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Chapitre 1

Plenaries

1.1 Introduction

L'ensemble des conférences plénières de la CIEAEM 66 ont été filmées et sont disponibles en ligne sur le site de la CIEAEM :

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Les conférenciers : / plenarists :

- Michèle Artigue, Université Diderot, Paris, France
- Paul Drijvers, Freudenthal intitute, Utrecht, The Netherlands
- Tamsin Meaney, University of Malmö, Sweden
- František Kuřina, University Hradec KRÁLOVÉ, Czech Republic,
- Alejandro Gonzales-Martin, Université du Québec à Montréal, Canada.

In these proceedings are included the text of the plenary given by Gail Fitzsimons (University of Melbourne) during CIEAEM 65 in Turin.

La conférence plénière de Gail Fitzsimons (Université de Melbourne) donnée à Turin (CIEAEM 65) est incluse dans ces actes.

1.2 It's not just a matter of pink blocks : The provenance of mathematical learning resources

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Abstract : In this paper, the role of learning resources in the teaching of mathematics to young children with diverse backgrounds is examined. In particular, learning resources are examined for their historical, social and cultural meanings which can and do impact on their potential to promote learning with different groups of students. For this research, Kress and van Leuwen's work on multimodalities is used as a theoretical frame in order to illustrate the alternative meanings that learning resources may have. Implications from a speculative example are used to suggest a broader understanding of how learning resources could contribute to different groups of children gaining foregrounds which utilise mathematical thinking.

Introduction

"*Girls are scientists too. I don't know why there are boy toys and girls toys or boy colors and girls colors. That doesn't make any sense. Blue is my favorite color.*" Rylee, aged 6

As I began this paper, there was a raging debate about LEGO and its impact on girls learning mathematics and their desire to enter science, technology, engineering and mathematics (STEM) careers [1]. Then a month away from the conference, LEGO announces the production of a range of girl scientist toys as a result of an online campaign [2]. So why this interest in gender and toys and what has it to do with mathematics education ? My interest stems from research results which suggest that teachers in diverse classrooms often feel that teaching everyone exactly the same is the most equitable approach (Norberg, 2000). Often teaching everyone the same actually means that some groups of children must learn to learn like children from dominant groups. For example, Kersh, Casey and Young (2008) recommended that "encouraging young girls to spend more time on the structural aspects of block building may be particularly helpful in reducing the gender gap in math and science achievement by promoting and nurturing the development of spatial skills and mathematical competency" (p. 247). This can be re-stated as "if girls just play with blocks like boys then they will gain the same mathematical competency and desire to enter STEM careers as boys". However, Riegle-Crumb, King, Grodsky, and Muller (2012) suggest that simplistic explanations such as these need to be rethought so the complexity of the situation is better understood. Consequently, I want to investigate why using some learning resources may be counterproductive to supporting children to learn mathematics.

Although mathematics education has a long history of using learning resources, how different kinds of resources contribute to learning is still under discussion. At the beginning of the nineteenth century, Pestalozzi advocated the use of artefacts. Since then support for their use has come in waves (Sowell, 1989). With each wave, the kind of learning resources that are advocated and their purposes that they are supposed to achieve alter. For example, Fröbel's philosophy behind advocating that children worked with his "gifts" was that they would promote mediation between the individual and God (Meaney, 2014). Recently, digital resources have been promoted as supporting children to work with more abstract concepts than are generally thought to be connected to concrete resources (Resnick et al., 1998). However, there remains questions about who determines the purpose of learning resources in mathematics teaching and how this purpose can be achieved. On the surface, the colour of resources may seem to have little to add to this discussion. However in this paper, I argue that some children perceive and respond to different attributes of learning resources, including colour in ways that designers and teachers do not recognise or intend.

Although many have investigated the role of learning resources on mathematics education learning (Sowell, 1989 ; Clements, 2000 ; Adler, 2000 ; Moyer, 2001 ; Ahmed, Clark-Jeavons, & Oldknow, 2004 ; Paek & Hoffman, 2014), the investigations have assumed that the learning resources were either carrying the meanings that researcher felt they had or no inherent meanings (see for example, Sarama & Clements, 2009). Moyer (2001) reiterated a point made by others to say in relationship to mathematical meanings, "manipulatives are not, of themselves, carriers of meaning or insight" (p. 176). Although accepting that manipulatives do not magically convey mathematical meanings to children,

1. Based on this article : http ://www.telegraph.co.uk/education/educationnews/10578106/Gender-specific-toys-put-girls-off-maths-and-science.html the International Organisation of Women in Mathematics Education (IOWME) have been discussing this issue on their facebook page.
2. https ://ideas.lego.com/blogs/1/post/11

I argue that some attributes of the learning resources may invoke children to ascribe meanings to the resources that could affect detrimentally their learning of mathematics.

Generally it is assumed that resources used appropriately can contribute to children being able to mediate between concrete and the abstract concepts and representations (Skoumpourdi, 2010). Physically manipulating learning resources is supposed to help children to gain, from a specific problem, a more abstract, generalised principal which can be used in a range of contexts (Ahmed, Clark-Jeavons & Oldknow, 2004). From this perspective, if there are difficulties with children learning mathematics through using the materials, it is generally the children or their teachers who are at fault. For example, Gravemeijer (1990) stated "concrete representations remain problematic as long as children keep seeing them as concrete material, rather than in relation to mathematics" (, p. 30). Viewing the failure of learning resources to produce the desired learning as the fault of the children or their teachers, generally from not explaining them sufficiently (Skoumpourdi, 2010), simplifies a complex situation.

In a written version of a keynote given at CIEAEM 2003, Afzal Ahmed made the following statement "The interplay among and connections between objects (structured or unstructured), images, language and symbols that lead to mathematical reasoning and the stating of mathematical propositions of very wide generality is well worth a closer study" (Ahmed, et al., 2004). In this keynote, I want to continue this study by investigating how the meanings attached to learning resources can be counterproductive for learning mathematics for some groups.

In this investigation, I use the work of Kress and van Leeuven (2001/2010) on multimodalities to discuss how mismatches in children and teacher's meaning ascriptions to artefacts could result in some groups of children not being able to use learning resources to gain the mathematical thinking anticipated by resource developers and teachers. Rather than contributing to a deficit explanation in which children or teachers are blamed for lack of learning, understanding the potential different meanings connected to modes contributes the mismatch can be used to identify alternative approaches to learning resources. However, before considering multimodalities, I want to discuss how meaning ascriptions are connected to dispositions to learn.

Dispositions to learn

In order to understand the relationship between using learning resources and inequitable participation in learning opportunities, it is important to recognise that all children learn something when engaged in activities, it is just that what they learn may not be about mathematics. This will have an impact on the meanings that that the children see as possible to ascribe in future situations. Whilst engaged in using learning resources, children draw upon and ascribe meanings, mathematical and otherwise, to their manipulations of those resources. These meanings then become potential meanings which can be ascribed to future situations where the children engage with the same or similar, from their perspective, learning resources. Their previous experiences therefore have an impact on their dispositions to learn in future situations. This relationship has been discussed by Skovsmose (2005a ; 2005b) from the perspective of children's backgrounds and foregrounds. Skovsmose's (2005b) viewed a child's disposition to learn as being affected by perceptions of past and potential future experiences. "Meaning in learning comes to refer to a relationship between the dispositions of the learner, the intentions of the learner, the intended and unintended effects of learning activities, and the learner's reflections on these effects" (p. 93). The dispositions of a learner "embody propensities that become manifest in actions, choices, priorities, perspectives, and practices" (Skovsmose, 2005a, p. 7). These propensities may be contradictory because the child may conceptualise different foregrounds and backgrounds simultaneously in the same situations.

By the foreground of a person I understand the opportunities, which the social, political and cultural situation provides for this person. However, not the opportunities as they might exist in any socially well-defined or 'objective' form, but the opportunities as perceived by the person. Nor does the background of a person exist in any 'objective' way. Although the background refers to what a person has done and experienced (such as situations the person has been involved in, the cultural context, the socio-political context and the family traditions), then background is still interpreted by the person. Taken together, I refer to the foreground and the background of a person as the person's dispositions. (p. 6-7) A child's foreground is determined by their current understandings about social, political and cultural situation in which they are situated. Thus, reflections on their past experiences form their interpretations of future possibilities. Most work that has used Skovsmose's ideas about foregrounds and backgrounds have not considered interactions with learning resources but learning more generally (for example, Gorgório & Planas, 2005 ; Lange 2009 ; Alrø, Skovsmose, Valero, 2009). However, it seems relevant that the impact of children's responses to engaging with different learning resources be considered as a component of their backgrounds which contributes to perceptions of their foregrounds and thus their dispositions to learn.

Multimodalities

Thus it can be considered that children's dispositions to learn will be affected by the meanings that they ascribe to the learning resources and to the situations in which they are used. Meanings can be expressed through verbal language but also through other modes :

Children use the full range of material and bodily resources available to them to make and express meaning . . . Language is only one tool in a range of human semiosis, and . . . individuals choices of semiotic modes are motivated by a complex web of interconnecting personal, institutional and social factors. (Flewitt, 2006, p. 46)

Drawing inspiration from Halliday's (1985) work on Systemic Functional Linguistics, Kress and van Leeuwen (2001/2010) described how meanings come to be attached to different modes, which included a range of physical and ethereal forms of communication. These included speech and writing, but also colour and sound. In their discussion, they suggest two semiotic principles, provenance and experiential meaning potential. Provenance is of interest when a sign is transferred into a new situation and the meanings that surrounded that sign in its original context remain with the sign. When LEGO blocks were coloured pink, meanings attached to pink, such as it being "a femininity marker" (Koller, 2008, p. 402) remain with the blocks. On the other hand, experiential meaning potential is about using information given to a sign that comes from a bodily experience. Kress and van Leeuwen (2001/2010) stated "material qualities can also acquire meaning, not on the basis of 'where they come from', but on the basis of our physical, bodily experience of them" (p. 74). For example, talking about something being heavier can include sensual meanings about the mass of an object that come from experiences of holding different objects.

Meanings can be expressed in different modes, such as the choice of a theme colour for a magazine article. This colour would be obvious in the pictures but also in the written language describing the pictures. In considering Kress and van Leeuwen's theory in regard to learning resources, their shape, colour and size, for example, could all articulate different meanings. The traditional linguistic account is one in which meaning is made once, so to speak. By contrast, we see the multimodal resources which are available in a culture used to make meanings in any and every sign, at every level, and in any mode. Where traditional linguists had defined language as a system that worked through double articulation, where a message was an articulation as a form and as a meaning, we see multimodal texts as making meaning in multiple articulations. (Kress & van Leeuwen 2001/2010, p. 4 ; italics in the original).

The meanings expressed by different attributes may be in harmony, or in conflict or somewhere in between depending on what the interpreter brings with them to the interpretation. Using their background knowledge, an interpreter can decide that one meaning is more important than other meanings. However, in doing so the interpreter's valuing of this interpretation is also reinforced because insight from other interpretations are not utilised, thus solidifying the importance of a particular meaning, in a two way feedback loop.

When one interpretation has become normalised as "the" only appropriate interpretation, then issues of power come into play. In explaining how grammar books come to represent how language should be judged, Kress and van Leeuwen (2002) wrote : In this case we can have grammar books, which become authoritative sources of information on the practices, even though they are simply records of these. However, here power intervenes inasmuch as these records tend to be of the practices of those who are regarded as belonging to a group whose usage can be accepted as definitive, and may as a result be imposed on other groups as well. (p. 344)

Thus, when expectations of how learning resources will contribute to developing children's mathematical thinking are based on the meanings assigned to them by one particular group then it is likely that other groups' mathematics learning will be impeded as their interpretations are disregarded or remain unrecognised.

Two examples

In order to exemplify how Kress and van Leeuwen's views on multimodality can be operationalised, I provide two examples from research into the development of games or apps for ICT that will support young children's engagement with mathematics. The first example illustrates how children's interpretations of meanings embedded into the game through its design contributes to the children learning about conjecturing. In the second example, the provenance attached to representations of the apples and numerals produce responses that are unlikely to contribute positively to mathematical learning of any kind for the child involved.

The first example comes from a six and a half minute video of two Swedish preschool children playing on an interactive table (see Lembrér & Meaney, 2014). The game was designed by first-year university, gaming students. As can be seen in Figure 1.1, the game consisted of a virtual set of scales, with different combinations of cubes, available at the bottom of the screen, which could be dragged to the scale pans. The balance beam can tip to suggest that one

Figure 1.1 – Making the scales even

side is heavier than another. When the pans are considered to be equal, a light in the centre of the balance beam glows green. A game such as this is known as a virtual replication (Paek & Hoffman, 2014). Unlike the difficulties identified by Paek and Hoffman (2014) in regard to virtual replications which relied on children using a computer mouse, the touchscreen technologies meant that the children intuitively determined how to move the cube blocks around.

(0 :01 :16.6) Anna : Annars kan vi göra så här. Att man tar av den.	Anna : Otherwise, we can do so here. That, you take off it.
Anna tar bort gul kub4.	Anna removes yellow 4 group of cubes.
Anna : Så tar han av den.	Anna : So he takes it.
Anna pekar på vågskålen på Albins sida.	Anna points to the scale pan on Albin's side.
Anna : Så tar han av den (hela innehållet på Albins sida vågen)	Anna : So he takes it off (the entire contents of the pan on Albin's side of the balance) and
och då tar han denna från mig.	then he takes it from me.
Anna pekar på gulkub2x2.	Anna points to yellow group of 2x2 cubes.
Läraren : Funkar det ?	Teacher : Does it work ?
Albin drar ner gul stapelkub2x3 och drar upp gul kub2x2 som ligger på golvet på Annas sida.	Albin pulls down a yellow 2x3 group of cubes and pulls up yellow 2x2 cube lying on the bottom on Anna's side.
Anna och Albin tittar tiden på vågen. Det blir grönt.	Anna and Albin watch the centre of the scales. It becomes green.
Albin : Jaaaaaa	Albin : Yessssssssss !
Läraren : Ooooohhhhh (gör ett förtjust läte). Det klarade ni ju !	Teacher : Ooooohhhhh (make a delighted cry). It says you have succeeded !

This game relies on the children interpreting the meanings embedded into the different objects on the screen to play the game. Having had some experiences with an actual balance, it was possible for the children to make use of the resemblance to a virtual balance. At different times, the teacher and the children used terms such as "heavier" or "lighter" to describe what they saw (Lembrér & Meaney, 2014).

Making use of the experimental meaning potential (Kress & van Leeuven, 2001/2010) connected to experiencing what heavier and lighter looks like on a real balance allowed the children to produce and learn about conjectures, while playing the virtual game. Their dispositions to learn was supported by positive previous, related experiences, even though the learning was about mathematical conjectures, rather than balances.

Although a teacher was present and interacted with the children, the game's green light came to gain the role of expert, in that it judged the children's success in balancing the scales. In Western society, the colour green has a provenance which connects it to something positive and moving forward. If the balance beam had glowed red, it was likely to be interpreted as meaning something else. Nevertheless, green does not have the same provenance in other cultures. For some children, the clues embedded into this game would make little sense to them and not produce the kind of conjectures that these young Swedish children did. This may lead to some groups of children feeling that mathematics makes little sense to them and thus reduce their dispositions to learn.

In the second example, the meanings that come with fruit and the presentation of the equations interfered with a 6 year old child, Miguel, my nephew, learning mathematics from playing games on a tablet (Lange & Meaney, 2013).

On the third and final day of recordings, the following interaction occurred around an app called Knattematte (see Figure 1.2).

