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Chapitre 5

Realities, technologies and mathematical experiences / Réalités, technologies et expériences mathématiques

5.1 Working group 3 : Realities, technologies and mathematical experiences / Réalités, technologies et expériences mathématiques

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Plan for paper discussions in the WG

Monday 21, 16 :30 –18 :30

Mathematics and concerts at stadiums

Alejandro Rosas, alerosas@mail.ipn.mx ; Jorge Luis Rosas, jrosas@fis.cinvestav.mx Leticia del Rocio Pardo, rociopardo2000@yahoo.com.mx

The street lamp problem : discovering the triangle centres starting from a real situation

Gentile Elisa, elisa.gentileicloud.com, Monica Mattei, mattei_monica@libero.it

Proving processes in a Dynamic Geometry Environment : A case study

Madona Chartouny, Iman Osta, Nawal Abou Raad

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Tuesday 22, 10 :30 –12 :30

La pensée arithmético-algébrique dans la transition primaire-secondaire et le rôle des représentations spontanées et institutionnelles

Fernando Hitt, Mireille Saboya et Carlos Cortés

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Algebraic interactions emerging from a ICT school experience

Pili Royo ; Joaquin Giménez, quimgimenez@ub.edu

La calculatrice comme milieu expérimental

Ruhal Floris, Ruhal.Floris@unige.ch

Resources for teaching trigonometry on teachers' training

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Wednesday 23, , 10 :30 –12 :30

Primary graphs
Daniela Ferrarello, ferrarello@dmi.unict.it
Early childhood spatial development through a programmable robot
Cristina Sabena, cristina.sabena@unito.it
The use of technology when teaching about the equal sign
Anna Wernberg, anna.wernberg@mah.se
The activity of programming on the continued education of the mathematics teacher
Maria Elisabette Brisola Brito Prado, bette.prado@gmail.com ; ; Nielce Meneguelo Lobo da Costa, nielce.lobo@gmail.com ; Tânia Maria Mendonça Campos, taniammcampos@hotmail.com

Synthesis of WG3

Traditional technology : pencil and paper
– How researcher can help the teacher to integrate traditional tools and new tools'
– Research must be done taking into account this problematic.
– Which kind of cooperation could be done between researchers and school teachers.
The role of thinking time
– Different tools imply diverse time management to investigating a mathematical problem.
– Process of modeling taking into account real phenomena.
Potential and limits of technology
Giving children the "power" :
– on the one hand, technology gives children more power
– on the other hand, technology makes things so easy that it goes too fast and you do not have the time to think, to really learn.

=>

"Slow math" with paper and pencil could help.
Technology is changing reality
– The children's perception of reality is changing, often faster than for the teacher and the parents (because for them technology is "less" a part of reality).
– "Google glasses reality" will be more and more different than "our" reality. . .

=>

but still we will have the problem of thinking about what is a triangle from a theoretical perspective (properties, . . .).
Reality inside mathematics
– Are some parts of mathematics more real/realistic/concrete than others (geometry, statistics, arithmetics) ?
– Even algebra may become realistic to students with the use of technology (because of formal/symbolic language of communication with ICT)

Synthèse du WG3

Technologie traditionnelle : papier/crayon

- De quelle façon les chercheurs peuvent-ils aider les enseignants à intégrer les outils traditionnels avec les nouveaux ?
- Des recherches prenant en compte cette problématique devraient être faites.
- Quel type de coopération peut-elle être développée entre chercheurs et enseignants'

Rôle du temps pour la réflexion.
- Selon les outils il y aura différentes façons d ?organiser temporellement la recherche de problèmes.
- Modélisation prenant en compte des phénomènes réels.

Potentialité et limites de la technologie

Donner du pouvoir aux élèves :
- D'un côté, la technologie donne plus de pouvoir aux élèves.
- D'un autre côté, la technologie rend les choses si faciles que cela va trop vite et il n'y a pas de temps de réflexion, permettant de "réellement" apprendre.

=>

"Maths lentes" avec papier et crayon peuvent aider.

La technologie change la réalité
- Avec le développement technologique, la perception de la réalité par les enfants évolue, très souvent plus vite que pour les enseignants et les parents (pour lesquels la technologie fait moins partie de la réalité).
- Et la réalité des lunettes google sera encore plus différentes de la notre.
-

=>

Mais nous avions, avons et aurons toujours à réfléchir sur les triangles d ?un point de vue théorique propriétés, argumentation,. . .)

