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Exploration of geometry by prospective mathematics teachers in Turkey with Geometer’s Sketchpad

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Abstract. In Turkey, only Euclidean geometry is studied in primary and secondary education. Non-Euclidean geometries are covered only at university level. This study was conducted with those students taking Geometry course offered in Primary Education Mathematics Teaching program. The content of Geometry course covers Euclidean and non-Euclidean types of geometry. The concepts of *similarity, difference, incongruity* and *opposition* are extremely important in teaching. In order to be able to distinguish between two *very* similar things, the differences between them need to be identified. In this study, therefore, it is suggested that only mentioning the presence of non-Euclidean types of geometry is insufficient to teach those types of geometry as well as to teach Euclidean geometry. Accordingly, the aim of this is to have Mathematics teacher candidates discover the difference between a non-Euclidean geometry – Hyperbolic Geometry – and Euclidean geometry by using Geometer’s Sketchpad, a dynamic geometry program.
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1. Introduction

Geometry is a branch of mathematics that is concerned with the examination of the relationships of points, lines, curves and surfaces and studies of space. Geometry, which also means knowledge of shapes in a sense, has an outstandingly irreplaceable role and significance in mathematics teaching. In Turkey, only Euclidean geometry is studied in primary and secondary education. Non-Euclidean geometries are covered only at university level. What non-Euclidean geometries mean, hyperbolic geometry and a special model of it – Poincaré disk model – and Geometer’s Sketchpad hyperbolic software, which allows studying on this model, are briefly mentioned below. The main objective of this study is to help Mathematics teacher candidates find out the existence and properties of a non-Euclidean geometry, hyperbolic geometry, by discovering the similarities and differences between Euclidean and non-Euclidean geometries.

1.1 Non-Euclidean geometries

In geometry, point, line and plane are undefined concepts. On the other hand, though not a mathematical definition, what is meant by Euclidean plane or plane in short is a straight smooth surface expanding in every direction. In the concept of plane, consisting of points and lines, some statements concerning points and lines are accepted without a need for proof of their validity. The proof of these statements, which are called axioms and considered to be self-evident, is not possible (as they are postulates). In geometry, the implications of the accepted axioms are examined. The five axioms of the Euclidean Plane, covered and studied in detail in primary and secondary education mathematics course syllabus, are listed below (Kaya, 2004):

1. For every point P and for every point Q not equal to P there exists a unique line l that passes through P and Q.
2. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E.
3. For every point O and every point A not equal to O, there exists a circle with center O and radius OA.

4. All right angles are congruent to each other.
5. Only one parallel can be drawn to a line from a point not lying on this given line.

The 5th axiom in Euclidean Geometry, known as Playfair axiom, states that in plane, only one parallel line can be drawn to a line from a point not lying on this given line. However, by the end of 1820s, Bolyai and Lobachevsky showed that a new type of geometry could be formed by taking the statement that "H: through any given point not on a given line, two (or infinitely many) lines can be drawn parallel to that given line" and some other Euclidean axioms. That is how hyperbolic geometry and therefore non- Euclidean geometry emerged. There are other geometries which negate the parallel postulate. Two of them are spherical geometry and elliptic geometry. Among non-Euclidean geometries, the one that is closest to Euclidean geometry is hyperbolic geometry, in that only one axiom of hyperbolic geometry is different from Euclidean geometry (Dwyer & Pfiefer, 1999). On the other hand, the essential difference is that while there is only one plane for Euclidean axioms, there are many models giving Hyperbolic Geometry (Bolyai-Lobachevsky) axioms. Some of these models are the Klein Model, Maximum Plane Model, Poincaré Upper Half-Plane Model, and Poincaré disk model. These models can be used to visualize Hyperbolic plane and to explore the geometrical properties of plane (Dwyer et al., 1999). This study was conducted by using Hyperbolic geometry and Poincaré disk model.

1.1 Poincaré disk model

Henri Poincaré (1854-1912) developed a disk model where the points of hyperbolic plane were defined as interior points of an Euclidean circle. In this model, lines are not merely straight lines like the ones they see in Euclidean plane. Instead, the lines of the geometry are formed by segments of circles contained in the disk orthogonal to the boundary of the disk, or else diameters of the disk (see figure1).