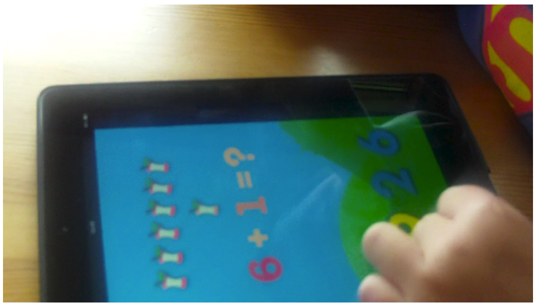

FIGURE 1.2 – Addition and eating apples

T : I wonder what this one does ?
M : Mmm, apple. I don't see any apples. (apples start to fall from the top of the screen)
T : How many apples can you see ?
M : Three. You just have to eat them.
T : You don't press four ? No. It's a very strange game. Two, three.
M : You've just eat them.
T : Four, five, six, seven.
M : I think that was eight.
T : Was it eight ? (new game)
T : Seven, one, two, three, four, five, six, seven. (new game)
M : Red apple
T : So how much is six plus one ?
M : Six
T : One, two, three, four, five, six. Six plus one makes ? And which ? (points to the numbers at the bottom of the page)
M : Seven.
T : There you go. (Miguel successfully answers the next question by himself)
T : Well done. (In the next go, the answer is 6 but Miguel pushes 9 instead. It just shakes. He then pushes the 6)
M : Six, five plus one. Two plus seven ? Nine.
T : Yep, you know these ones that's pretty good. (M in the next game, just pushes a range of numbers till the correct one moves into position. Then the next game comes up).
M : Nine
T : One plus six is ?
M : Seven
T : Mmhmm. One plus.
M : This is a maths game. (Miguel returns to the main menu)

Knattematte gave the player a choice of three different fruits. Although the fruit turned out to stand for different levels, Miguel chose apples initially because they were his favourite fruit. Correspondingly, the experimental meaning potential for him was the taste of the apples. Therefore, it is perhaps not so surprising that Miguel considered that virtual eating of the apples was what he should do. Although I, with my teaching experience, tried to steer him towards counting, he found that he could make the apples look like they were eaten by tapping on them.

Later, a simple addition was given with an appropriate number of apples and a choice of four numbers provided at the bottom of the screen. The correct number needed to be tapped, but if a wrong number was tapped, it just shook. Miguel soon learnt that finding the correct answer to move on to the next game could be achieved by simply tapping on each number until the correct one was identified. Regardless of finding this solution strategy, he labelled Knattematte "a maths game" and promptly left it. Although only 6, Miguel had grown to dislike mathematics as represented by worksheets. His emotional reaction to worksheets counterbalanced his liking of apples so he opted out of the game. However, he played other apps for a long period of time, when they did not resemble artefacts from school mathematics, even if I could see that they involved different mathematical understandings (Lange & Meaney, 2013).

Thus, the provenance of this game and its experiential meaning potential as interpreted by Miguel resulted in possible learning opportunities being discarded. He had no disposition to learn through this game.

These two examples indicate how designers, teachers and children's interpretations of learning materials, whether they are physical or digital, can be in harmony and thus contribute to children's learning about mathematical thinking or in dissonance resulting in a lack of learning from these materials. These are individual examples. In the next section, I hypothesise how groups of children with similar background experiences may come to form different interpretations of learning resources than was anticipated by adults and how this might contribute to these groups being less likely to take up certain kinds of learning opportunities.

The issue of pink blocks

In this section, I describe briefly what is known about the connection between children's early childhood building with blocks, particularly LEGO and later mathematics interest and achievement in high school. From there I consider how spatial thinking is related to having a STEM career. I also consider what is known about gender differences in regards to spatial thinking, which has been linked to playing with blocks. From this, I argue that the provenance of the colour pink might contribute to girls building more LEGO and thus gaining spatial thinking but it may also make girls feel that unless STEM careers also have an aura of femininity then these are not for them. Thus making LEGO bricks pink could reinforce girls' perceptions that only feminine careers are possible. This is a speculative argument which requires research. Nevertheless there is sufficient existing evidence to use this as an example, even if speculative, of how the foregrounds of some groups of students can be reduced.

The issue of pink bricks is a speculative example of how some groups of students may be channelled into making choices about their mathematics learning through subtle influences such as the meanings ascribed to particular learning resources because of their outward appearance. This is not to say that dispositions to learn are linked exclusively to the use of learning resources. Instead I argue that because learning resources are often considered in research literature to have limited if any meanings embedded into them, the non-mathematical meanings associated with them may go an unrecognised which could contribute to the reasons for some groups disengaging from mathematics not being understood which allows the children or their teachers' to be blamed for a societal problem.

In their longitudinal study of the impact of block play on school mathematics achievement, Wolfgang, Stannard and Jones (2001) found that the complexity and adaptability in children's block play in preschool correlated with their mathematics achievement in high school. As well, in relation to LEGO play, they found :

Since all other outcome variables at the middle school and high school levels such as number of classes taken, number of honors classes taken, average mathematics grades, and a combined weighted value of all mathematics courses taken were all significant, we may clearly state that there is a statistical relationship between early LEGO performance among preschool and achievement in mathematics, not during the elementary school years, but later at the middle and high school level. (Wolfgang, Stannard, & Jones, 2003, p. 473)

The data that they had collected could not be used to determine whether the correlation was causal. However, other researchers such as Piccolo and Test (2010/2011) suggest that block play leads to increased spatial thinking and having high levels of spatial thinking is linked to students choosing STEM careers (Uttal & Cohen, 2012). If LEGO play also contributes to improving spatial thinking, then Wolfgang et al.'s (2003) result would seem to be in alignment with these other studies. There has been some research on gender aspects in regard to spatial thinking. Newcombe (2010) stated that males had superior spatial skills and while training was unlikely to result in closing the gap in results for genders, it would improve the skills of both males and females. It has been suggested that boys are more involved in block play in preschools and this may contribute to better visual-spatial skills (Calder et al., 1999). However, in their own study with preschool children, Calder et al. (1999) did not find any significant differences in play type according to gender or to spatial-visual skills. More recently, Ferrara, Hirsh-Pasek, Newcombe, Golinkoff, and Lam (2011) found that there was no gender differences in a study of the language used by preschool children and parents involved in block play. It thus seems that gender differences in block play and the development of spatial skills is not clear cut.

In a study on adults from two geographically but culturally different communities in rural India, Hoffman, Gneezy and List (2011) found that gender-related differences in solving a spatial problem were present in the patrilineal society but not in the adjoining matrilineal society. Although it cannot be presumed that there is a causal correlation, it does seem that cultural norms may play some part in whether gender differences in spatial thinking occurs.

It is therefore interesting to speculate how colour might affect whether or not gender differences arise in regard to developing spatial thinking from block play. Studies such as Ferrara et al. (2011) and Wolfgang et al. (2003) used blocks in a range of colours. However, the new version of LEGO for girls is predominantly pink and purple in colour. The figures are also different, less androgynous and use pink and purple in the clothing. The building sets for LEGO Friends are of poodle parlours, beauty shops and shopping malls. Without a thorough analysis it is difficult to know if the construction demands differ between the traditional kits and LEGO Friends. Figure 1.3 provides a photo of different LEGO friend constructions.

FIGURE 1.3 – Building with LEGO friends

The complexity of the building skills needed for LEGO Friends may provide opportunities to gain the same spatial thinking skills as from constructing the traditional kits which could result in the higher mathematical achievement noted in Wolfgang et al.'s (2003) study. A similar expectation could be that girls can develop superior spatial thinking from constructing LEGO Friend kits and this could contribute to more of them seeing STEM careers as part of their foregrounds.

However, colouring the bricks pink with its provenance connected to femininity may counterbalance the desire to adopt a STEM career. As "gender differences in math performance, even among high scorers, are insufficient to explain lopsided gender patterns in participation in some STEM fields" (Hyde, Lindberg, Linn, Ellis, & Williams, 2008, p. 495), there is a need to think about other aspects of this complex situation to consider causes and thus solutions. Although the introduction of pink LEGO bricks would not yet have any impact on gender disparities, there is a need to think about why simply improving mathematical skills may not lead to more women taking up STEM careers.

Pink had originally been a male colour, associated with the red of blood and fighting. In contrast, blue was considered a girl colour because of its association with the Virgin Mary's cloak (Koller, 2008). However in the last half century, the gender associations with colours have swapped. Koller (2008) found that as a result many companies marketed products perceived as being unfeminine in pink, such as cars or electronic goods. Those marketing these products judged, rightly in most cases, that the products would take on the qualities of femininity that pink portrayed. Although Koller (2008) also noted that "many positive associations with pink can easily tip over into negative associations such as naivety and stupidity", pink has come to "communicate fun and independence, financial and professional power without conforming to masculine norms, as well as femininity and self-confidence" (p. 416). However colouring cars and electronic gear in pink has not necessarily contributed to women wanting to understand the technology on which these things are built.

STEM careers are situated as primarily masculine occupations (Riegle-Crumb, et al., 2012), and thus in opposition to the pink provenance of femininity. Even the post-feminist associations with pink that situate women as financially independent and competent (Koller, 2008) may not contribute to girls associating their exploratory play with pink bricks with foregrounds that include having STEM careers. The choice by the LEGO company to produce LEGO Friends was a commercial one and it has been extremely successful[3]. However, it cannot be considered to have the same potential outcomes as LEGO's choice in the 1990s to focus on kits for boys. This focus allowed playing with LEGO to be seen as a masculine activity which can be regarded as encouraging a seamless move into seeing STEM careers, with the same provenance, as available in their foregrounds. It is not clear that girls will see such a smooth path ahead of them. Girls may gain more spatial thinking skills but this may not contribute to them taking up STEM careers as it seems to have for boys. This is in contrast to suggestions that increased building block experiences will provide a smooth path towards STEM careers as suggested by researchers such as Kersh, Casey and Young (2008). Riegle-Crumb et al. (2012) suggest that associations of STEM careers with masculinity will continue to be the main contributor to why girls do not take up these career choices -"yet because of the societal pervasiveness of gender essentialist beliefs and the accompanying socialization and micro-level interactions that support them, gendered patterns in choice of major will not shift accordingly" (p. 1067-1068). Thus, the focus on pink and its associations with femininity might result in restricted or at least not expanded foregrounds for girls in regards to STEM careers, even if more girls improve their spatial skills. Rylee's point, made at the beginning of the paper, echoes the concerns of researchers such as Riegle-Crumb et al. (2012). Colouring toys particular colours to indicate whether they are for boys or for girls flies in the face of her belief that "girls are scientists too".

Ways forward

The meanings that are ascribed into learning resources reflect the wider cultural values of the dominant groups within a society. Some groups of children who do not share the background experiences of children from dominant groups may have difficulties interpreting the meanings attached to learning resources that designers and anticipate. Having a green light to indicate that a correct result has been achieved only makes sense because in many Western countries green has this provenance. Although its use in learning resources will reinforce this often undiscussed meaning making, if children initially do not make the connection then they are unlikely to be able to reinforce this meaning. Consequently mathematics may take on the provenance of being nonsensical and other mathematical experiences are reacted to as though they have the same nonsensical provenance.

On the other hand, the provenance of aspects of learning materials such as colour can be problematic in its own right and result in some groups of children having their foregrounds reduced. Teachers and resource designers may anticipate that making a learning resource pink will encourage girls to use it but the other connotations that come with that colour may offset these advantages. Thus, the neutrality of learning resources needs to be problematised so that the interaction between them and children's dispositions to learn mathematics, both in present and future opportunities, are better understood. Although teachers in classrooms and designers of learning resources can take into account the provenance and experiential meaning potential of different aspects of learning resources, there is also a need for mathematics educators and researchers to challenge stereotypes at the societal level. If more equitable representation of different groups in STEM careers is to be achieved then rhetoric alone, about the need for such a change, will not produce it. Colours which have gendered, class or ethnicity connotations are only one aspect of learning resources which can reinforce the idea that certain skills can be developed only by particular groups of people. Changing colours will not change outcomes unless societal beliefs are fundamentally challenged. In 1916, after reviewing research on women and their possibilities for taking up different careers, Lowie and Hollingworth wrote "no rational grounds have yet been established that should lead to artificial limitation of woman's activity on the ground of inferior efficiency" (p. 284). Yet in 2014, we still have a psychologist gaining newspaper headlines by suggesting that innate differences between women and men will mean that few women will ever become engineers[4]. Such headlines suggest that research results alone will not lead to more equitable foreground perceptions by children. It also means that aware teachers and children, such as Rylee at six years old, cannot be held responsible for overcoming societal value systems by themselves.

3. http ://www.businessweek.com/magazine/lego-is-for-girls-12142011.html
4. http ://www.dailymail.co.uk/news/article-2689540/Stop-trying-make-girls-science-It-goes-against-human-nature-claims-psychologist.html

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1.3 Mathematics in and for work in a globalised environment

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Résumé : Cet article commence par une brève analyse de quelques conséquences de la mondialisation pour la politique d'éducation, qui dispose actuellement d'un fort accent sur l'économie. En particulier, il problématise des documents pédagogiques encadrées en termes de compétences et d'aptitudes. La vie professionnelle est de plus en plus exigeant : apprentissage continu et l'innovation qui, en partie, dépendent de la connaissance disciplinaire compétent. Cependant, contrairement à l'école, les problèmes et les solutions ne peuvent pas être connues et enseignées à l'avance. Localement, de nouvelles connaissances doivent être élaborées, de façon créative et en respectant les contraintes, pour trouver une solution viable. Je donne un exemple paradigmatique de la façon dont les mathématiciens et les ingénieurs collaborent dans l'industrie de l'acier. Je vous propose ensuite quelques exemples de l'enseignement des mathématiques problématique pour le lieu de travail et l'examen de la façon dont l'enseignement des mathématiques peut être différente. Enfin, je présenterai un projet de recherche en milieu de travail novateur qui adopte une approche socio-mathematique.

Abstract : This article opens with a brief analysis of some consequences of globalisation for education policy, which currently has a strong focus on the economy. In particular, it problematises curriculum documents framed in terms of competences and skills. Working life is increasingly demanding continuous learning and innovation which, in part, depend upon relevant disciplinary knowledge. However, unlike school, problems and solutions cannot be known and taught ahead of time. Locally new knowledge must be developed, creatively and within constraints, to find a workable solution. I give a paradigmatic example of how professional mathematicians and engineers collaborate in the steel industry. I then offer some examples of problematic mathematics education for the workplace, and consideration of how mathematics education might be different. Finally, I introduce an innovative work-place research project which adopts a sociomathematical approach.

How might globalisation be understood in relation to education ?

> Contemporary globalization is defined as the intensification of cross-national interactions that promote the establishment of transnational structures and the global integration of cultural, economic, environmental, political, technological and social processes on global, supranational, national, regional and local levels. (Dreher, Gaston, & Martens, 2008, quoted in Milana, 2012, p. 782)

Educational activity is shaped simultaneously in global, national, and local spheres, and today's education systems are no longer immune to the effects of globalisation. Educational identities of teachers and students alike are becoming more standardised, while at the same time there is unprecedented access to, and sharing of, knowledge and information, ideas and products, real or virtual –witness the global uptake of Web 2.0, social media, etc.

One of the consequences of globalisation, and the rapid and unpredictable flows of money and people, as well as advanced technologies and communications, is that the historical model of full-time education followed by full-time work until a dignified retirement no longer applies to the vast majority of people. The likelihood is that there will be several changes of jobs and even occupations, in different geographical locations ; periods of unemployment or under-employment ; casual or contract labour or relatively more secure employment ; promotion or demotion. Although learning is an integral part of living and being (informal education), inevitably there will be an unprecedented need for ongoing learning, formal, and non-formal education in and for work and beyond. Yet, there is also the likelihood of gaps between periods of formal education, and returning to study can be a challenging experience, particularly in mathematics with its constantly evolving use of new technologies.

In terms of education policy, there has been a marked trend toward responding to the needs of the economy ahead of other global concerns such as the environment and human rights (Milana, 2012). This is reflected in the introduction of various mechanisms of quality assurance originally designed for business, including Total Quality Management models (e.g., ISO 9000), benchmarking, and rankings (FitzSimons, 2011). The politicisation in many countries of PISA and TIMSS results, along with national testing, directly influences mathematics curriculum and assessment, even though few people are actually willing or able to question the epistemological basis on which these assessments are formulated (Lundin, 2012).

Salling Olesen (2010) briefly outlined the history of work from traditional societies, where working and learning were an everyday part of life, through industrial capitalism to the present, post-industrial workplaces. He identified shifting conceptualisations of work, from a focus on the *individual* to a *human capital* perspective or else an *individualistic learning* perspective. He noted that, in recent times :

> a good deal of industrial work has turned into planning, control and adjustment, as well as communication and logistic tasks. Reporting has become an integrated aspect of manufacturing and service work. The quantitative proportion of office work and different types of business services is increasing, the difference from manufacturing on the other hand is becoming less obvious. Office work is undergoing many processes of industrial reorganisation similar to those that took place in manufacturing. Automation and semi-automation by means of information technology is shaping data processing, accounting and text processing in a way which has similarities with the industrial development. (p. 7)

He observed that there are new divisions of labour, offering new opportunities for learning ; while, at the same time, the notion of *workplace* has also taken on new meanings as a societal institution on one hand, and involving new spatial and sensual realities, so that older stereotypes of work and learning should be avoided.