Realité à l'intérieur des mathématiques
- Certaines parties des mathématiques sont-elles plus réelles/réalistiques/concrètes que d ?autres (géométrie, statistiques, calcul) ?
- Même l'algèbre peut devenir une réalité, plus facilement avec la technologie, étant donné l'aspect symbolique des langages informatiques.

5.2 Mathematics and concerts at stadiums

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Résumé : Dans ce travail, nous présentons une activité didactique dans lequel les étudiants dans un cours de calcul intégral sont confrontés au problème de compter les sièges d'une section d'un stade de football. La difficulté est que la section est incurvée de sorte que les lignes ont des numéros de sièges, ce nombre augmente à partir de la première ligne sur le terrain de la rangée du haut. Deux groupes d'étudiants ont résolu le problème, il fallait la restriction qu'ils ne devraient utiliser un crayon, du papier et des livres, mais pas d'ordinateurs ou Internet. Le deuxième groupe a été en mesure d'utiliser toutes les ressources technologiques qu'ils voulaient. Lorsque nous avons analysé les réponses que nous avons trouvé une plus grande variété dans les moyens de résoudre le problème dans le groupe qui n'a pas la technologie, par exemple ils ont utilisé la géométrie, les progressions arithmétiques, sommes de Riemann et de l'intégration. Dans le groupe qui a eu l'occasion d'utiliser les ressources technologiques et internet nous avons trouvé deux types de solutions : ceux fondés sur des feuilles de calcul et les y compris l'utilisation de mathématiques de logiciels spécialisés. Une de nos conclusions est que les étudiants qui n'ont pas la technologie devait se rappeler les connaissances acquises dans d'autres cours y appliquer ces connaissances pour résoudre le problème

Abstract : In this work we present a didactic activity in which university students in a course of Integral Calculus face the problem of counting the seats of a section of a football stadium. The difficulty is that the section is curved so that the rows have different numbers of seats, this number increases from the first row on the field to the top row. Two groups of students solved the problem, one had the restriction that they should only use pencil, paper and books but no computers or internet. The second group was able to use all the technological resources they wished. When we analyzed the responses we found greater variety in the ways of solving the problem in the group that did not use technology for example they used geometry, arithmetic progressions, Sums of Riemann and integration. In the group that had the opportunity to use technological resources and internet we found two types of solutions : those based on spreadsheets and those including the use of specialized software math. One of our conclusions is that students who did not use technology had to remember the knowledge acquired in other courses y applied that knowledge to solve the problem.

Introduction

Often students when faced with an activity away from repetitive exercises and traditional math class, have different types of problems. First you must understand the nature of the activity, then they should find out who are the main variables in the studied phenomenon, then must represent the phenomenon in mathematical terms and eventually solve and explain your solution.

We present a didactic activity based on the context of a concert organized in a football stadium. The problem is that the stadium has two straight sides and its ends are curved causing the number of seats per row will change. Therefore students must find some way that allows them to calculate the number of seats in a given row.

Although activity was presented to students in the field of integral calculus second semester of college a specific topic of integral calculus is not involved.

School Context

In (IPN) National Polytechnic Institute in Mexico students of engineering degrees must take different courses in mathematics, including differential calculus and integral calculus, in one real variable and with many variables. In Differential Calculus is used to provide the student lists of exercises where different derivation rules are applied, lists of 20, 30 or more functions to derive. Similarly the student will be provided for different sets of ffunctions to integrate with the correct integration rule. In the new educational model of the IPN, it seeks to develop different mathematical skills in students so far focus on the use of learning activities that address students with situations that are close to their daily life.

Thus the aim is for students to apply their mathematical knowledge according to what they guess (by themselves) and check their results with real world.

The activity

Students were given a sheet of paper with the following instructions :

Consider that you are the Concert Logistics Manager of Alpha Rocks Company, and you must organize a concert where a famous rock singer will perform.
By the number of attendees, the concert will be held in a stadium, but the stage on which act the artist will be placed in one of the curve sections of the stadium, so that all those seats may not be used, so to calculate the cost of the tickets you should know the total number of seats that are in that section.
The problem is that as this section of the stadium is not rectangular, each row has a different number of seats.
The first row has 50 seats, the second row has 52, third row has 54, etc. Please answer the following questions.

1. How many seats are in row 45 ?
2. Now, because we need to know the total number of seats in the curve section, how many seats are there in total if the section has 50 rows'
3. Find a formula that lets to to calculate the number of seats at any row.
4. Find a formula that lets you to calculate the number of seats in the curve section if the section has n rows.