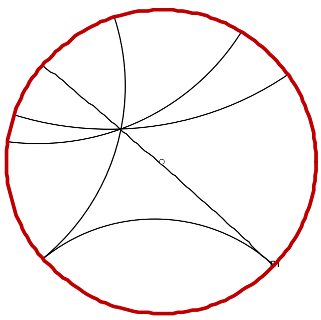

Figure 1. The lines of hyperbolic plane

Also, the boundary of the disk is not included and is different in distance. All the interior points of the circle form this plane. In order for two points to be collinear in this plane, they either need to be in the form of the arc of a circle perpendicular to C or need to be on a diameter. The angle between two lines in this model is the angle between the tangents drawn to the lines on intersections of these lines (see figure 2).

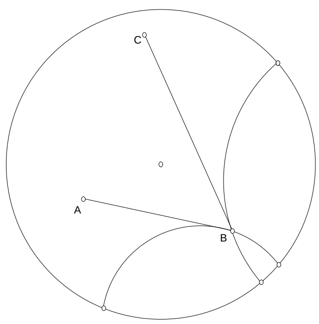

Figure 2. The angle between two lines in hyperbolic plane

2. Method

The sample study population is composed of 15 mathematics teacher candidates. After being introduced to the technical properties of Geometer's Sketchpad software in a total of 10 hours in 3 weeks, the sampled students were asked to complete the activities by using special Sketchpad tools so that they explore the hyperbolic geometry modeled by Poincaré disk model. The research activities were prepared under the subject headings points-lines, angles and triangles. Some examples of these research activities are presented below.

3. Findings

The students actively participated in the process of the discovery and development of the similarities and differences between Euclidean geometry and hyperbolic geometry by manipulating shapes and changing their forms(see figure 3). In Angles in Triangle Activity 1, for example, interrogative questions were asked so that the students examine and construct the idea that in hyperbolic geometry the sum of the interior angles of any triangle is less than 180 degrees. Accordingly, it was expected that the students would reach the generalization that "the sum of the interior angles of a triangle is less than 180 degrees" by manipulating the drawings they created and performing measurements on them.

Figure 3. Students explore the differences between the two geometries

In general, the students were asked to show geometrically which of the Euclidean geometry theorems presented in each activity are also valid in hyperbolic geometry. When the students claimed that a theorem was not a valid one in hyperbolic geometry, they were asked to show a sample not proving the theorem in hyperbolic geometry model by using Sketchpad program. If they couldn't find an example not giving the theorem, they were asked to reach the generalization that "this is a valid theorem in hyperbolic geometry, too" by showing that it was proven for at least three samples and by recording them. Two of these activities and student samples are presented below.

3.1. Angles in triangle

1) The sum of the measures of the interior angles of a triangle is equal to 180 degrees.
2) The sum of the exterior angles of a triangle is equal to 360 degrees.
3) The measure of an exterior angle of a triangle equals the sum of the measures of its two nonadjacent interior angles.
4) The angle between the bisector of the interior angle on a vertex of a triangle and the bisector of the exterior angle on the same vertex is equal to 90 degrees.

After all of the students completed the first question, they were asked the question "What is the sum of the interior angles of the triangle you have drawn?" Although they all found got the right sum, they did not give an immediate answer because they thought they made a mistake. However, they did give their answers for the question when they realized that they all got the same result. The students were all surprised to find out that the sum of the interior angles of a triangle is less than 180 degrees in hyperbolic geometry unlike the sum in Euclidean geometry, a well-known fact for all of them. Then, they all answered the question “How does the sum of the interior angles of the triangle change when you change the triangle by dragging the vertices?” in the same way; again they all gave the right answer that the sum remained less than 180 degrees. They realized that the result would still be less than 180 degrees when they were asked to draw different triangles and reached the conclusion that in hyperbolic geometry the sum of the interior angles of a triangle is less than 180 degrees. In figure 4, the results gained by one of the students are presented. Concerning the first question, this student wrote the following conclusion: “It is evident that while the sum of the measures of interior angles of a triangle is equal to 180 degrees in Euclidean plane, the sum of the interior angles is less than 180 degrees in hyperbolic plane.” The results gained by the other students indicate similar conclusions.