In summary, one consequence of globalisation is greater interconnectivity worldwide, which has resulted in less control by national governments over critical policy areas such as education and work. In addition to environmental phenomena such as climate change threatening survival of flora and fauna (including humans), there are also rapid and unpredictable flows of people and finance, generating uncertainty at societal and individual levels. Consider, for example, the dire effects on large cities, such as Detroit, of volatile demand (due, in part to global oil price fluctuations and global financial crises) alongside the technological substitution for labour or shifts to low-wage economies in the automobile industry ; and this is played out in smaller cities and towns throughout the industrialised world, with consequential flow-on effects for the su-rounding population.

Knowledge, competence, and qualifications

The global flows of knowledge, people, capital, and jobs have given rise to the issue of recognition or validation of skill levels, acknowledging people's non-formal and informal learning. Formalised as National Qual-fications Frameworks, they often include the implementation of a *learning outcomes-based* approach to curriculum. At an *individual* level, these learning outcomes are generally taken to be the knowledge, skills, competences, attitudes, and experience gained as an inherent part of learning and work processes. *Socially*, the anticipated benefits are to enable greater mobility of workers, to promote lifelong learning, and to assist employers from a Human Resources perspective (Bohlinger, 2012 ; EACEA, 2010). However, there is no universal agreement on the definitions of any of the key terms used to describe learning outcomes. The critical literature claims there is not sufficient research to support the political changes from an epistemological perspective, as it is argued that learning outcome statements are not capable of capturing complex curricula and knowledge satisfactorily : Learning outcomes can inhibit useful learning processes, fail to recognise explorative and unintended learning, and create a target-led culture (Souto-Otero, 2012). Bohlinger (2012) points out the problem of assessment of competence (as a performance) can only ever reflect part of that competence, and is clearly dependent upon the skills (or competence) of the assessor.

In recent years the majority of European countries have revised their mathematics curricula to bring into effect a stronger focus on competences and skills, an increase in cross-curricular links, and a greater emphasis on the application of mathematics in everyday life. The Education, Audiovisual & Culture Executive Agency (EACEA) (2011) states :

> ...mathematical competence will be understood to go beyond basic numeracy to cover a combination of knowledge, skills and attitudes. Mathematical competence will refer to the ability to reason mathematically, to pose and solve mathematical questions, and to apply mathematical thinking to solve real life problems. It will be linked to skills like logical and spatial thinking, the use of models, graphs and charts and understanding the role of mathematics in society. (p. 8)

However, for the five identified areas of mathematical competence –mastering basic skills and procedures, understanding mathematical concepts and principles, applying mathematics in real-life contexts, communicating about mathematics, and reasoning mathematically –the EACEA report found that specific teaching and assessment methods were lacking, and made recommendations for support and reinforcement considered desirable in line with EU policy. Clearly, policies can affect all involved in education. But, do they achieve what they intend –especially in relation to work ? As Gal (2013) reminds us, there has been little interest shown by the mathematics education community in the external perspective : in what actually happens to students after they leave school, in how people actually use mathematics in the workplace or elsewhere, or in understanding how the mathematical knowledge and dispositions developed in school play out in adults' lives.

What kinds of skills and competence are needed in the workplace ?

In workplaces, apart from academic knowledge such as mathematics, there is generally a wealth of codified knowledge in the form of manuals, charts, instructions, etc., in print and online. This codified knowledge is combined with the pervasive non-codified *cultural* and *personal* knowledge of workers which Eraut (2004) considers as *skills*. By cultural knowledge he means knowledge that is uncodified and acquired informally through participation in social activities, often taken for granted. Personal knowledge is defined by Eraut (2004) as :

> what individuals bring to situations that enables them to think, interact and perform ... it includes not only personalized versions of public codified knowledge but also everyday knowledge of people and situations, know-how in the form of skills and practices, memories of episodes and events, self-knowledge, attitudes and emotions. Moreover, it focuses on the use value of knowledge rather than its exchange value in a world increasingly populated by qualifications. (pp. 263-264)

Workers bring three interrelated kinds of *skills*(or logics) : (a) technical (related to equipment & work organisation), (b) behavioural (non-codified cultural & personal qualities), and (c) cognitive (education & training) skills (Mounier, 2001 ; Wedege, 2000). The capacity to exercise these skills is developed from personal, social, formal and non-formal educational, and working life experiences. In post-industrial workplaces there is a rich mixture of various forms of knowledge, and hence opportunities, or imperatives, for learning. However, learning at work is of a qualitatively different kind from learning at school. There is always a specific purpose, or a social motive, *integrating* learning into meaningful social and communicative contexts.

Working for a living is inevitably a social and cultural activity, mediated by tools and artefacts, material and non-material, including an inescapable need for communication in many forms, verbal and non-verbal. Eraut (2004) prefers a social-centred definition of *competence* because it reflects both the everyday role on the job and the mediating role between workers within and between levels, internally, and between the workplace and clients, externally. He notes that "judgements of competence are still very situation specific, depending on the context of the performance and also the expectations of each individual performer." He added that "what counts as competence will change over time as practices change and the speed and quality of work improves. Thus, from a learning viewpoint, competence is a moving target" (p. 264). Compare this with the static concept of competence used in education discourse.

In order to stay in business, workplaces need to produce and use new forms of knowledge or recontextualise existing forms. Underlying this knowledge creation is "the interdependence between different forms of knowledge –theoretical and tacit" (Guile, 2011, p. 4), a consideration often neglected by education policy makers. Engeström and Sannino (2010) argue that "traditional modes of learning, deal[ing] with tasks in which the contents to be learned are well known ahead of time" (p. 3) are inadequate when faced with the complexities of shorter life cycles of entire production, with their constantly shifting or accelerating transformations in conceptualisations. Workers are constantly having to learn things that do not currently exist, and for which they have no prior experience. (For a graphic example of this in the highly complex work of structural engineers working only from architects' plans, see Gainsburg, 2006.)

Innovation at work

Particularly in a globalised economy, responsiveness to continual change is essential, requiring different forms of innovation, and hence learning, at work. Ellström (2010, p. 27) portrayed practice-based innovation as a cyclical process of adaptive and developmental learning (see figure 1.4). It is driven by contradictions and tensions between *explicitly* planned change and ongoing change arising *implicitly* through the constant "variations and modifications in performance that arise in response to unforeseen events, disruptions and problems" (p. 37).

Apart from formalised Research and Development, innovations may be "viewed as a function of the learning and knowledge creation that takes place in the production of goods and services in organizations" (Ellström, 2010, p. 27).

> A basic assumption ...is that the interface and the interplay between the explicit and implicit dimensions of work may be driving forces for learning and innovation processes. The underlying idea is that tensions and contradictions between work processes as officially prescribed (the explicit dimension) and as perceived and performed in practice (the implicit dimension) create potentials for learning and practice-based innovations in an organization. (p. 32)

This process "begins with questioning, a disturbance or the emergence of a problematic situation ...[which] leads to routinized patterns ...being broken and a search for new ways of dealing with the disturbance or the problematic

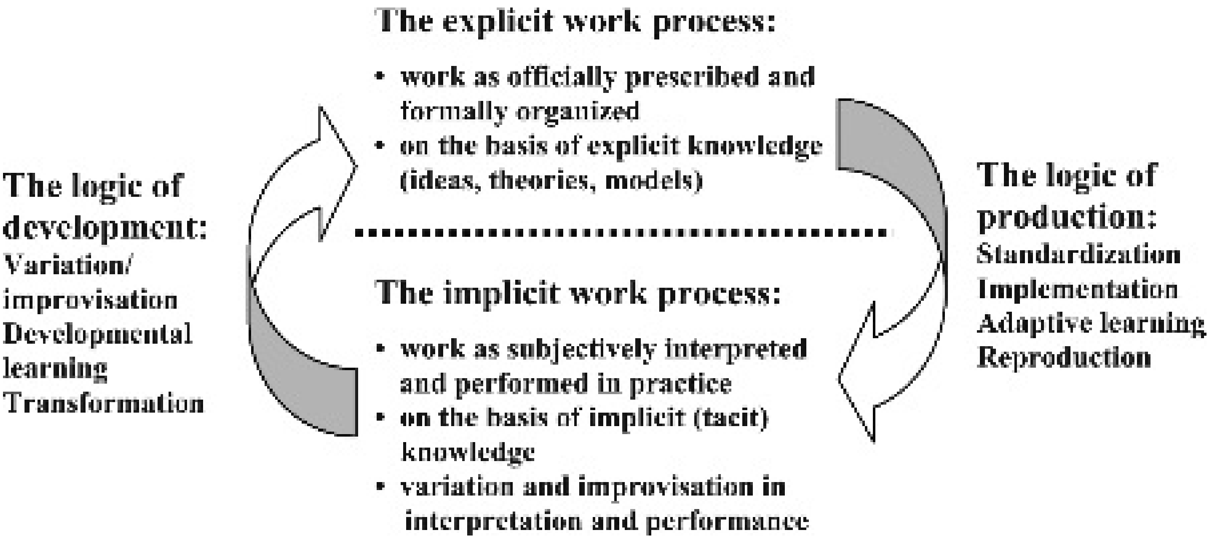

FIGURE 1.4 – Practice-based innovation as a cyclical process of learning (Ellström, 2010, p. 32)

situation at hand" (p. 36). Ellström explains how the interplay between these two operational dimensions (explicit & implicit) takes place in accordance with two complementary processes or logics. The logic of production has an emphasis on the mastering and reproduction of prescribed work processes, while the logic of development is mainly focused on exploration and re-conceptualisation (or reconstruction) of the operations that are performed in practice. Following a similar line of thought to Eraut (2004) regarding personal or individual factors, Ellström (2010) continues that the implicit work process depends in part upon these which could include "knowledge, values, attitudes to work, emotional and personality-related factors ...There is typically a considerable creativity and an ability to improvise when it comes to finding solutions to unexpected problems that arise" (p. 31).

Focusing specifically on workplace learning, the logic of production requires a "process of adaptive or re-productive learning [with] a focus on establishing and maintaining well learned and routinized action pat-terns. [The aim] is to reduce variation so that the task concerned can be performed rapidly and with a low percentage of error" (Ellström, 2010, p. 33). On the other hand, the logic of development presupposes learning with "a strong emphasis on the subjects' capacity for self-management and their preparedness to question, reflect on and, if necessary, transform established practices in the organization into new solutions or ways of working" (p. 34). The importance of each kind of logic will vary with the kind of work being undertaken : for example, nurses responsible for administering drug dosages would be expected to routinely use their mathematical knowledges and skills at the highest level of accuracy and be ready to question possible misunderstandings in communication ; whereas nurses responsible for maintaining a patient's health and well-being from social and emotional perspectives would tend to vary their practices according to the specific needs of that person.

There are a number of different barriers to innovation which may be related to subjective factors (e.g., an individual's competence), organisational, cultural, or structural factors (e.g., the division of labour). The in-dividual's subjective capacity is related to, for example, "previous experience with similar tasks, [their] knowledge and understanding of the task at hand, self-confidence and occupational identity" (p. 36). However, organisational and structural factors may exert a greater influence on learning and innovation where mathematics and technology use are concerned (Smith, 1999) as management decides the level of responsibility for individual workers.

In summary, innovation is the essential core of workplace activity in a globalised environment. The ques-tion arises : How well does today's mathematics education prepare students for this world of work which re-quires continuous development, creativity, a blend of theoretical and tacit knowledges, as well as complex personal and cultural knowledges ? In relation to creativity and innovation, how meaningful is the learning that takes place in formal mathematics education ?

What does the literature tell us about mathematics IN work ?

Across a range of industries and sectors –manufacturing, trades, creative and performing arts, agricultural production, minerals and energy, etc –mathematical concepts, thinking and reasoning are frequently an implicit, if not explicit, aspect of planning, production (taken in its broadest sense), and communications. A summary of recent workplace mathematics literature (FitzSimons, 2013) indicates that workers at all levels may need to operate with advanced technological tools. For example, they might measure, collect, enter, and/or interpret data (in graphs, tables, spreadsheets) ; they might (re-)program and even adjust the operations of machinery in use ; or keep electronic records of human and material resources and stock. These technologies radically extend human thinking and learning capabilities, and embody the often unique, idiosyncratic, conceptual systems of their designers, rendering the underlying mathematics invisible. However, most forms of technology in the workplace, material and communicative, can only function in combination with human activity (Wedege, 2000) through what has come to be known as *instrumental genesis* (Rabardel, 1995/2002 ; Trouche, 2004).

Acknowledging the complexity of the so-called *transfer* of knowledge from academic institution to workplace, Eraut (2004) identified five interrelated stages, claiming that the 2^{nd}, 4^{th}, and 5^{th} tend to be ignored in formal educational processes :

1. of knowledge from academic institution to workplace, Eraut (2004) identified five interrelated stages, claiming that the 2nd, 4th, and 5th tend to be ignored in formal educational processes :
2. understanding the new situation –a process that often depends on informal social learning ;
3. recognizing what knowledge and skills are relevant ;
4. transforming them to fit the new situation ;
5. integrating them with other knowledge and skills in order to think/act/communicate in the new sit-uation. (p. 256)

In a sequence strikingly similarity to Eraut's, Nakagawa and Yamamoto (2010, 2013), gave an account of their observations of the Japanese steel making industry that involved collaboration between industrial engineers and themselves as academic and industrial mathematicians, respectively. All participants, engineers and mathematicians, met to form an understanding the context of the problem in all its complexity, including relevant historic and current information. They then discussed the problem from their respective knowledge bases until they reached a common understanding, and this required not only mathematical and scientific knowledge but also, significantly, personal and social skills. The mathematicians then recognized what knowledge and skills were relevant, and analysed the data logically to interpret their previous observations. Transforming their mathematical knowledges and skills to fit the new situation needed to be integrated with the operational and economic realities of the particular site. This analysis and transformation of mathematical knowledges and skills in the resolution of the original problem then offered a starting point for innovation in the field of mathematical research.

In this account, the mathematicians have brought to bear their cultural knowledge of the workplace and its processes as well as their personal skills ; they have developed a collaborative and holistic approach, from understanding the nature of the problem in the beginning, through to communicating the outcomes at the end. Nakagawa and Yamamoto (2010, 2013) emphasised that while the culture of education has traditionally implied teaching to be uni-directional, from school to industry, it is critical that teachers and academic mathematicians also learn from industry. As I will discuss further below, this means adopting a humble and respectful stance towards the holistic enterprise of work, in all its complexity, contingency, and uncertainty, with codified and non-codified knowledges, going well beyond the familiar simplistic portrayals found in school mathematics texts and prominent international assessments (discussed further below).

In my experience, non-mathematician workers at all levels may go through similar kinds of processes, on-ly less mathematically sophisticated, with less intensity, and *possibly* less serious consequences. Yet the mathematical and other knowledges they generate may be locally new. Clearly, doing the actual calculations, quantifications, and visualisations, are only one aspect of the total work process for any worker, with levels of accuracy and precision determined by the context, assisted wherever possible by available technologies and artefacts –along with the invaluable learning from experience (Björklund Boistrup & Gustafsson, 2014 ; Johansson, 2014, Smith, 1999). Very few workers will actually admit to "doing any mathematics" but, from a researcher's point of view, mathematics is often involved in their actions, even though the work-ers are not necessarily conscious of it, and nor do they necessarily wish to be (see, e.g., Wedege, 2010b).

Observing a non-mathematician at work : Paul the painter

Close observation of a professional painter shows just how much mathematics can be involved in such a task. Paul is a self-employed house painter, in his mid-thirties. After various periods of employment, unemployment, and retraining in different fields, he recently completed a mature-age apprenticeship, and now works as a sub-contractor for others or as a sole tradesman. Now that he has full responsibility for his own business, he has had to draw upon his own mathematical skills in order to stay in business. It is usual for customers to ask for a quote from several businesses before accepting one. The preparation of a formal, and legally binding, quote places considerable mathematical demands on people who typically have not studied mathematics beyond the compulsory requirements of the education system. The quote should contain details of fixed costs, such a non-consumable materials, and variable costs, such as paint and labour, and include government taxes on services such as labour. Paul must ensure that all non-consumables, such as the various sizes of ladders, brushes, rollers, groundsheets, and safety equipment, are available and costed into the quote, as well as the requisite personal and public insurance covers. He must also have some means of assessing the physical dimensions of the job so that he can estimate quantities of materials for preparation of the surfaces, along with the various types of paint coats to be applied at different stages, and even the appropriate type of paint base (oil, acrylic, external-use, internal-use, all-purpose, etc.) and cleaning materials for the brushes and rollers at the end of each work day. He also needs to make a realistic estimate of his time actually on-the-job, as well as off-the-job in purchasing materials, completing official legal documentation, and advertising his services. The mathematical demands in this stage of the work are non-trivial. Preparing a quote that is too high risks losing the job ; quoting too low risks making a loss and working for nothing at some point.