Please write a report including every step you did to find your solution.

FIGURE 5.1 – Stadium with a curve in the head.

As can be seen, we decided not to include directions about how students should solve the problem.

This activity was applied in two different environments. In a first application activity was in a classroom in a two hours sesion and the students used everything they wanted except computer and internet. This time the work was done individually. In a second group we applied the same activity but now we allowed students to do the activity at home, they also had to work individually but this group had the freedom to use computers and the internet. Finally some students interacted but they were only five out of 30 students in the group.

Some solutions

First we discuss some solutions we got from the group that did not use computers or internet.

In general this group of students began their work by the method of trial and error, several students began making a table with the number of seats per row : Row 1, 50 seats, Row 2, 52 seats, Row 3, 54 seats, Row 4, 56 seats and so on. Using one of these tables a student made the following arrangement : row 1, 50 seats, row 2, 50 +2 seats, row 3, 50 +2 +2 seats, etc. This arrangement led to find another arrangement : row 1, 50 +2 (0), row 2, 50 +2 (1), row 3, 50 +2 (2), etc.

She finally obtained the expression : row n, $50 + 2(n - 1)$ seats (for n from 1 to 50). With that result she answered questions 1 and 3 of the problem. To calculate the sum of rows 1 to 50, the student wrote the number of seats per row in the style of accommodation used to calculate the sum of the first n integers :

$$50 + 52 + 54 + \cdots + 146 + 148 = x$$

$$148 + 146 + 144 + \cdots + 52 + 50 = x$$

$$198 + 198 + 198 + \cdots + 198 + 198 = 2x$$

from where she got $50(198) = 2x$ and the final answer $x = 4950$ seats. But she could not obtain a formula for the case of an arbitrary number of rows.

Other students used a similar arrangement but adjusting the initial value, so the expression they obtained was : row n, 48 +2 (n) seats (for n from 1 to 50). To find the solution of questions 2 and 4 a student applied the formula corresponding to the sum of an arithmetic progression.

Two students used geometry because they observed that the shape of the curved section drawn on paper corresponds to a trapezoid, as you can see in figure 5.2.

FIGURE 5.2 – Curve section looks like a trapezoid.

To calculate the number of seats in the section one of the students applied the formula for the area of the trapezoid considering a smaller base of length 50, a larger base of length 138 and a height of 50. The other student was a girl who used analytic geometry to calculate the number of seats for this case : she used half of the curved section and divided into a rectangular area and a triangle. As shown in figure 5.3.

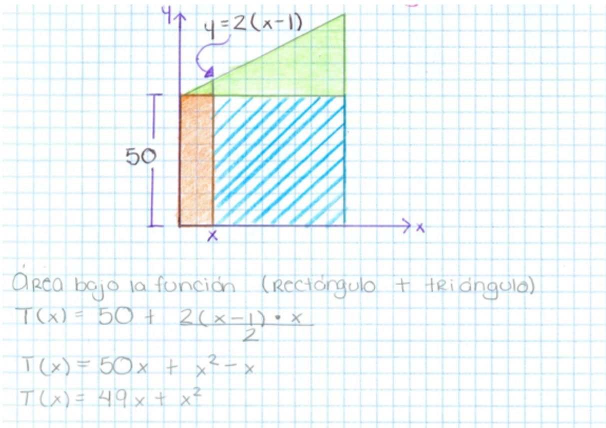

FIGURE 5.3 – Formula calculated using analytic geometry.

Many students obtained similar expressions by various means, for example a student judged to have two points : (1, 50) and (2, 52) and used these points in analytic geometry formula that provides the equation of a line through two points. The function obtained is : $f(x) = 2x + 48$. This expression produced the number of seats in row 45. To answer the question of the formula for the sum of any number of rows she used Sums of Riemann.

Another student from the expression $f(x) = 2x + 48$ integration used to calculate the total. In this case the student did not obtain the correct value of the total seats in the section. When we asked why he thougt he had not the correct value he could not respond.

There were some other solutions but they were a combination of those solutions we have explained.

The second group showed two kinds of solutions. The first one is based on the use of a spreadsheet. Students only wrote a table like the one in figure 5.4 using formulas like $= B2 + 2$ to calculate the number of seats in the next row. But they did not solve question 4.

	A	B	C
1	fila	asientos	
2	1	50	
3	2	52	
4	3	54	
5	4	56	
6	5	58	
7	6	60	
8	7	62	
9	8	64	

FIGURE 5.4 – Spreadsheet used by some students to calculate the number of seats.