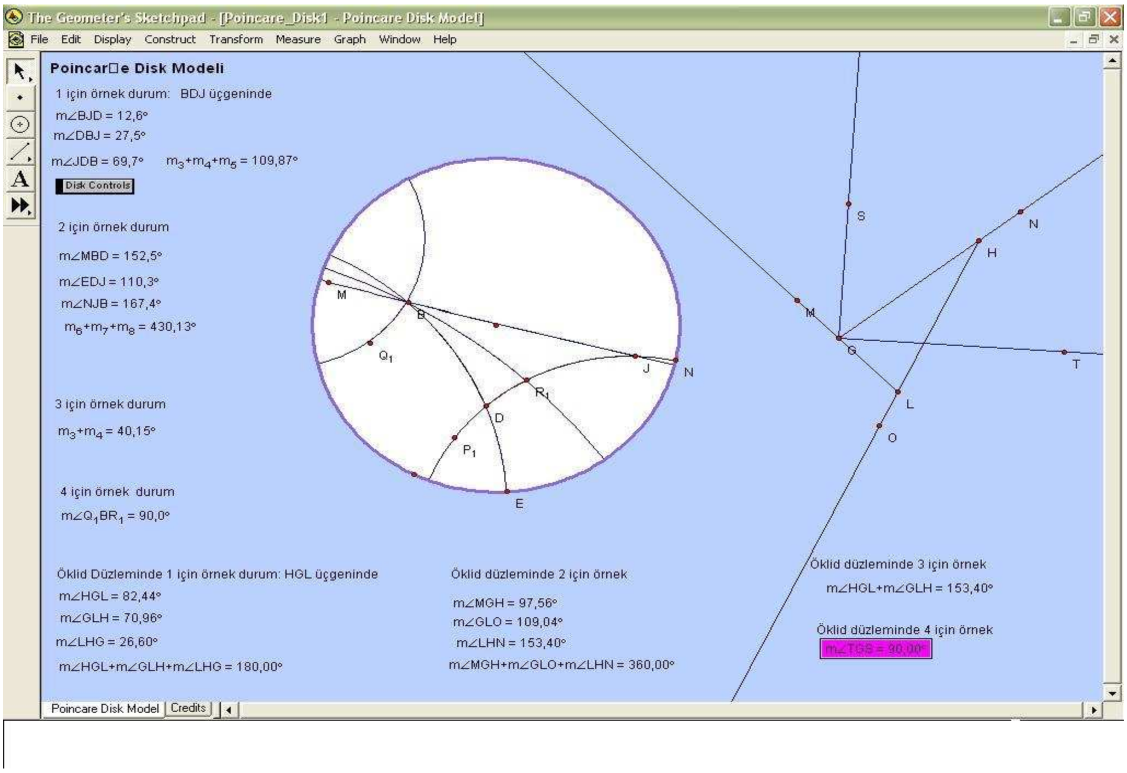

Figure 4. One of the students answer

3.2. Right triangles

1. It is possible to draw a right triangle.
2. Pythagorean Theorem: In any right triangle, the area of the square of the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs.

A student sample for this activity is given in figure 5. As can be seen in the student’s statement, they were again surprised to realize that the Pythagorean Theorem, well-known by them in Euclidean plane, is not proven in hyperbolic geometry.