Once the quote is accepted, starting and completion times need to be negotiated, and purchasing of consumable materials, as well as any additional non-consumables, takes place. In 2013, house paint in Australia was sold according to the size of the can : 1 *litre* @ $ 40, 10 *litres* @ $180, 15 *litres* @ $240, for a reputable brand. At first glance, the largest can is the best buy. However, there are other considerations : If all the can is not used up, then money is wasted ; if not enough paint is purchased this means yet another trip to the paint store (which can be a considerable distance away), and travel costs must be added to time costings. On the other hand, it is risky to purchase too little paint at the higher unit cost and need to return for more smaller cans at the relatively higher prices.

Once the job has begun, there are decisions about the organisation of the order of work, in terms of which parts to start on, preparation of groundsheets, physically preparing the surfaces to be painted, followed by undercoat and possibly multiple topcoats of paint (see figure 1.5). Paul also needs to make on-the-spot deci-sions, such as what to do in the event of contingencies such as rainy weather, running out of paint, and so on.

The question of painting is rich in mathematics and could form part of an holistic community project for school students, in association with other practical school subjects. However, questions for young students, commonly found in school mathematical texts, trivialise and demean the work of tradesmen such as Paul who also takes great pride in doing "a good job" in terms of aesthetics and ethics :

"If you can paint

$$\frac{3}{4}$$

of a wall using only 2/3 of a can of paint, how much will you be able to paint with a full can of paint ?"

I assert that no professional or amateur house painter has ever asked themselves this question ! Further, that few if any young children learning fractions would have the experience or interest in serious house painting, nor the motivation to find the solution, let alone evaluate it for its reasonableness. This is the type of question that Palm (2008, p. 42) described as a "less authentic task."

Problematic representations of mathematics FOR work

At CIEAEM 63, I discussed the work of operators in the pharmaceuticals manufacturing industry (FitzSi-mons, in press). The following problem is taken from an official training manual, obviously written in traditional mathematics education textbook form :

The machine breaks down at 12 noon and starts again at 1.05 pm.
The average amount produced is 20 items each minute.
How much production was lost while the machine was stopped ?
You can fill in the second column of Worksheet 5 for extra practice at multiplying numbers.

FIGURE 1.5 – Paul the painter at work

What is the probability of a breakdown on the stroke of 12 noon ? Where is any consideration of the reality of the situation ? Where is the recognition of the complex contextual knowledge which applies to the food and pharmaceuticals manufacturing industry concerning legal requirements ? In the highly regulated food and pharmaceutical manufacturing industries there are strictly enforced Standard Operating Procedures requiring a *line clearance* (i.e., a complete cleanout) for certain products before the process can restart. What does the wording of this question say about respect for the workers who are actually doing the job ? What does it say about the "researchers" who produced this text ? How does it position the workers who are competent adults with many years of practical experience here and elsewhere ?

Many non-trivial mathematical questions could arise from such an incident if it were treated seriously. Organisational questions arise, such as whether the workers have complained repeatedly to management about poor maintenance processes, ageing machinery, lack of appropriate staff available, etc. Even though this problem was ostensibly set in an industrial context, it immediately destroyed any connection with the workers' reality by confounding their valued identities as adults with those of school children. Instead, why not just ask the workers :

What can go wrong on the production line ?
What are the possible causes ?
What are the possible legal and other consequences ?
What might be the economic cost of a stoppage between 12.00 & 1.05 pm ?
What factors do you need to consider ?

A mathematics teacher familiar with the work processes of the industry would also consider using a Fish-bone Diagram (see figure 1.6) in order to help tease out the issues, and also to assist the workers to become familiar with conventional industrial process control tools in order to enhance their democratic empowerment through gaining powerful knowledge in the ongoing struggles for labour rights (FitzSimons, in press).

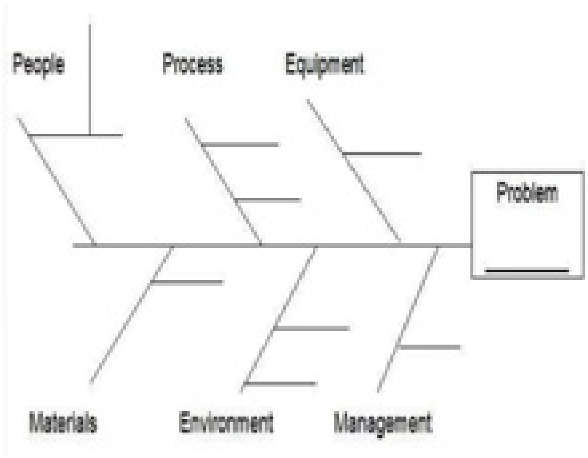

FIGURE 1.6 – A fishbone diagram

Roth (2012) illustrated the disparity between the mathematics taught to apprentice electricians in the form of formal trigonometry, complete with "magic circles" to help them find the correct ratios, and the practical reality of working on-the-job with an inscribed tool which removed the need for any formal mathematical calculation whatsoever. Because there was so little connection between the two classrooms, mathematics and trade, completing their mathematics training was reduced to a rite of passage due to the fact that the apprentices could not complete their formal qualification without it. As Roth (2012) noted :

> The requirements for doing well in the two locations are very different : exhibiting knowledge of trigonometry, on the one hand, and doing a good job that makes bending and subsequent pulling of wires practical. Formal trigonometry is the reference in the classroom, whereas rules of practice are the main references on the job. (p. 1)

Roth focuses our attention on the disconnection between the theoretical mathematics curriculum specifically developed for a designated trade and approved by key industry personnel, and the remainder of the apprenticeship program where the curricula are at least recognisable on-the-job, even if the rules are bent or broken in the interests of successfully completing the task within the constraints (generally time & money). Vocational mathematics curriculum designers and textbook writers need to move beyond their traditional school education orientation to become well informed about, or even personally engaged in the field of workplace mathematics education research (see Section 9), as well as keeping up to date with evolving theories and practices of mathematics teaching and learning, including its new technologies.

International assessment programs, such as PISA or PIAAC conducted under the auspices of the OECD, tend to have an undue influence on school and adult education curricula, respectively. Wedege (2010a) of-fered a critical analysis of PISA :

> PISA claims that the starting point is societal and labour market demands. However, the frame-work is based on a conceptual construct of academic mathematical knowledge in terms of competencies and not on empirical research on people's needs of mathematics in society. (p. 43)

She explains that the OECD (2003) exemplar question posed in relation to building a garden bed has nothing to do with the actual practice of carpentry, and, in fact, practical experience of completing such a task (prac-tical competence) could well lead to an incorrect mathematical conclusion according to the academic con-ceptual framework of the question. Wedege (2010a) concluded that

> in many educational documents, like policy reports and curricula, the discourse is not guided by a *logic of competence*where the dualism between the individual and the situation is constitutive, but rather by a *competency logic* like the one we find in the international surveys. (p. 41)

The former, logic of competence, refers to properties inherent to a concept (i.e., what a person can actually do in a given situation), while the latter, competency logic, is intended to apply to a given context, once a concept of competency (i.e., certain mathematical knowledges & skills to be demonstrated) has been speci-fied.

Referring to PIAAC, the *Project for the International Assessment of Adult Competencies,* linked to the PISA project, using similar definitions of skills, Tsatsaroni and Evans (2013) were also concerned by the problematic at-tempt at recontextualising adult numeracy practices in test situations of either hand-written or computerised modes, resembling school contexts as distinct from adults' actual workplace and everyday contexts. In this way, they claim, PIAAC is likely to limit adults' responses to the scholastic tasks because the two situations, of actual practice and formal assessment, are completely differently structured activities. Tsatsaroni and Evans support Wedege (2010a), noting that :

> What distinguishes competency from earlier understandings of the concept of competence is the fact that competency draws on behaviourist notions of "performance", while ignoring other tradi-tions of social science research which have more complex (implicit or explicit) definitions of com-petence.

Tsatsaroni and Evans (2013) were also concerned that the PIAAC survey is heading down the path of *generic* curricula which are not actually based on any particular work practice, but wish lists from powerful employer voices : One implication of this is that workers will be constantly in need of re-training. In the absence of any recognised curriculum for adult numeracy internationally, the fear is that PIAAC could actually form a *de facto* curriculum for adult numeracy internationally. In FitzSimons, 2002, chapter 6, following Bernstein (2000), I discussed the power of business and industry in policy formulation of what has now eventuated in Australian vocational education curricula (actually known as *learning outcomes*) as a collection of generic numeracy (and other vocational) skills that must ONLY consist of those observable in a given industry setting (see also, Wheelahan, 2007, 2009). The problem with generic curricula is that they remove the theoretical curriculum coherence necessary for making decisions in times of uncertainty when locally or globally new solutions must be found. (This will be discussed further below.)

Differences between mathematics at work and mathematics at school

The discipline of mathematics is characterised as the science of quantity and space, together with the various symbolisations of each sub-discipline (e.g., algebra, geometry, statistics). It is taken as an objective reality, neither subjective nor physical. Nevertheless, it is generally regarded as fallible (Davis & Hersh, 1980/1983). As a *vertical discourse* (Bernstein, 2000), mathematics is described as being theoretical, conceptual, and generalisable knowledge ; coherent, explicit, and systematic, with strong boundaries between itself and other disciplines. School mathematics content is based on an arbitrary, and currently conservative, selection from the academic discipline of mathematics, but is then recontextualised by teachers drawing on available, sometimes mandatory, texts, in an effort to make it accessible to learners through the pedagogical means at their disposal.

Workplace mathematics is practical knowledge, informed by the accumulated mathematical knowledge and ex-perience of workplace and other diverse cultures in society throughout history, often undertaken in complex and/or contradictory contexts. It is developed by people in response to an experienced or potential, imagined reality. Much of the mathematical work done by non-mathematicians could be described as a *horizontal discourse* (Bernstein, 2000) : specific, locally useful knowledge ; a set of strategies which are local, segmentally organised, context specific and de-pendent. Compared to school mathematics, there are weak boundaries between mathematics and other workplace knowledges. One consequence of this is that mathematics in workplace is often invisible to outsiders, even to (prospec-tive) mathematics teachers (Nicol, 2002). (This is discussed further in Section 9.) Using it does not necessarily make it visible, and most workers claim not to use anything they learned at school. It often appears as *common sense* ; be very simple in relation to school mathematics –and usually looks very different when presented in written or graphical form (Williams & Wake, 2007a, b). It is totally embedded in contexts : historical, cultural, social, political, economic, etc. It is said to be *crystallised* in technological and other artefacts : in tools of production and communication, in work practices and organisation. (See FitzSimons, 2013, for a review of several large-scale studies on mathematics in the work-place.)

For professional academic and industrial mathematicians, mathematics is both a tool and the object of their work. For non-mathematicians, including engineers, mathematics is but one of many resources available in the resolution of ever-evolving, contextually complex workplace problems. Using existing mathematical knowledge, derived from formal and informal learning, generally goes unnoticed. However, in breakdown or problematic situations, creative reasoning is called for and locally new solutions must be found. Importantly, the "best" mathematical answer according to the discipline may be of *no* practical use in the workplace : Mathematics at work is contextually situated and dependent, unlike the practices of school mathematics where contexts are generally imaginary or grossly over-simplified. As is the case with school mathematics, the purposes, structures, and products of work frame the mathematics that is carried out (Nicol, 2002) –so that the practices of these two different genres of mathematics are understandably distinct from one another, and also from the work of professional mathematicians. Steen (2003) summarises the differences :

> Mathematics in the workplace makes sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics. Work-related mathematics is rich in da-ta, interspersed with conjecture, dependent on technology, and tied to useful applications. Work con-texts often require multistep solutions to open-ended problems, a high degree of accuracy, and prop-er regard for required tolerances. None of these features is found in typical classroom exercises.
> …Numbers in the workplace are embedded in context, used with appropriate units of measurement, and supported by computer graphics …Employees need statistics and three-dimensional geometry, systems thinking and estimation skills. Even more important, they need the disposition to think through problems that blend quantitative data with verbal, visual, and mechanical information ; the capacity to interpret and present technical information ; and the ability to deal with situations when something goes wrong …(p. 55)

In the workplace meaningful communication is of the essence. Workers need to communicate about mathematically relevant concepts and results with internal or external customers or suppliers of goods and services, with others at different geographical locations, or with people working at different levels of authority in their own organisation. They need to have the confidence to question the mathematical assertions and assumptions of others in their workplace contexts, and this can ultimately be a question of life and death, even survival of the business (FitzSimons, in press). High level mathematical communication skills are also vital for people responsible for on-the-job training of apprentices or other newcomers.

Where is the mathematics ?

The study by Nicol (2002), which focused on prospective teachers visiting workplaces, showed that they found it difficult to see anything but elementary mathematics, and even more difficult to recontextualise workplace mathematics in forms appropriate to the kinds of senior classes they were expecting to teach. This is probably not surprising after at least 12 years of experiencing conventional mathematics education, where mathematical theories and rules are generally abstracted from any semblance of reality, and answers are inevitably found at the back of the book, or else adjudged by the teacher for their adequacy. Although the teachers were able to make general observations on the need for flexibility, collaboration, problem solving, proficiency with technology, creativity, and good communication skills, they were unable to foreground the mathematics actually embedded in workplace activities and how it was used, or to determine the kinds of mathematical understandings necessary. When attempting to design materials for students, they either tended to remove the workplace context in an attempt to make the mathematics more accessible, or else to recontextualise the actual mathematics in use at work at a much higher academic level than would ever be seen or needed at the workplace. Many other authors over the last two decades have accepted the futility attempting to map workplace mathematics either onto school-based curricula or to employers' impressions of the (school) mathematical skills necessary for the (generic) workplace (FitzSimons, 2002, 2013 ; Wedege, 2010c).

In his study of the mathematical competence of workers involved in routine production work at 16 auto-mobile production sites in the USA, Smith (1999) identified three broad domains of mathematical context : (a) measurement, (b) numerical and quantitative reasoning, and (c) spatial and geometric reasoning (including spatial visualisation, orientation, and translation in two or three dimensions). The first two were defined in terms of observable mathematical actions, but the third required inferences to be made. Smith provided an exemplary coding scheme as a table, with definitions and examples for each of the three categories, and elaborated on these so that the reader could form a reasonable impression of the mathematical and other work performed in this context of production. He also provided tables of results, along with discussions, ac-cording to two levels of mathematical complexity of the knowledge demands that he found in this study : low and modest. His thorough description of his methodology and findings, along with his comparative analysis of his findings with the demands of schooling in mathematics at the time in the USA, offers a model for others interested in this area of research. Gainsburg (2006) also provided a detailed description of the methodology she used to study the high level mathematical activities of structural engineers, including her auditing

Find a "friend" who is not a teacher. The person could be a member of the non-teaching staff (e.g., administrative worker, librarian, laboratory technician, canteen manager), or a local shop-keeper, tradesperson, nurse, cab driver, manager, farmer, glass blower, analyst, information technology/computer technician, golf pro... any paid worker! They should have access to and regularly use some techno-mathematical equipment : e.g., computer, calculators. **Observing** 1. As they do their work, look for evidence of the use of numeracy or mathematical ideas and techniques. 2. Ask them to explain what they are doing and why. [You may have to wait until after the person has finished the task/s at hand; maybe even come back later. In this case, make sure to take specific notes to help you recalling the moment/event] 3. If you can see that numeracy would be involved : ask them about possible breakdowns in equipment or communication or how they managed to solve an unforeseen problem in the past. 4. Use Engeström's activity theory framework to analyse what numeracy is involved in this job. (a) Ask : who, what, why, how, where,when (b) Identify : the subject, the object, the tools used [both material and verbal], the rules, the community, and the division of labour. (c) Consider each of the *Fundamental Capabilities* [for school mathematics] and identify any instances you observed or that were reported by the worker.

TABLE 1.1 – Workplace mathematics observation task for teachers

of a relevant university subject. She highlighted the challenges for the engineers, as well as herself, to keep track of the various lines of complex hypothetical reasoning that they used.