Second kind of solutions were based on the use of specialized software. For example, two students used Mathematica to solve the questions, we do not know if they were aware of what kind of mathematics were involved in their solution.

Conclusions

We have just started to analyze these solutions so that our conclusions are preliminary. First we found a greater variety and richness of the solutions proposed by the students who did not use computers or the Internet, we also found that the students were a little worried about the answers they got. However, when each student gave his answer that matched the responses of other students they began to have greater confidence in their work. Some students commented on how different solution methods led to the same answers. We assume that for further analysis should pay more attention to the computer skills that students were able to develop in the group using computers and the internet. This is the first activity of a set of three and we are now applying activity 2.

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5.3 The use of technological resources for teaching trigonometry on teacher's training/L'utilisation des ressources technologiques pour l'enseignement de la trigonométrie sur la formation des enseignants

Nielce Meneguelo Lobo da Costa, Maria Elisa Esteves Lopes Galvão,Maria Elisabette Brisola Brito Prado

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Résumé :Dans cet article, nous presentons la recherche développée dans un processus de formation dont l'objectif était l'enseignement de la trigonométrie à l'utilisation des ressources technologiques . Une étude de recherche a eu lieu dans le cadre de la formation continue au Programme d'éducation Observatoire (Programa Observatório da Educação) et l'autre a été développé dans le cadre de la formation des futurs enseignants de mathématiques. La méthodologie utilisée dans les deux études était qualitative, avec une analyse interprétative. Dans le premier étude , nous avons discuté d'une situation de programmation avec le logiciel GeoGebra, qui a permis aux enseignants d' obtenir des outils pour discuter avec leurs élèves les variations des paramètres des fonctions impliquant des sinus et de cosinus et de leur impact sur le graphique. Dans le deuxième étude, nous présentons un exemple de discussion contextualisée associée à une fonction périodique pour lequel il a été créé un modèle pour une situation réelle. Avec l'aide de Cabri Géomètre - à simuler le mouvement d'une roue, un modèle a été construit, et, de l'élaboration d'un tableau de hauteurs, les futurs enseignants construitla representation grafique dune fonction périodique, en articulant les deux représentations. Dans les deux études, le dynamique fournies par le logiciel a permis une expérimentation beaucoup plus riche que celle classique, sur l'enquête sur les propriétés et le comportement des fonctions trigonométriques.

Abstract :In this article we discuss research developed in training processes whose focus was the teaching of trigonometry with the use of technological resources. One research study occurred in the context of continuous training at the Education Observatory Program (Programa Observatório da Educação) and the other one was developed in the context of training future mathematics teachers. The methodology used in both studies was qualitative, with interpretative analysis. In the first research study, we discussed a situation involving programming with the Geogebra software, which enabled the teachers to get tools to discuss with their students the variations of parameters in functions involving sines and cosines and their impact on the chart. In the second research study, we present an example of a contextualized discussion associated to a periodical function for which it was established a model for a real situation. With the help of Cabri-Géomètre to simulate the movement of a Ferris wheel, a model was built, and, from the elaboration of a table of heights, future teachers built a draft of a periodical function chart, articulating both representations. In both studies, the dynamics provided by software enabled a much richer experimentation than the conventional one on the investigation about properties and behavior of trigonometric functions.

Introduction

The Mathematic and Reality theme that guides the CIEAEM 2014, in the context of our work, namely the training of teachers for Basic Education, led to reflections on pedagogic and didactical issues involved in the act of teaching and learning mathematics in the XXI century, the reality of teaching mathematics and, in particular, about the educational resources available to the teacher today, especially digital resources, which join traditional concrete materials and tools (pencil, ruler and compass). Naturally such resources were and are considered relevant in teaching and learning. Among the educational resources for teaching, the teacher can still resort to the use of games, problem solving, history of mathematics and digital information technology and communication DITC. However a simple insertion, at school, the resources of any kind, does not guarantee a mathematic teaching with active student participation, characterized by an exploratory, investigative and useful for their practical life character. This means that the proposition of mathematical tasks connected with the real world, which make the student learn from research, from raising hypotheses, from exploration and research, in addition to further discussion of the solutions, is a challenge that goes far beyond the insertion of DITC on teaching.