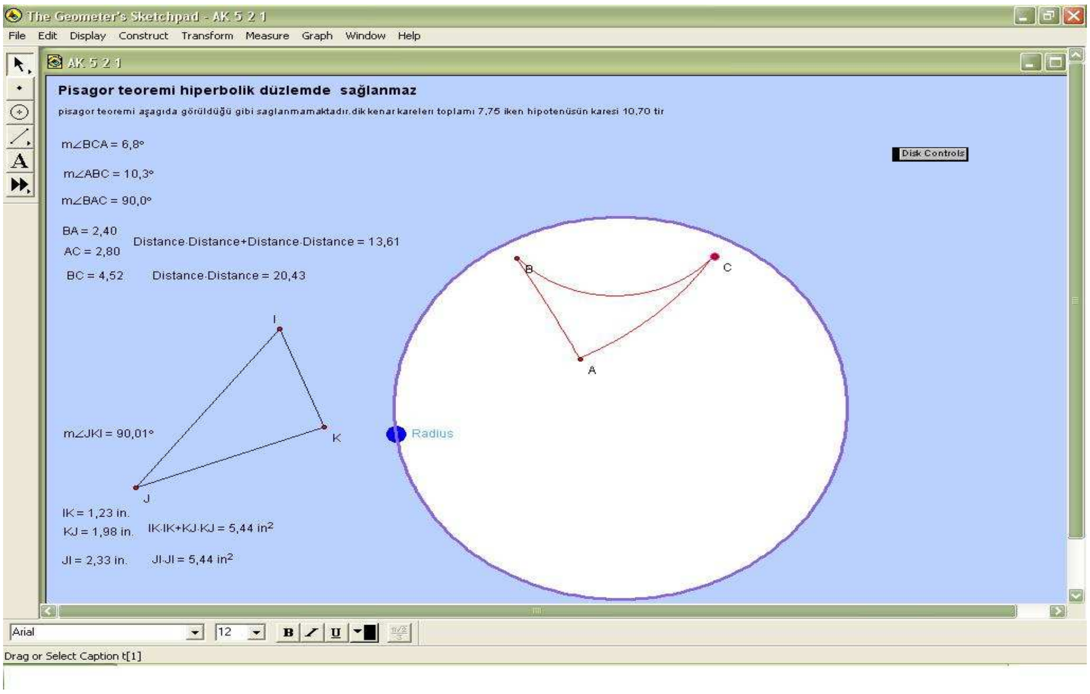

Figure 5. A student sample for right triangles

3.3. Special theorems

1. If two or more straight lines which are parallel to each other intersect two other straight lines, the lengths of corresponding line segments determined on the secants are proportional. (1st Thales’ Theorem)
2. When two parallel straight lines cut two intersecting straight lines, the angle sides of the triangles formed are divided into proportional segments. (2nd Thales’ Theorem)
3. Menelaus’ Theorem.
4. Ceva’s Theorem.

The result gained on the 1st Thales’ Theorem by one of the students checking whether the special theorems are proven in hyperbolic geometry or not is presented Figure 6.