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The following is a list the kinds of people observed as a possible source of ideas for others :

1. An ex-teacher, now conference facilitator.
2. A laboratory technician responsible for running and maintaining a school's science laboratory.
3. 2$^{\text{nd}}$ Year apprentice fitter and turner as part of is part of the maintenance team of the equipment at a crude oil processing plant.
4. A lecturer in management with teaching, research, and co-ordination duties.
5. A site manager for a shop fitting company whose daily work is to review all plans and drawings.
6. A pharmacist working part-time at one of a chain of suburban pharmacies.
7. A graphic designer, working for an advertising company, whose responsibilities include designing, composing and modifying images and artefacts.
8. An instrument technician working in a small hospital with 3 operating theatres, cleaning, preparing and sterilising medical equipment after surgery and ensuring that stock levels are maintained.

9. A professional punter, generally concentrating on thoroughbred racing but often including gambling on cards and other sports betting.
10. A person making a table, whose top alone weighed upward of 200 kilograms, in his workshop shed at home and organising transport to the customer.

Even though the teachers only spent 1-2 hours doing their observations, following a sociocultural approach they were able to appreciate the complexities of the work done, and how mathematics was an integral part of the whole activity, along with technical skills, and personal and cultural knowledges of the people they ob-served. Below is my general feedback to the whole group :

Together, your accounts have captured very well the often unexpected complexities of people's work. I imagine that most school students, not to mention the community at large, are unaware of the role that numeracy actually plays in the lives of people in everyday life and at work. The detailed descriptions of typical jobs that workers are expected to do "routinely" as well as the insights offered by breakdowns in those routines have provided a rich source of material for drawing out implications for school mathematics/numeracy education. I am sure that the students in your institution –and your colleagues –who are able to share in your findings will be much better prepared for life beyond their formal education. Most importantly, there can be some answers to the eternal question in mathematics classrooms : "Why are we doing this ?" "What do we need it for ?"

From prospective and practising teachers learning about mathematics at work through formal pedagogical activities in higher education requiring them to conduct basic observational research, I now turn to examples of teachers, school students, and industry personnel interacting in the boundary crossing work of forming partnerships where they temporarily share a common goal. (See Damlanian, Rodrigues, & Sträßer, 2013) for further details.

One project (Bonotto, 2010, 2013) involved secondary students working with local industry with the intention of changing school students' images of mathematics and to encourage them to continue their studies of mathematics at higher levels. In the Veneto region, problems were suggested by local industry, public ad-ministration, and other non-scholastic institutions, and were clearly of importance to these bodies, with the results from the school students actually regarded as being of value. Workplace managers visited the schools and offered data, themes, statistics, linear programming, Operations Research, and modelling ; also cryptography tasks. They included the following mathematical topics :

Statistics : data collected from a provincial tourist office and a pharmaceutical manufacturer ; also industrial quality control.
Linear programming and operations research : optimization problems for a garbage-collection service, a telephone call-centre, and share trading.
Modelling : the growth of tumour tissues, and Fourier analysis tasks suggested by the cardiology department of a local public hospital.

A positive outcome of this project was, contrary to international trends, an increase in enrolments in mathematics courses at the local university.

In a second project, Hana et al. (2010, 2013) described a boundary crossing exercise where school pupils acted as industry consultants, supported by student teachers who worked between the school, the university, and the workplace, to organise mathematical projects. In one lower-secondary school pupils acted as statistical consultants for a local company producing valves, where stock control organisation had broken down. In this project the workplace manager participated respectfully, as with adult consultants, giving professional critical feedback to the pupils, who, in return, were able to ask critical questions of the company. Communicative competence was an essential aspect of this interface between mathematics and industry. Through this school-industry partnership, the student teachers were enabled to learn and teach in and between different contexts. The authentic industrial context clearly altered the regular conditions of learning and teaching, and influenced the outcomes for all concerned.

In summary, it is very important in an era of globalisation, that prospective and current mathematics teachers learn to *see the mathematics* –not only in work, but in the world around them –in order to motivate their students and, more importantly, so that their students might be more prepared, mathematically at least, to participate in the world beyond school. Internationally, there are different forms of partnership between education and industry, with benefits for all participants who are able to respectfully learn from each other (see, e.g., Damlanian et al., 2013 ; Nicol, 2002).

How might formal mathematics education be different with respect to a globalised environment ?

Let us return to the five areas of mathematical competence listed by EACEA (2011) : mastering basic skills and procedures, understanding mathematical concepts and principles, applying mathematics in real-life con-texts, communicating about mathematics, and reasoning mathematically. Clearly, they are regarded as im-portant and their intention appears to be related to workplace mathematics. How well might they meet the actual needs of mathematics in and for work, and beyond, as described above ? Apart from begging the ques-tion of exactly *what* "basic skills" need to be mastered and *who* will decide this, based on *what criteria*, , there appears to be an underlying assumption that historic definitions of mathematical skills will be adequate for future generations. Authors including Artigue (2011), D'Ambrosio (2012), Gal (2013), Lesh (2010), and Steen (2003), have called for different approaches to mathematics curriculum and pedagogy in relation to the preparation of young people preparing to enter the workforce in the future. They refer to changing the mathematical content and the relative emphasis on skills and processes vs. conceptual understanding of big mathematical ideas, mathematical ways of working, and the use of technologies of communication as well as mathematical labour-saving and exploratory devices.

Working in a globalised environment will require creativity in mathematical reasoning (see, e.g., Lithner, 2008), transformation and development of locally new knowledge needed to work within constraints to solve unforeseen problems. Solving the ever-evolving workplace problems requires *abductive* reasoning, in the first place, as well as deductive and inductive reasoning. Eco (1983, cited in Arzarello & Sabena, 2012, p 192) described abduction "as the search for a general rule from which a specific case would follow." Unlike deductive and inductive reasoning, the conclusion drawn from abductive reasoning is not necessarily true, but must be plausible. Consider, for example, how the doctor diagnoses your disease or the automotive mechanic diagnoses the fault with your car ; or the detective work done, not only in the legal professions, but beyond that, every day, on scales ranging from the micro to the macro, in workplaces around the world.

Arzarello and Sabena (2012) recognise all three kinds of reasoning as playing a role in the process of inquiry, but abduction is the only one to introduce new ideas. From a workplace perspective, this is what so many workers do each day as a normal, unremarked part of their jobs, when confronted by a disturbance, a questioning, or the emergence of a problematic situation (Ellström, 2010). They consider all of the relevant information available to them, and possibly their colleagues, through every possible sensory channel, including touch, sound, and smell, and then make –often in very short time –plausible hypotheses to be tested, mentally and/or physically, supported by inductive or deductive reasoning. The study by Arzarello and Sabena focused on semiotic and theoretic control in the context of an elementary calculus class. They presented a three-fold model of semiotic actions, based upon the study of students attempting to sort three unlabelled graphs which depicted a function, its derivative, and a possible anti-derivative. In the process, according to Arzarello and Sabena, successful students passed through the following stages :

1. Interpreting relevant signs ; perceiving with respect to knowledge and cultural knowledge dimensions
2. Identifying and describing the relationships between the interpreted signs
3. Justifying the relationships with respect to mathematical theory.

Arzarello and Sabena described this as a shift from semiotic control to theoretical control of actions, as the evolution from truth because of data to truth because of theoretical reasoning. For me, there are parallels with the processes that many workers pass through when confronted with problematic situations, even though they are unlikely to be specifically mathematical problems per se, and in spite of the fact that theoretical mathematics is most likely to be used as a tool rather than the goal. This means that, in practice, the justification process is likely to be that "it works.". However, due to the pervasive invisibility of mathematics for non-mathematicians, it is highly probable that the theoretical mathematics actually learned in formal education at any level would increase the options available and support judgements made, albeit unconsciously for the most part. As it happens, many papers presented in the CIEAEM 65 Working Group 1 : *Knowledge and Competences* offered theoretically well informed accounts of practical teaching which addressed issues of creative reasoning and control by students. The Group also raised the difficulties of assessing mathematical competence, with its complementary goals of preparation for work and life, in the school situation. (See the WG 1 Report by FitzSimons & Kazadi, this issue.)

However, beyond mathematical competence, as defined by EACEA (2011) and others, there needs to be an ac-knowledgement of the personal and cultural skills that are essential aspects of communication in work and beyond, in an acknowledgement of : (a) the possible negotiation of task parameters with clients or super-visors, (b) the de-velopment of shared understandings with others who do not share the same mathematical background, and/or (c) development of shared understandings when using mathematical representations which are "foreign" to traditional academic usage (computer-generated or otherwise). The EACEA's expression "applying mathematics in real-life contexts' seems to imply simplistic assumptions made about transfer and the value of word problems typically found in school mathematics texts and tests (Evans, 1999 ; Lundin, 2012 ; Pais, 2013 ; Palm, 2008), as well as workplace training manuals (FitzSimons, 2002, 2013). Transfer of mathematical ideas and techniques from school to workplace is far from simple, as evidenced by regular reports in the mass media of employers experiencing "problems with today's school leavers" (or even mathematics graduates !). Developing mathematical expertise, even competence, within an education environment can never be taken as equivalent to the repertoire of personal, cultural, and technical skills, along with the worker's ongoing development through practical experience, that are essential workplace requirements for competence (Eraut, 2004). And, certainly, innovation as described by Ellström (2010) requires much more than notions of application based on the simple transfer of school mathematical skills.

It is well known that employers facing an over-supply of applicants will use mathematics as a crude proxy for intelligence, and require applicants to complete tests which may be completely unrelated to the work that will eventually be undertaken. However, one aspect of assessment that is rarely addressed seriously in formal education is the consequence of errors. Human activity inevitably involves making mistakes. In school, nowadays at least, errors may be used as a source for mathematics teachers to assist in the teaching/learning pro-cess, rather than treated as a source of ridicule and shame for unfortunate learners –as many older adults will recall. Workplace supervisors may also use mistakes made by novices as a learning opportunity (FitzSimons & Wedege, 2007). However, once workers are given full responsibility for their work tasks, mathematical mistakes can be catastrophic. Consequences of errors in calculations or quantifications such as misreading powers of 10, or mistaking the prefix *milli-* [m] for *micro* [μ] can be devastating : for example, in pharmaceutical dosages to people or other animals, or in the application of chemicals in fertilisers or pesticides or herbicides. The environmental consequences of ignorance of, or deliberate disregard for, careful use of, for example, manufactured chemicals in terms of spoilage or pollution of earth, air, and water are with us all to see on the mass media, if not in our own local environments. Similar considerations apply to mining, drilling for oil, de-forestation, monoculture crops, and so on. Returning to the theme of quoting or tendering for work, a family member of ours had to suffer the consequences of a typographical error when his company's quote for removing some large and dangerous trees for local government omitted a zero, and was ultimately the one accepted as a legally binding document. In my experience as a teacher of vocational students intending to become laboratory technicians, it was not uncommon for them to confuse the processes of multiplying and dividing when calculating dilution factors using authentic but unfamiliar numbers with positive and negative exponents, and many decimal places appearing on the calculator display. Since they had little practical experience, they were not even aware that they had made such apparently simple mistakes when using their calculators, which would have actually resulted in huge mistakes in practice. Lack of conceptual understanding of probability was a factor in the USA Challenger Space Shuttle disaster, when engineers wrongly made the assumption of independence of *O-Ring* failure in the abnormally cold conditions before the launch. Mathematical error detection and remediation in contextual situations could be a valuable and valued skill to develop in formal education.

As discussed previously, mathematical competence is the policy mantra of the present time. But, what exactly is this competence and how might it be assessed ? Definitions have been given that involve mobilising a variety of resources including mathematics to solve a so-called real problem. In mathematics education, individual skills have to be learned and practised, and eventually become an unconscious part of a person's repertoire. When the skills have become refined and part of a routine, and used appropriately, the person is said to be competent. When a breakdown occurs, a competent person can draw upon their accumulated mathematical knowledge and experience to determine possible causes and possible alternative actions, which may include changing the goals or resources and eliminating or by-passing the mathematical problems altogether. Clearly, mathematical skills are easier to assess in educational settings, and, as pointed out by Tsatsaroni and Evans (2013), asking adults to undergo formal education tests with facsimiles of artefacts as prompts (even when conducted in friendly, informal settings) can never determine what they would actually do outside of that setting in other situations with the actual range of possible resources to hand, and their normal parameters of independence in decision making regarding the range of possible actions and the im-portance of finding a specific answer, to what degree of accuracy, and, above all, the consequences of making an error !

Conclusion

Globalisation has led to a questioning of the autonomy of disciplines such as mathematics, as learners worldwide demand contextual relevance. Recent curriculum frameworks identify the supposed qualifications (skills and competencies) that will be needed at work, yet these are based on a static and outdated assumption that everything can be known ahead of time and taught accordingly. Not all learners will need to understand in detail the highest levels of mathematics currently taught in schools, but they will need to understand the big ideas behind the major areas of quantity and space, in terms of deep conceptual understanding : e.g., randomness, variability, rates of change and optimisation, periodicity, infinity, alternative geometries, and so forth. In the world outside of education, the discipline of mathematics continues to develop, vertically within recognised disciplines and horizontally into new sub-disciplines, both as a result of current workplace problems and in anticipation of as yet unknown problems, even though this may not be the mathematicians' actual intentions, and there may be a considerable time lag between the original research and eventual practical outcomes.

Learners of all ages will need to understand that there are many mathematics and that what they encounter in school is but one, albeit universally recognised and valorised, version (Knijnik, 2012). However, few people who need to innovate, even to progress personally, in the global economy can afford to be without relevant abstract, theoretical knowledge, including mathematics, and rely solely on contextual knowledge. A strong, meaningful disciplinary knowledge foundation is essential for critical decision making. However, the institution of mathematics education needs to re-address the question of what is valuable knowledge, in and for work and beyond, in a complex and fast-changing globalised world. To do this, there needs to be respectful, informed debate between mathematics educators, applied mathematicians, and other people with expertise in environmental, economic, scientific, social, and technological trends.

Above all, mathematics education must eschew the imaginary pseudo-contextualisations of mathematics at work that ultimately destroy its own credibility. The insertion of words such as *plumber, painter, tiler,* etc. into mathematical tasks, without any of the associated constraints and practicalities of actual activities, demeans both the workers in their valued occupations for which they have trained, qualified, and are justly proud, and the students *apparently* fooled by the context. In the workplace, every mathematical action, well-founded or not, has a consequence which can affect people's lives, livelihoods, social and environmental wellbeing, and so on. For some workers, negotiation of the parameters of a task is an everyday occurrence, including costs and benefits, along with critical questioning, justification, and clarification, to a variety of stakeholders within and beyond their workplace. It is worth reiterating that the best mathematical solution may not be feasible or even desirable in practice.

It is generally acknowledged that the mathematics used in the workplace by non-mathematicians is largely at a relatively low level, in comparison with school mathematics curriculum documents ; also that it is con-textually situated and meaningful to the people involved in the outcome. Mathematics (thinking, acting, communicating) is but one of many resources available in the pursuit of an action, goal, or outcome that has potential value to product creators, business owners (from local to multinational companies), ultimate users, or other beneficiaries (e.g., people with medical problems and the researchers who are working on solving these problems ; or users of new technological devices and their developers). In the workplace, situations of breakdown can engender new learning, or innovation (Ellström, 2010 ; Engeström, 2001). Sometimes it is the mathematical aspects of a task that are the cause of the breakdown : for example, discrepancies between calculated and the values expected based on historical data or available as codified theoretically founded information. Once serious discrepancies are noticed by workers or supervisors, action must be taken to resolve the conflict. Gainsburg (2006) provided a detailed account of structural engineers grappling with hypothetical extreme values in the safety-critical process of building design, and eventually finding workable resolutions to their problems. Wedege (2000) gave examples of what she termed *semi-skilled* workers making decisions, within the limits of their expertise, on quality control in an electronics manufacturing factory and in the loading of cargo at an airport where safety is a prime concern : Once major changes are required, the workers also know *when* to shift responsibilities upward. If it should happen that the mathematical decisions of workers themselves are the cause of the breakdown, they can expect that their employment will be terminated or at least under close scrutiny.