In fact, with the resources of the paper and pencil environment it is possible to promote a teaching that has reported the above characteristics; we believe, however, that with the use of technological resources the possibilities may be extended. If we, as teachers and students, use, with the software, exactly the same logic of confrontation of mathematical tasks done with pencil and paper, we will take the risk of making an impoverished use of technology. We believe that, if properly integrated to teaching, technology can be a constructive thought, i.e., one can think of mathematics in alternative ways with the aid of technology.

The question that is presented for researchers and trainers of mathematics teachers is : What changes, from the student's perspective, in this movement from a mathematics teaching centered on the use of concrete materials and resolutions of tasks on paper and pencil to the use of technology? This is an issue that should be present in discussions on training courses for future teachers or currently practicing teachers.

Artigue (2000), reflecting on the problematic of using technology in the mathematics classroom warns that we must be aware about its dual role, a pragmatic one, contributing to the production of responses and an epistemic one, assisting the understanding of mathematical objects involved. Artigue also highlights the problem of underestimating the complexity of the process of instrumentalization of teachers (adaptation of the instrument by the user for specific uses) and instrumentation (how the instrument shapes strategies and knowledge of the user). He also emphasizes the idea of instrumental genesis from Rabardel (1995), i.e., the process from which an artifact (the object - i.e., a map, a software, a computer, a tablet, etc.) becomes an instrument for the individual. When starting to use an artifact, the individual constructs his own utilization schemes and, thus, starts to enrich his mental outlook.

The purpose of this paper is to discuss researches developed in training processes that focused on the use of technology for teaching trigonometry, one in the context of continuing education in the Education Observatory Program (Programa Observatório da Educação) and the other in the training of future mathematics teachers, respectively, the research of Poloni (2014) Miashiro (2013). Both deal with trigonometric functions and have used aspects and resources from two different softwares for dynamic modeling; in addition, both presented situations favoring the instrumental genesis, as understood by Rabardel.

Theoretical foundation

The researches discussed here are based on studies of Mishra and Khoeler (2006) and regarding the technological pedagogical content knowledge (TPACK) and, in relation to the construction of this knowledge, on the Theory of instrumentation from Rabardel.

The TPACK (Technological Pedagogical Content Knowledge) includes the understanding by the teacher of how to represent concepts using technologies; of how to pedagogically address the use of technological resources constructively to teaching and to the student learning on curricular concepts, in this case, mathematic concepts. It is this integration of technological, pedagogical and content knowledge that enables to make use of digital technologies as a new form of representation of thought.

One of the aspects of the TPACK construction process relays on the ownership and instrumentation of the technologic resource, which should facilitate the reconstruction of teacher's pedagogic practice. In this sense, we believe it is crucial in the formative processes to promote the instrumentalization and instrumentation of teachers, in order to assist the expansion of technological pedagogical content knowledge (TPACK). When considering the teacher, this process of instrumental genesis is part of the construction of TPACK, since in it, other knowledge besides the technological knowledge are mobilized and amalgamated.

The Researches

In the research documented by Poloni (2014), the Geogebra software was used to discuss with teachers the learning possibilities through investigation. The activity presented below aimed to equip teachers for programming with the Geogebra software to enable them to discuss with their students the variations of parameters in functions involving sines and cosines and the impact caused by them in the graph of each function. It was necessary for teachers to build the function in GeoGebra and, for the programming, besides the knowledge about the available tools in the software and also about the type of programming, it was necessary to lead the teacher to conclude that any point of the graph of this function is of type P This means that the intrinsic features of the software need mathematical and technological knowledge.

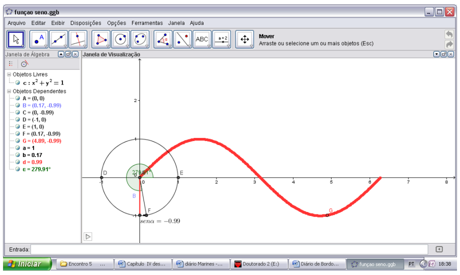

FIGURE 5.5 – Function constructed on the Geogebra software Source : Poloni (2014)

After the generalization of the coordinates of the point of the function $f(x) = \sin x$, the construction of graphics for $f(x) = \cos x$; $f(x) = \tan x$ and others using the software was immediate. This leads us to think that the teacher began the process of instrumentation, in addition to the instrumentation process. Besides, participating teachers were able to construct the graphs of trigonometric functions involving sine, like $f(x) = a + b \sin x$. From the constructions it was possible to vary the values of the parameters a and b and to identify the changes in the graphs of functions, articulating an algebraic expression for each function with the graph that corresponds to it. Conversions of registers (Duval, 2006) were present and the teacher built utilization schemes and began to realize what kind of strategy was possible to develop in the software, i.e., the way of modeling in the software the situation under study.