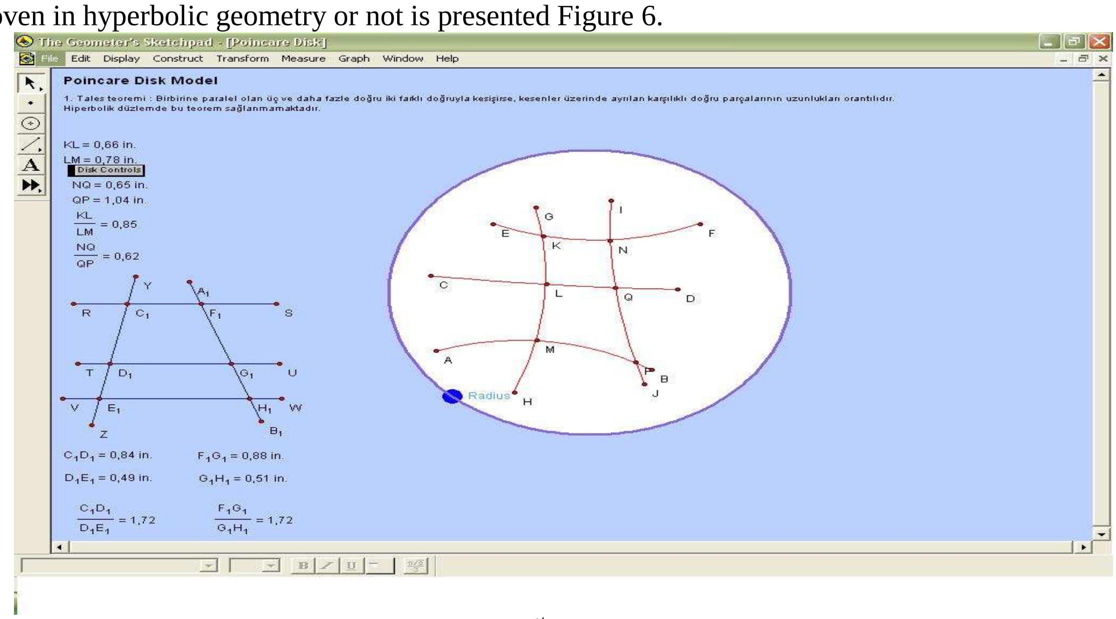

Figure 6. The result gained on the 1st Thales’ Theorem by one of the students

By different examples, this student and all of the others indicated the conclusion that the 1st Thales theorem was not proven.
One of the students obtained the model of Ceva theorem in Euclidean and hyperbolic geometry on the same screen(see figure 7). This programs is dynamic. The student tried different positions and compared her results.

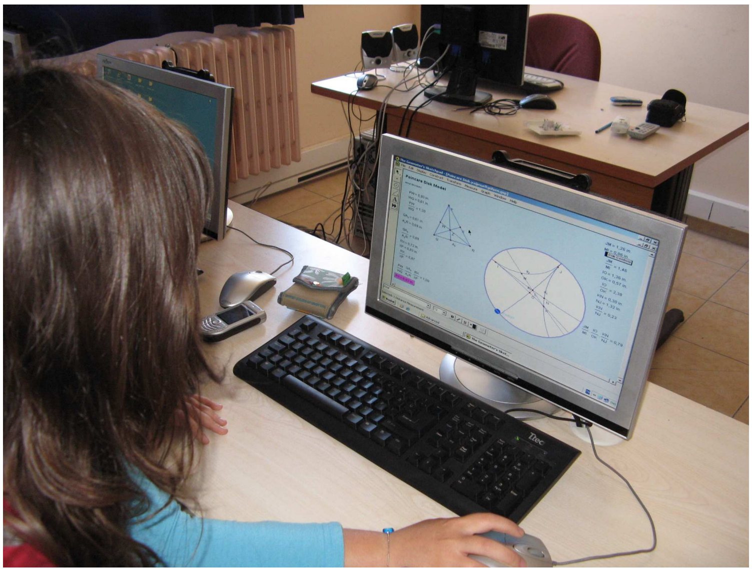

Figure 7. The result gained on the Ceva’s Theorem by one of the students

The result gained on the Ceva’s Theorem by the student checking ,the Ceva’s theorem is not proven in hyperbolic geometry presented figure 8.

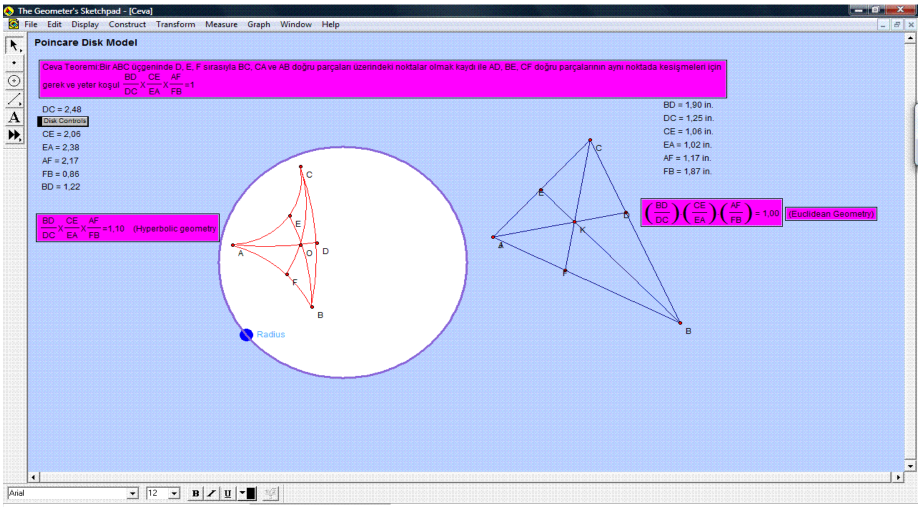

Figure 8. The result gained on the Ceva’s Theorem by the student

One of the students obtained the model of Menelaus’ theorem in Euclidean and hyperbolic geometry on the same screen. The sudent reached the conclusion after making the necessary calculations using the features of the program (see figure 9).