Many workplace mathematics studies in the past have sought to identify the mathematics actually used in practice, even how it might be improved (e.g., Coben et al, 2010 ; Hoyles, Noss, Kent, & Bakker, 2010), and many have made comparisons between this genre and that of formal mathematics education (e.g., Williams & Wake, 2007a,b). However, few if any have enquired into the relationship between an individual's previous formal education and their current mathematical, or mathematics-containing, practices ; that is, integrating the work as a societal process, the mathematical and other knowledge required to do that job, and the individual's subjective experiences as a learner of mathematics and a person doing a responsible job. *Adults' Mathematics : In Work and for School* School is an innovative research project which aims at analysing and understanding adults' mathematics-containing competences. Rather than adopting the usual one-way assumption of moving from school or academic mathematics to the workplace –apparently based on the traditional life trajectory of young people moving into adulthood –it aims to revise this assumption, emphasising the two-way relationship between mathematics education and the workplace (cf. Nakagawa & Yamamoto, 2010, 2013). This research adopts a *sociomathematical* approach (Wedege, 2010c), and addresses the societal context of knowing, learning and teaching mathematics, with the intention of informing future curriculum and teaching. The research is framed by Salling Olesen's (2008) heuristic model which attempts to capture the complexity of workplace activity and allows for examination of the dynamics of workplace learning situations in general (see figure 1.7).

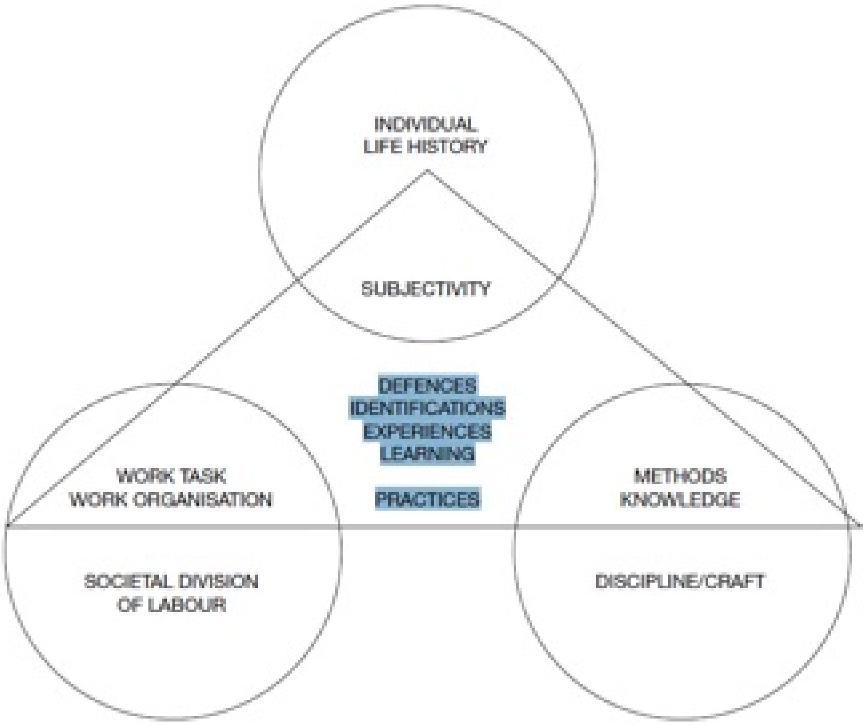

FIGURE 1.7 – Workplace learning (Salling Olesen, 2008, p. 119)

The model, according to Salling Olesen (2008) :

> suggests that learning in the workplace occurs in a specific interplay of experiences and practices, identifications and defensive responses. It also suggests that learning in the workplace is not a response to technical and organisational conditions only ; it mediates the specific relation between three relatively independent dynamics : the societal work process, the knowledge available and subjective experiences of the worker(s). Based on professions, the model pays particular attention to the cultural nature of the knowledge and skills with which a worker approaches a work task, whether they come from a scientific discipline, a craft, or just as the established knowledge in the field. (p. 118)

Building on this model, researchers Björklund Boistrup and Gustafsson (2014) and Johansson (2014) have each adopted different methodologies in order to learn more about how adults utilise structuring resources in different workplace measuring activities. They have also been able to gain insights into the workers' ration-ales for their efficient and functional outcomes, even though there were few visible connections with the formal mathematics that might have been expected in a school mathematics education context. Björklund Boistrup and Gustafsson adopted a multimodal approach, where all forms of communicative resources (e.g., body, speech, tools, symbols) were taken into account, as well incorporating the institutional norms of work-place activities into their analysis of two case studies : (a) lorry loaders in a road-carrier company, and (b) a nursing aide in an orthopaedic department of a hospital. In both cases, complex tasks were accomplished without the visible use of conventional measuring tools. In the first case, the pallets were the structuring re-source (as well as a communicative resource) :

> ...the lorry loaders could estimate the weight of the pallet while they were seated in the forklift. With this information they knew where to place it in the trailer in order to get the right pressure on the three different axes along the trailer which can carry 24,000 kg. (Maria C. Johansson, personal com-munication, 18th September 2013)

In the second, the plaster itself was the structuring resource (Björklund Boistrup & Gustafsson, 2014) :

> the nursing aide measured up with the dry plaster wrap directly on the patient's arm, prior to the actual plastering process. The aide then repeated the measure several times when folding the plaster before finally adhering it to the patient's arm.

Johansson adopted Bourdieu's concepts of capital and habitus as framework to analyse a nurse's transition between the mathematical practices and school and work which inevitably involve learning. In this case, the nurse was taking readings of mathematical outputs of various body function tests and interpreting them ac-cording to her own experience of doing this work. This transition between two complementary knowledge systems "requires a habitus with the "third eye," but also demanded mathematics as an educational capital"(Johansson, 2014). Both articles highlight the difficulty in recognising potential mathematical activities in work, and how these can be overshadowed by other competences and components in work –such as caring in the case of nursing, and saving the customer money through unorthodox (by conventional understandings) but practically justified methods of loading pallets. Both studies also highlight the importance of embodied knowledge, which of course contributes to a lack of visibility to an outsider (as discussed in Section 8).

Acknowledgement

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1.4 Mathematics as a part of culture

František Kuřina

University Hradec KRÁLOVÉ

Thank you very much for the opportunity to speak at your conference. It is a great honour for me.

I am not a mathematician, I am only a mathematics teacher, but the connections between mathematics and other parts of human culture are very important for me. It is from the position of a mathematics teacher that I would like to speak here –very subjectively and with poor English. Please be patient with me.

Although I don't speak French it was the great French mathematician **Jacques Hadamard** (1865-1963) who influenced my way of looking at mathematics and mathematics teaching. In 1957 I studied in parallel with theoretical mathematical lectures at Charles University in Prague his classical Lessons in Geometry (2008) and after fifty years, in 2007, his masterpiece *The Mathematician's Mind* (1996).

I would like to recall here Hadamard's words :

"Practical applications are found by not looking for them, and one can say that the whole progess of civilisation rest on that principle."

When Greeks, some four centuries BC consider the ellipse and found remarkable properties of it, they did not think and could not think of any possible use of such discoveries. However, without these studies **Kepler** could not have discovered, two thousand years later, the laws of motion of planet and **Newton** could not have discovered universal atraction.

Giorolamo Cardano (1501 - 1576) is not only the inventor of a well-known joint which is an essential part of automobiles, but has also fundamentally transformed mathematical science by the invention of imaginaries. The whole development of algebra and analysis would have been impossible without that fundament ? (Hadamard 1996).

Mathematics is surely a part of culture, but school matematics is often a "theory" without connections with technology, sciences, literature and art. Of course there are exemptions : realistic education of **Freudenthal's Institut** in Utrecht or Michal Serra's **Discovering Geometry.**

The main problems in mathematics education connected with my theme are :

1. How to teach problem solving and how to cultivate creativity.
2. How to connect mathematics with other parts of culture.

My university teacher *Bohumil Bydžovský* (1880-1969) emphasized the psychological point of view in education. This means also to cultivate connections with all parts of human culture and to see and support the students success, satisfaction and perhaps also some unexpected results –namely surprise.

Roger Penrose wrote in the book *Shadows of the Mind* (1995) : "For our remote ancestors, a specific ability to do sophisticated mathematics can hardly have been a selective advantage, but a general ability to understand could well have". I would like to replace the words "remote ancestors" by words "contemporery students". For them also the understanding of mathematics is much important than formal knowledge of sophisticated mathematics. The understanding of mathematics is fundamental in mathematics education. One of the ways to cultivate understanding is to see mathematics in the state of creation and in the context of human culture as a whole. To show some possibilities in this sense, is one of the aims of my lecture, in which I understand : mathematics is elementary mathematics.

Mathematics in reality and reality in mathematics

Four elements of reality are, in my opinion, extraordinarily connected with human culture and mathematics :

– (N) numbers as measuring of parts of reality,
– (S) shapes as the scenes of reality,
– (M) motion as demonstration of changes in reality,
– (D) dimension as nature of reality.

All the elements N, S, M, D are present in the worlds of our children, all are subjects of our life –not only in history, but also in the present, all give stimuli for mathematics.

(N) *Numbers as the measuring of reality*

Although **Leopold Kronecker** (1823-1891) said that whole numbers are created by God, I believe (together with **Karl Popper** (1902-1994)) that natural numbers are human products. They are products of human speech, invention of counting and counting without limit (Fig. 1.8).

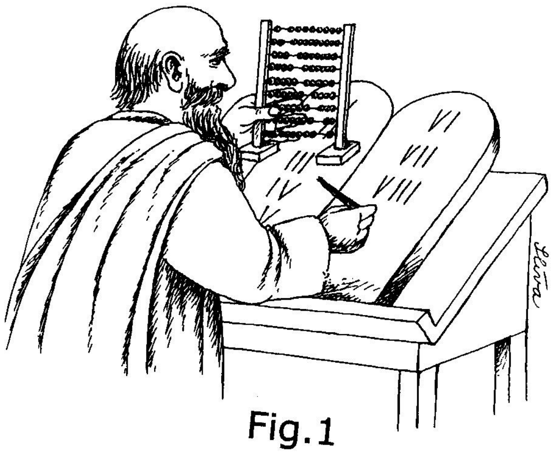

FIGURE 1.8 – Counting and counting

Numbers are of course abstract notions, but in spite of this they are connected with reality which we percieve with all our senses :

– three stones we can see and touch,
– three beats of a bell we can hear,
– three different tasks we can relish,
– three different perfums we can smell, . . .

These numbers are parts of everyday reality, they are measures of finite sets. Our children aquire the conception of small natural numbers in their normal life in the family and preschool with the development of their mother language. At schools the old abacus is the proper instrument for the representation of pure quantity.

To measure lengths human society created a number of apparatus. The results of such measuring are expressed by means of positive real numbers and plenty of units (m, km, yard, mile, foot, . . .). It is a shame of our civilisation that in 1998 the *Mars Climate Orbiter* wrecked as a consequence of mistake in applications of units of lengths (Barrow 2002).

Two school problems

Problem 1 (for students aged 10 years)

In how many ways it is possible to sew on a button with four holes ?

Some results are in the Fig. 1.9

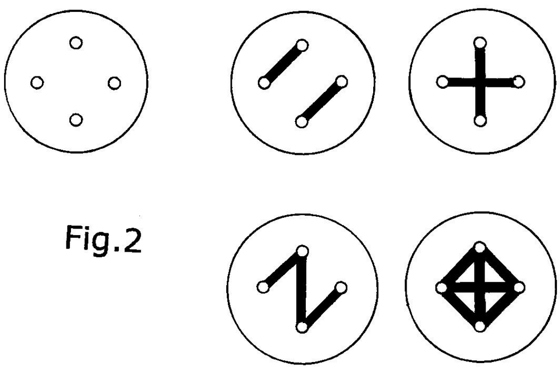

FIGURE 1.9 – How many ways it is possible to sew on a button ?

Already on such a simple problem we can present some mathematicaly interesting questions (the role of definitions, problems of equality,. . .)

Every child can draw some results of this problem –SUCCESS is an important motivating power in good mathematics education.

Problem 2 (for students aged 18 years)

How many ancestors do you have since the beginning of our era ?

If we will count four generations during one century I am product of 80 generations of my ancestors. According the fig. 3 is the number of my ancestors 2^{80}. This is more than the number of all persons living in this time. This is a suprise. How is it possible ?

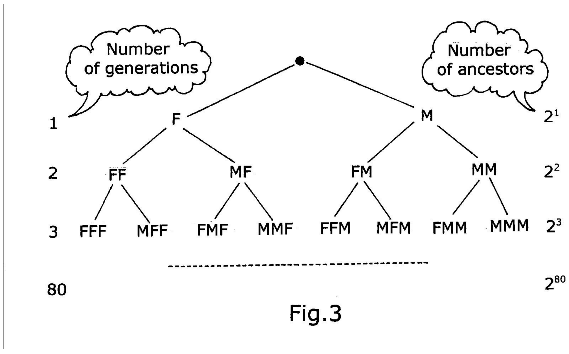

FIGURE 1.10 – Ancestors

SURPRISE is a further motivating power in education.

A very interesting part of the set of natural number is the non finite set of prime numbers.

Although there are still many unsolved problems with prime numbers, these numbers have applications in different areas of contemporary society, for example in coding, ciphering, geometry, technology, . . .The prime numbers occure also in nature.

$$\ldots 1944, 1961, 1978, 1995, 2012, \ldots$$

is a part of an arithmetic sequence with difference 17. These numbers are the years of emergence of cicada *Magicicada septendeci* in certain areas of the US. This insect with 2 large and 3 small eyes spend most of their 17-year lives underground. After 17 years, mature cicada nymphs emerge at a given locality, synchronously and in tremendous numbers. After such a prolonged developmental phase the adults are active for about 4 to 6 weeks. Within two months of the original emergence, the life cycle is complete, the eggs have been laid and the adults cicadas are gone for another 17 years.

(S) Shapes as the scenes of reality

Our nature offers a great abundance of shapes in the realm of zoology, botany and mineralogy (Fig. 1.11). Many of them have geometrical properties, for example *symmetry.* Symmetry is connected with balance, not only in nature, but also in technology. It is possible to describe the regularities of flowers by means of groups of isometries. Nature designed Platonic solids ages before *Platon* (428-347 BC) (Fig. 5).

Art emerged out of efforts made to map nature, and descriptive geometry from the needs of design engineers. Art has of course a very long and interesting history. According the book *5000 Jahre Geometrie* (Scriba, Schreiber 2001) there are geometrical ornament that have been found and come from about 40 000 years BC (Fig. 1.13).

In 1926 *František Kupka* a very noted Czech artist with strong links to France published the series of wood engravings *Quatre Histoires de Blanc et Noir.* It is a summary of Kupka's work which is also a grand tour of Modernism (Fig. 1.14).

The connection of drawing and geometrical mapping of space was emphasized by the great German artist *Albrecht Dürer* (1471-1528).

The founder of descriptive geometry is of course the great French mathematician *Gaspard Monge* (1746-1818). Descriptive geometry is a subject with a long and rich history in my country, but at this time the methods of mapping are in decline.

Fig.4

FIGURE 1.11 – Shapes in nature

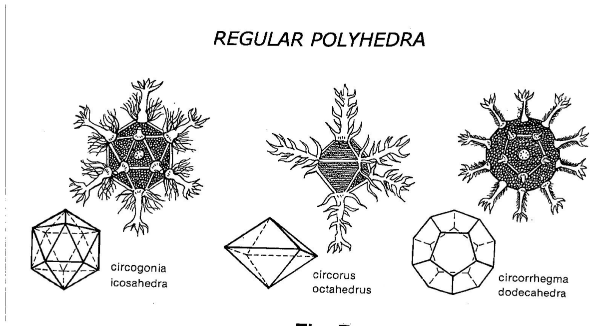

Fig.5

FIGURE 1.12 – Regular polyhedra

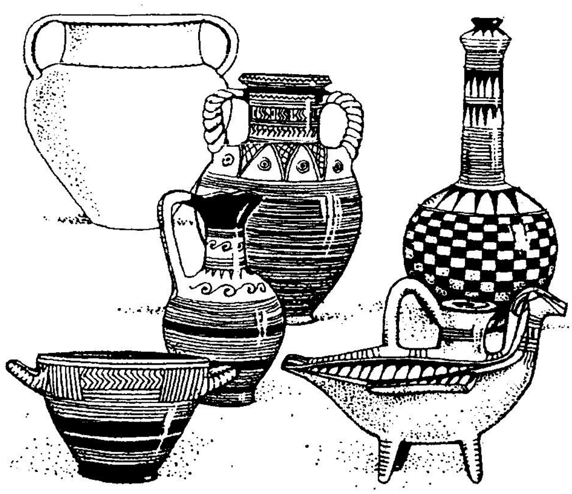

Fig.6

FIGURE 1.13 – Geometrical ornaments

One very simple shape which is frequented in nature and very useful in technology is the circle. It is not a surprise that the study of the properties of the circle is connected with well known names from mathematics history :

- **Thales** (624-548 BC),
- **Euclid** (365-300 BC),

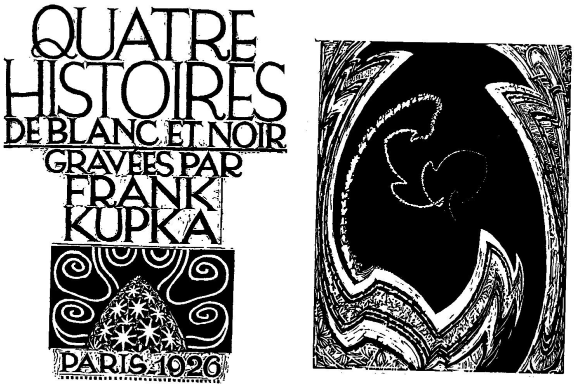

Fig.7

FIGURE 1.14 – Kupka's work

– **Apollónius** (262-190 BC),
– **Descartes** (1596-1650).

