to perform, when it is well known that, by contract, his objective is that of receiving a nod of approval and so is perfectly capable of producing a series of proposals often quite contradictory. The apparent absurdity (from the mathematical point of view) of the series of proposals gains a logic (from the point of view of didactics).

Many studies about the didactical contract have highlighted situations that implicitly were under everybody's attention, without being clearly expressed and understood: students give up taking risks, abdicate the burden of responsibility for their own learning and act only in terms of the contract. With reference to fractions this is rather evident.

Excessive semiotic representations

Opening any textbook shows immediately the immense number of semiotic representations available for expressing fractions.

Handling these registers choosing the distinguishing features of the conept we must treat and convert, is not learnt *automatically*. This learning results from a process of explicit teaching in which the teacher must render the student co-responsible.

Teachers often underestimate this aspect, ignoring the warning of Duval and passing from one register to another, believing that the student follows. The teacher is able to jump from one register to another without problems, because he has already conceptualized: while in fact the student does not so, the student follows at the level of semiotic representatives, but not of meanings.

Prematurely formed images and models

Dealing with fractions, often an image can be transformed too quickly into a mental model, when it should still remain an image. Let us take some examples.

(a) The image of a unit-whole divided into *equal* parts, taking *equal* to be identity, congruency, superimposability, creates an effective and durable concept of fraction which then transforms into a model and has to be respected on all occasions and thus impedes the noetics of the fraction.
(b) The image of dividing a unit-whole in equal parts and taking some of them suggests semantically that this "some" cannot be "all". The model is easily reinforced, given that it coincides with a strong intuition, but then impedes the passage to a unit as a fraction $\frac{n}{n}$ and to improper fractions.
(c) The use of geometric figures is seen by students to be specific and meaningful, whereas for the adult it is random and generic. The continuous use of only rectangles or circles compels to a way of thinking in which an image, instead of being open, ductile and modifiable, becomes a persistent and stable model. If the fraction is proposed using different figures (triangle, trapezium,…) the student no longer grasps the noetics of the fraction because the situation is not a part of his model

Misconceptions

There are numerous examples of misconceptions concerning fractions. Many of the examples we have seen are ascribable to students' misconceptions which have then become premature models when instead they had to remain provisional images. We have seen examples. I recall misconceptions linked to order between fractions, based on that between natural numbers, to the simplification of fractions, to the handling of equivalences between fractions, to operations between fractions, to the choice of figures on which to operate with fractions …

Ontogenetic, didactic and epistemological obstacles

Many of the things to be learnt concerning fractions can be considered as true epistemological obstacles and are easily recognizable in the history and in the practice of teaching.
(a) The reduction of fractions to minimum terms has for long been a specific object of study in history, as is shown by the fact that the Egyptians who cultivated fractions for many centuries used only fractions with unitary numerators
(b) The passage from fractions to decimal numbers required over 4,500 years of Mathematics.
(c) The handling of zero in fractions has often created enormous problems in history, even for illustrious mathematicians.

Excess of didactic situations and lack of a-didactic situations

The situations that teachers propose for the learning of fractions are mainly didactic whereas they rarely correspond to a-didactic situations. The result is of almost total failure in the learning of fractions and rational numbers, and this is clearly pointed out in the international literature. Today we know that the construction of meaningful learning must pass through a-didactic situations, but that these are by no means the most used in teaching practice, while it should be so.

Una conoscenza esplicita delle interpretazioni posibili degli errori degli studenti con gli strumenti della ricerca didattica, aiuta a capire sempre meglio il problema.
In altri scritti ho anche tentato di proporre all'insegnante possibili modelli di comportamento concreto, sul tema delle frazioni.

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