Figure 9. The student make the necessary calculations on the screen

The student tried different positions and compared her results. The result gained on the Menelaus' Theorem by the student checking ,the Ceva's theorem is not proven in hyperbolic geometry presented Figure 10.

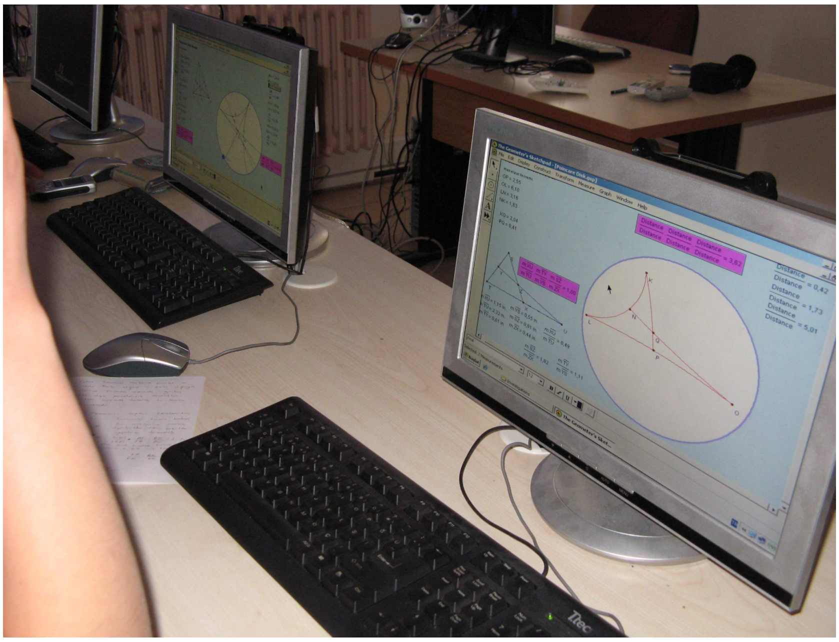

Figure 10. The result gained on the Menelaus' Theorem by the student

Once they completed all of the activities, the students stated that there were many differences between these two types of geometry, whose axiom systems were very close to each other (only parallelism axioms are different). They said they showed that many theorems proven in Euclidean geometry were not valid in hyperbolic geometry. They realized that the axioms of Euclidean geometry were just self-evident and that even a minor change in the axioms would yield a geometry quite different from Euclidean geometry. They also stated that the dynamic software helped them a lot realize these differences.

3. Conclusions

The aim of this study is to help students recall their previous knowledge on Euclidean geometry and, with the help of a computer software program, discover the similarities and differences between Euclidean geometry and hyperbolic geometry, which is one of the non-Euclidean geometries and which was chosen because it is really similar to Euclidean geometry. As hyperbolic geometry is abstract, the use of technology helped the students visualize the space. Also, because the software program used made it possible for the students to work on Poincaré model, a hyperbolic geometry model, the students were able to manipulate how the properties of Euclidean plane changed or remained unchanged in hyperbolic plane. Each of the students individually formed their own examples on computer and compared each others results. Through the observations carried out to determine the efficiency of the classes, it was concluded that they all obtained the same results by different examples. Moreover, they had the opportunity to see the different examples of each other since they all formed different ones. Above all, the teacher candidates who knew no geometry other than Euclidean geometry became aware of the existence of other geometries.

Non-Euclidean geometries can be included in secondary education geometry course syllabus at an elementary level. The use of dynamic computer software programs in teaching non-Euclidean geometries can help students visualize these abstract geometries. Finally, an experimental study might be conducted by expanding the study group.

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