The circle is the figure of constant width. Is the circle the only shape with this property ? German engineer **Franz Reuleaux** (1829-1905) constructed another such figure (Reuleax triangle) (Fig. 1.15) which has applications not only in technology, it occures also in art. The Wankel rotary engine has a triangle rotor with form near to Reuleaux triangle (Fig. 1.16).

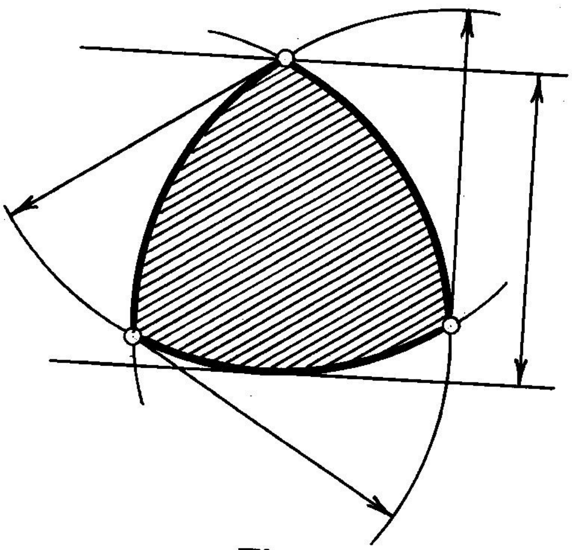

Fig.8

FIGURE 1.15 – Reuleax triangle

Problem 3 (*for students agend 13 years*)

Draw several circles arbitrarily and color the regions of this map only by two colors. Two regions with common frontiers must be colored by different colors.

One of many results is in the Fig. 1.17.

The beauty of the obtained picture may be a resource of SATISFACTION for the students.

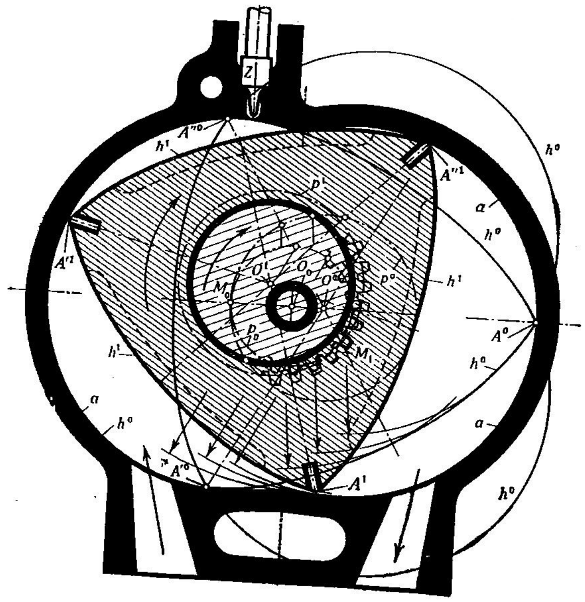

Fig.9

FIGURE 1.16 – Rotor

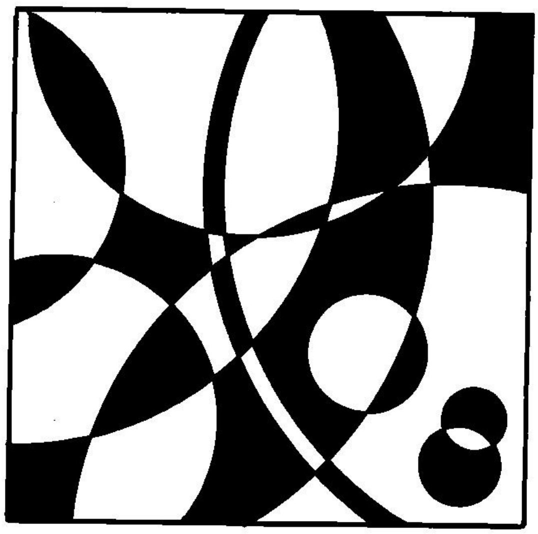

Fig.10

FIGURE 1.17 – Coloured regions

Problem 3 is of course a very simple modification of the *four color* problem which was solved by mathematicians **K. Appel** and **W. Haken** in 1976

(M) *Motion as demonstration of changes in reality.*

Motion of an animal in nature, motion of a car in the street, motion of a rocket in space, motion of a pencil tip while drawing, motion of an idea in history –all these are examples of changes in reality. The first visual representation of motion was realised by the invention of film here in Lyon by **August** and **Luis Lumiere** in 1895. In the painting of a cow from Lascaux the motion is expressed very sugestively (Fig. 1.18). An interesing record of motion is in Fig. **??** by **Strinberg**.

Fig.11

FIGURE 1.18 – Expressed motion

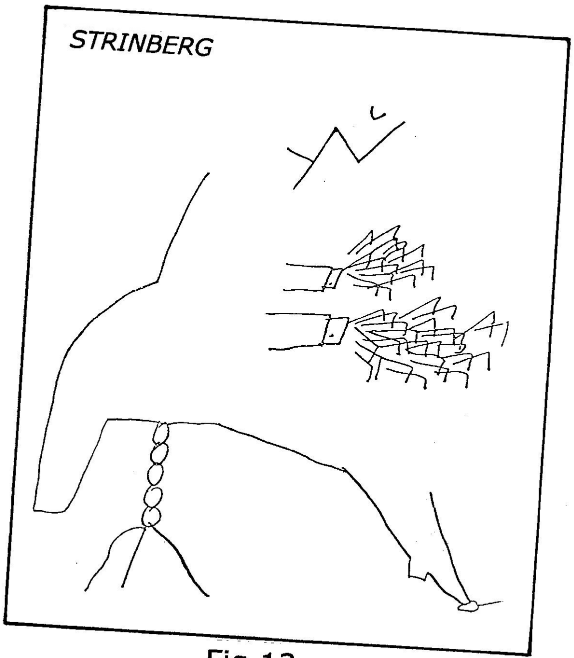

Fig.12

FIGURE 1.19 – Expressed motion

Many geometrical notions are connected with motion.

According to **Jacques Hadamard** (2008) we have for example :

"The *distance* AB is said to be equal to the distance A'B', if the first segment can be moved onto the second in such a way that A falls onto A' and B onto B'.

Every line contains infinitely many points. It can be viewed as beeing generated by a point which moves along it. This is what happens when we trace a line on paper with pencil or pen.

In the same way, a surface can be generated by a moving line.

If a point can occupy infinitely many positions, we call the figure formed by the set of these positions the *geometric locus of points*."

By means of motion we can introduce for example the circle, the ball, the cylinder or the cube. Shapes which we construct by means of motion in mathematics emerge also in nature : by plants or animal growing.

(D) *Dimension as nature of reality*

The dimensional point of view is presented in how children understand the world : the ball and its shadow, the shoe and its print, the mother and her photograph, ...

We can describe the position of a point in space by means of coordinates. The world we live in has three dimensions, the plane is two-dimensional, the line has only one dimension. If we think of our space as a set of triad of real numbers defined on the three-dimensional vector space, we can continue and construct four-dimensional space. In this space there exists the four-dimensional cube whose net is three-dimensional (Fig. 1.20). The Spanish artist **Salvador Dali** created a crucifix in the shape of the net of the four-dimensional cube (Scriba, Schreiber 2001) (Fig. 1.21).

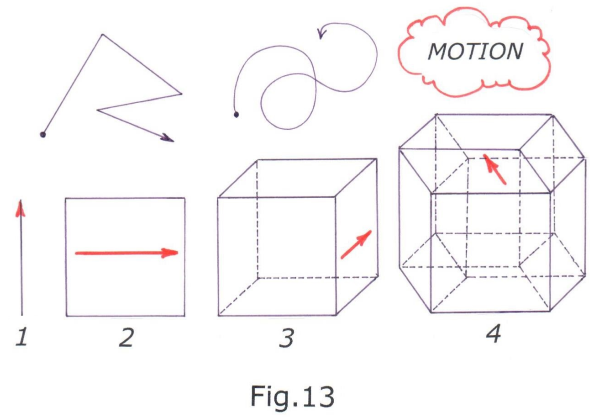

FIGURE 1.20 – Four dimensional cube

Two stimuli

Life and history are two most important stimuli for the cultivation of culture in mathematics education.

1 Dividing of space

The fence demarcates the garden, the frontiers define the teritory of a state, the hedge divides the area onto fields, the flat is divided into rooms, ...

Just like rooms in a house, so the cells of a plant are of different sizes, shapes and contents. Even quite an evident fact in our world (the coast is the frontier of continent and sea) has a very sofisticated expression in mathematics as the *Jordan curve theorem : a simple closed curve divides the plane into two connected regions, an "inside" and "outside"* (Fig. 1.22).

If the simple closed curve is a broken line, one of the regions is a polygon.

There are many interesting problems connected with polygons. I would like to remind only one.

Problem 4 (for students aged 15 years)

Find a polygon with this property : from a point in its inside (outside) it is not possible to see any of its sides in whole.

Some solutions are in Fig. 1.23.

2 Filling of the space

At the dawn of civilisation geometry was connected with solving practical problems.

For example the ancient Egyptians were concerned with the flooding of the Nile, food reserves and building the pyramids. Here the measuring of lengths, areas and volumes was important.

The base of the measuring of lengths is the **Archimedean axiom** :

$$\forall a \in R^{+} \forall b \in R^{+} \exists n \in N [na > b].$$

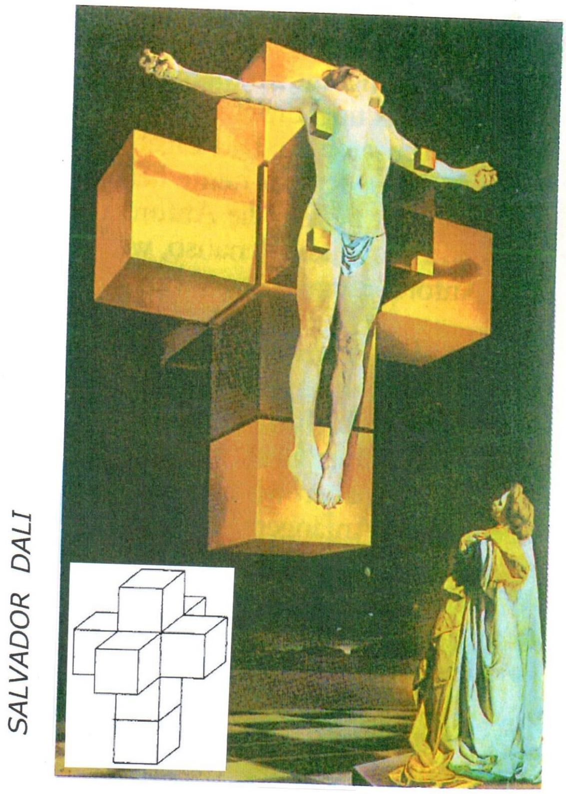

Fig.14

FIGURE 1.21 – Dali

The scale is a result of getting over a segment on a ray, measuring is the process of filling a segment by means of unit segments and their parts. The segments are of course the interpretations of positive real numbers, including irrational numbers.

Calculation of lengths, area and volumes are important parts of school mathematics.

During the early development of calculus, the Italian mathematician **Bonaventura Cavalieri** (1598-1647) formulated this principle :

If two solids have the same cross sectional area whenever they are sliced at the same height, then the two solids have the same volume.

Archimedes (287-212 BC) formulated this principle in physical interpretation about 2000 years in advance (Kordos 1994, Fig. 1.24) .

Mathematics and literature

Mathematics is strongly connected with all human culture. There are for example three pieces of literature which deal explicitly with the mathematical problem known as Fermat's last theorem.

The history is well known. It was in number theory that **Pierre Fermat** (1601-1665) made his greatest mark - literally.

While reading his Latin translation of Diophantus's Greek masterpiece Arithmetica, he wrote a deceptive simple comment in Latin text to a problem about finding squares that are sums of other squares (for example, $3^2 + 4^2 = 52$).

Fermat wrote :

"On the other hand it is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain" (Marilyn vos Savant, 1993).

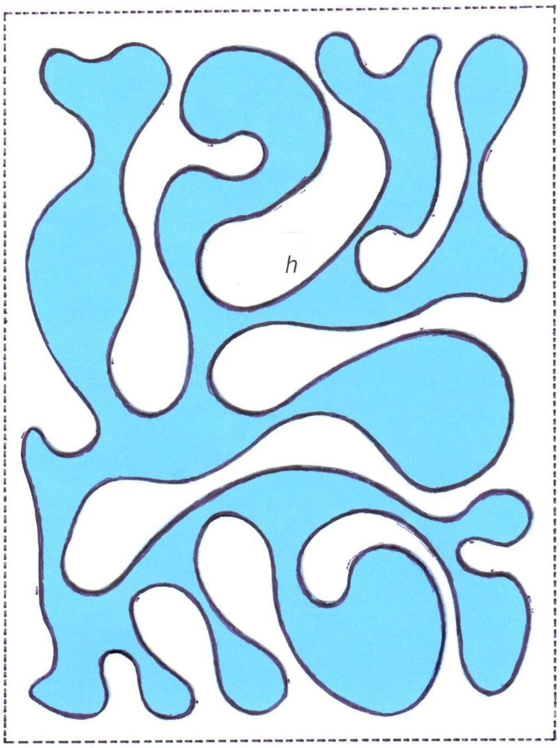

Fig.15

Figure 1.22 – Jordan's curve

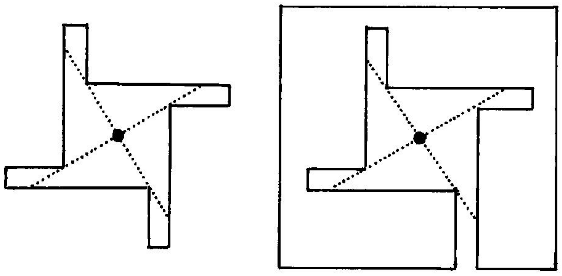

Fig.16

Figure 1.23 – A particular polygon

The Czech writer **Karel Matěj Čapek-Chod** (1860-1927) wrote the story "$x^n + y^n = z^n$" in which a mathematics teacher proved the Fermat's last theorem as a soldier in World War I. But his proof was destroyed together with his life.

The known French writer **Marcel Pagnol** (1885-1974) wrote a story "La petit fille aux yeux sombres" (Small girl with dark eyes) in which mathematician Lemeunier proves Fermat's last theorem for $n = 3$.

And the American poet **Cody Phanstiehl** wrote about Andy Wiles who "presented with smiles" the solution of this old problem.

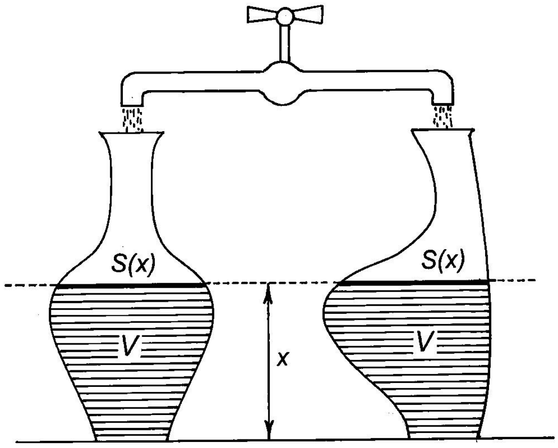

Fig.17

FIGURE 1.24 – Physical interpretation

Anglo –Irish satirist **Jonathan Swift** (1667-1745) touched on the idea of self –similarity, a notion of fractal geometry, over 260 years ago :

So naturalists observe, a flea
Hath smaller fleas that on him prey ;
And these have smaller fleas to bite'em,
And so proceed ad infinitum.

Benoit B. Mandelbrot (* 1924), Father of Fractals (Fig. 1.25), said :

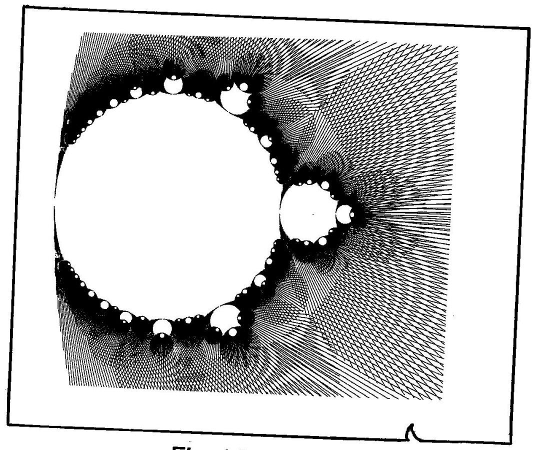

Fig.18 BENOIT MANDELBROT

FIGURE 1.25 – Mandelbrot' set

Clouds are not spheres,
mountains are not cones,

coastlines are not circles...
many patterns of Nature are...irregular and fragmented.

Technology, industry, agriculture, ...are of course the parts of human culture which can't work without mathematics. An important feature of mathematics is harmony, which is naturally the common feature of good creative and musical art and of course also literature.

Man as a creator

The lives of our children are full of creating new things. From building mud-pie out of sand, to children's drawings and building of complicated structures from cubes. Many activities which lead to new outcomes are common in all spheres of our life : new things, new ways, new products, new solutions to problems, new proof, new notions, new definitions, new knowledge, new understanding, new theories...

All such activities I will call constructions. According to the method of creation and according the results of constructions I will distinguish :

1. *Hard constructions* : the way and the result is determined.
2. *Soft or fuzzy constructions : both the method and result are open.*

Most problem solving in our school is hard constructions. Students solve typical problems such as application of known theory. If we want to develop creativity and inventiveness of students we would have to give them an opportunity to work also with soft and fuzzy constructions. The frontiers between hard, soft and fuzzy constructions are of course fuzzy and depend on many factors (previous knowledge, training, talent, ...)

To calculate $1\frac{3}{4} \div \frac{1}{2}$ is a hard construction, but to find a problem or story which is possible to solve by means of this calculation is a fuzzy construction. The author of this problem –Liping Ma-wrote that 90 % of a group of Chineese teachers solved this problem but only 5 % of a group of American teachers were successful. (Ma, 1999).

Construction is the proces of transformation of a state into a new state, it is a method of realization of a methamorphosis.

Our life is full of metamorphosis. We can find them in nature, technology, art, mathematics, ..., in all human activites (writing a novel, drawing a picture, constructing a machine, solving a problem, ...).

Metamorphosis in nature : from a caterpillar into a butterfly (Fig. 1.26).

Fig.19

Figure 1.26 – From a caterpillar into a butterfly

Metamorphosis in literature.

Franz Kafka (1883 –1924) one of the greatest writers of modernist and expressionist literature of the 20$^{\text{th}}$ century wrote the short story Metamorphosis in which we can read : "As Gregory Samsa awoke one morning from uneasy dreams, he found himself transformed in his bed into a gigantic insect...His numerous legs, which were pitifully thin compared to the rest of his bulk, waved helplessly before his eyes" (Fig. 1.27, 1.28).

Examples of constructions in technology and art are in in Fig. 1.29 and 1.30.

Metamorphosis in mathematics.

Find the value of the term $\sqrt{6+\sqrt{2}} - \sqrt{2}$

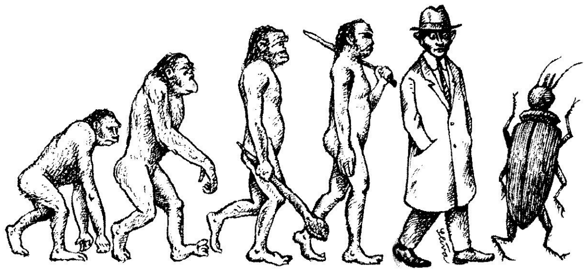

Fig.20

FIGURE 1.27 – Evolution

Fig.21

FIGURE 1.28 – Franz Kafka

We can calculate :

$$\sqrt{6+4\sqrt{2}}-\sqrt{2}=\sqrt{4+4\sqrt{2}+2}-\sqrt{2}=\sqrt{(2+\sqrt{2})^2}-\sqrt{2}=2+\sqrt{2}-\sqrt{2}=2$$

This is metamorphosis of symbols which is realized according to the laws of aritmetic, although for somebody this

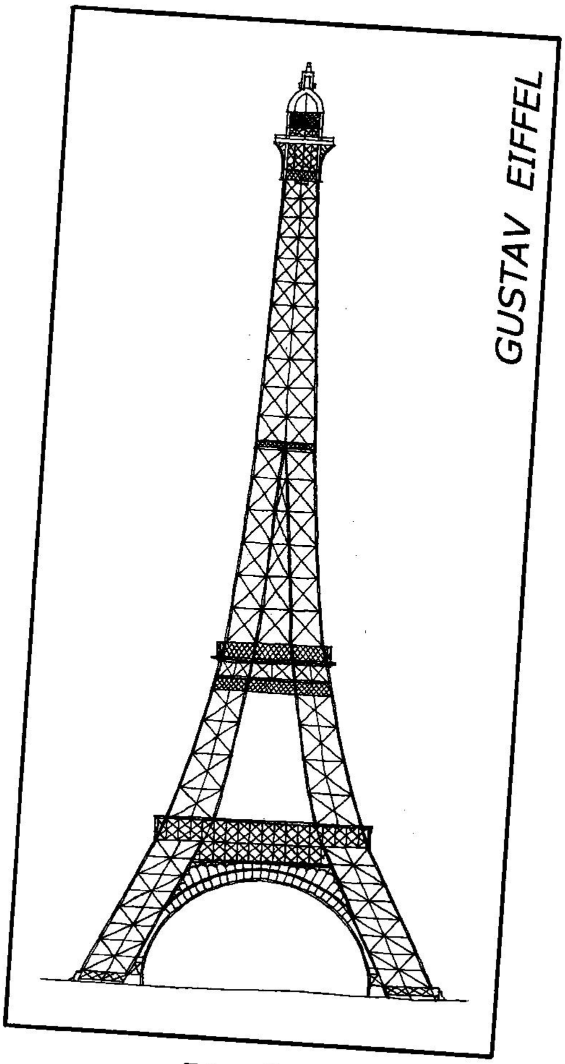

Fig.22

FIGURE 1.29 – Eiffel tower

claculation may be a miracle similar to the metamorphosis of caterpillar. For them who know elementary mathematics it is evident and simple.

James A. Garfield (1831 –1881), 20th US President proved Pythagorean theorem by the application of known formulas for area.

The area **A** of the trapeze ACDE (Fig. 1.31) is the sum of the areas of triangles ABC, BDE and ABE :

$$A = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab.$$

According to the formula $A = \frac{1}{2}(a+b)v$ for the area of trapeze with bases a, b and height v is $\mathbf{A} = \frac{1}{2}(a+b)^2$. From the equality

$$c^2 = a^2 + b^2$$

The construction of metamorphosis of known formulas into Pythagorean equality is not the application of logic itself.

Fig.23

[!h]

FIGURE 1.30 – Aubreye Beardsley

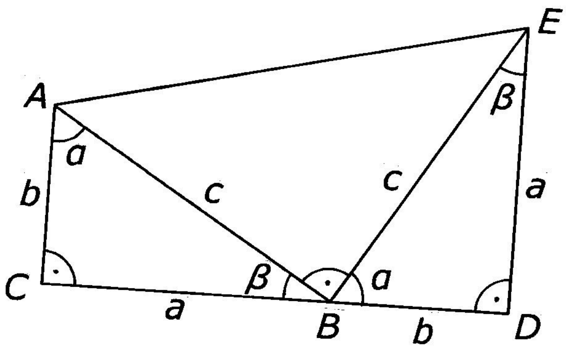

Fig.24

FIGURE 1.31 – Figure 24

The transformation of a part of mathematics into a new theorem, a new proof, a new definition or solving new problem is creative work usually hidden as the metamorphosis of the caterpillar to the butterfly, or Kafka to the insect, or transformation of a set of words into a poem, or the transformation of a set of colours into an artistic painting.

In all parts of culture *intuition, experience, talent and applications* are important.

Carl Friedrich Gauss (1777 –1855) referring to an arithmetical theorem which he had unsuccesly tried to prove for years writes : Finaly, two days ago, I succeeded, not on account of my painful efforts, but by the grace of God like a sudden flash of lightning, the riddle happend to be solved. I myself cannot say what was the conducting thread which connected what I previously knew with what made my success possible (Hadamard 1996).

How can we teach problem solving ?

There are some nice publications in this area, for example by **George Polya, Jacques Hadamard, Terence Tao,**. . .but in the school practice this problem, in my opinion, is still unsolved. Why ? According to Hadamard there exist four stages of problem solving :

1. Preparation.
2. Incubation
3. Illumination
4. Verification

All these stages require time. This is the main problem of school practice.

Metamorphosis of dimensions.

Our writer **Bohumil Hrabal** grasped literature as a metamorphosis of our multi-dimensional world into one-dimensional string of speech –sounds. Metamorphosis of the dimensions is of course an important mathematical idea.

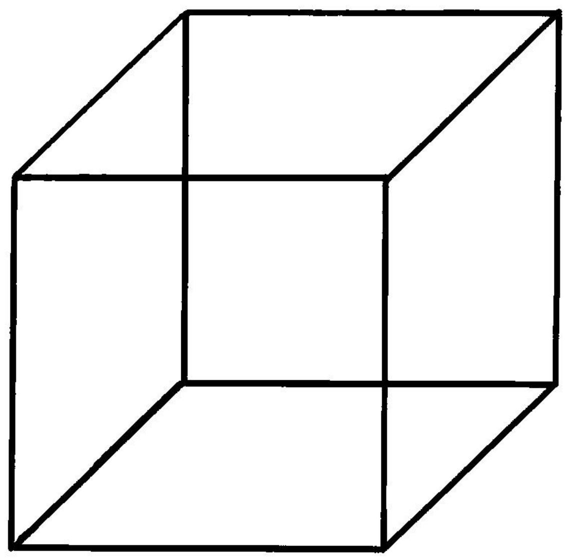

FIGURE 1.32 – Cube ?

Find the solid drawn in the Fig. 1.32. This is not a cube, because in the picture of cube there are not all edges visible. How can we imagin this polyhedron ? One of its faces is the hexagon $ABCDEF$ (Fig. 1.33), one face is the square S which is in front of the plane ABC. These two faces are connected with two triangles and four trapezes.

The visual lanquage of mathematics is, in my opinion, very important for understanding, but in our schools is often neglected

Culture of school mathematics

Culture is a very important phenomen in human development. Culture of school mathematics means, in my opinion, to see connections between the real world and mathematics, to have good orientation in individual parts of mathematics and also in mathematics as whole, to understand different mathematical languages and last but not least to solve problems by proper methods. Mathematical culture should be cultivated from the beginning of mathematical education. In our country the level of mathematical culture is not very high. I will show this with just one example.

My students aged 18 years solved this problem :

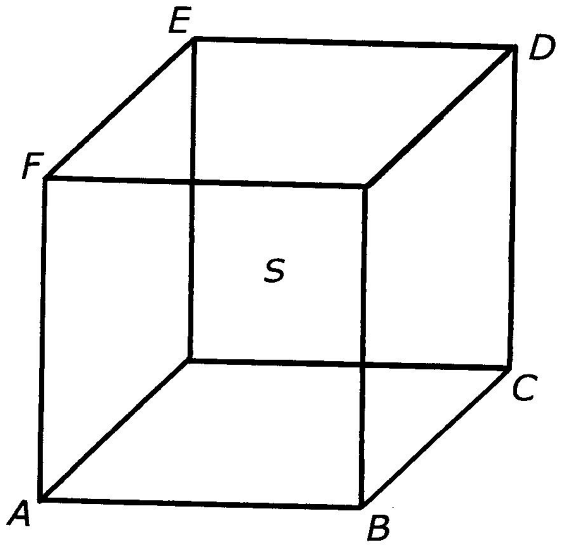

Fig.26

FIGURE 1.33 – Hexagon

Find the area of a regular dodecagon inscribed in the circle with radius r.

Only two students found simple solution shown in Fig. 1.34. It is possible to see complicated ways to "right" solution from these results :

$$A_{12} = 6r^2.\frac{\sin 30}{\sin 75}.\sqrt{1 - \frac{\sin^2 30}{4\sin^2 75}}$$

$$A_{12} = 12.\sqrt{\frac{r(2+\sqrt{2-\sqrt{3}})}{2}.\frac{r\sqrt{2-\sqrt{3}}}{2}.\frac{r\sqrt{2-\sqrt{3}}}{2}.\frac{r\left(2-\sqrt{2-\sqrt{3}}\right)}{2}}$$

74 % of students didn't solve our problem.

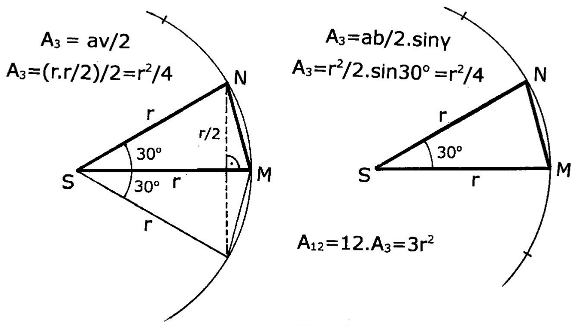

Fig.27

FIGURE 1.34 – A geometrical solution

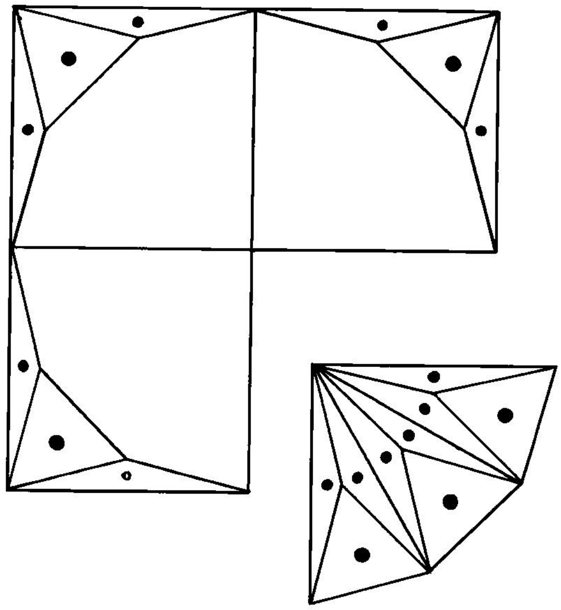

Fig.28

FIGURE 1.35 – Chinese solution

In the ninth century a Chineese mathematician solved our problem as shown in Fig. 1.35. This is nice mathematical culture.

Cultivation of mathematical culture is a very difficult task for teachers. We should turn our attention not only to the solution of problems, but also to the quality of this solution.

Conclusions

W. T. Gowers wrote in the book *Mathematics : Frontiers and Perspectives* (2002) about two cultures of mathematics. He means the distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories. He formulated the conclusion : mathematics needs both sorts of mathematicians.

In my opinion the two mentioned views on mathematics are important also in school mathematics. The balance between these two parts of mathematic education is an important part of mathematical culture. School mathematics grow out of the problems of reality and to these problems it should return.

The process of creating something new is similar in all areas of human culture –including mathematics. This is the foundation for the union of mathematics with other culture.

My recommendation : To construct system of simple but not evident problems connected with all areas of culture, to excite the interest of students for mathematics, to experience success with our students. But : mathematics is labour, mathematics is heavy business. If our students see, that it is possible to gain success without labour, our effort become vain.

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