In research Miashiro (2013), the Cabri-Géomètre II software was used in the training process to enable future teachers to interact with the trigonometric ratios, trigonometric cycle and trigonometric functions. To build a sketch graph of a periodic function, it addressed a context based on the real world. A model was developed during the formation for the simulation of the motion of a Ferris wheel, adapted from the book Functions Modeling Change : a preparation for calculus (Connally et al, 1998). The proposed model was related to the Ferris wheel, located in London. The Ferris wheel built with the dynamic geometry in Cabri was prepared to permit the measurement of the height of a point on the circle at intervals of 5 minutes, and to display on the left side of the figure, above the altimeter word, a measure in centimeters in the range of 1 to 9 cm. The point of the circle representing the position of a passenger could be manipulated with the "hand" tool.

From the obtained data, organized in a table, it was produced a sketch of the graph of the function that describes the variation of heights and a discussion was guided about their periodicity. The work involved a real situation, and the construction of its model, in which two conversions of registers of

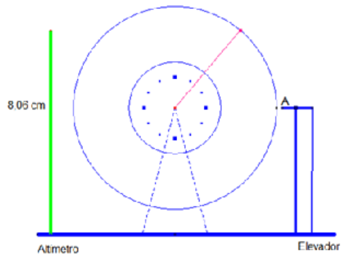

FIGURE 5.6 – Ferris wheel constructed on the Geogebra software Source : Miashiro (2013)

semiotic representations occurred, according to Duval (2006) it was possible to observe in several of the future teachers the process of instrumental genesis, especially when they proposed didactic adaptations to the activity.

Conclusions

Solving mathematical tasks is different when done on paper and pencil and technological environment, which allow us beyond observing and recording, to manipulate and to test.The instrumentation enables the development of technological pedagogical content knowledge by the teacher. In both studies, after investigations about measures and angles in the trigonometric cycle, the properties of functions were reached, in pathways that show the importance of the instrumentation process of teachers to integrate technology in the perspective of TPACK. Finally, we add that, regarding technological resources, each artifact that appears has its own characteristics and demand from the user a process of instrumentalization and instrumentation. From the schemes of use, evolving into the instrumental genesis, the individual evolves in the instrumentation process, which in turn drives the development of technological pedagogical content knowledge by the teacher.

Acknowledgements

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5.4 The street lamp problem : discovering the triangle centres starting from a real situation

Gentile Elisa, Monica Mattei

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Résumé : Cet article décrit une expérience d'enseignement : une activité de recherche de problème utilisant Geogebra est proposée à des élèves de l'école secondaire. Le problème amène à découvrir les centres du triangle en partant d'une situation de la vie réelle : le problème du lampadaire. Les élèves ont à décider de la meilleurs place pour placer un lampadaire de façon à ce qu'il allume toute la zone piétonne triangulaire. L'activité débute par des manipulations utilisant la photo de la zone et une torche permettant de simuler le lampadaire et se poursuit par une simulation de cette situation avec Geogebra. Le logiciel aide les élèves pour conjecturer et découvrir les centres du triangles à travers leurs propriétés. De plus, cette activité amène les élèves à expliquer leur approche du problème et à décrire dans le langage naturel les propriétés essentielles des objets mathématiques en jeu.

Abstract : This article describes a teaching experiment : a problem solving activity with the use of GeoGebra proposed to middle school students. The problem proposed involves the discovery of the triangle centres starting from a real situation : the street lamp problem. The students have to decide the best place to put a street lamp in order to enlighten the whole triangular pedestrian area. The activity starts with the manipulation of poor materials such as the picture of a pedestrian area and a torch to simulate the lamp and continues with the transposition of this exploration through technology. GeoGebra helps the students in conjecturing, discovering the triangle centres and recognising their properties. Furthermore this activity forces the students to explain their approach to the problem and to describe in natural language the main properties of the geometrical objects involved.

Introduction

The activity we proposed to our students is the adaptation to middle school context of an open ended problem, "The street lamp problem", that was presented during the PLS course "Problem solving with GeoGebra" we attended this year. PLS (Piano Lauree Scientifiche) is an Italian project for the professional development of the mathematics teachers based on the collaboration between university teachers and school teachers. In particular, the "Problem solving with GeoGebra" project is part of a huge research project that involves Italy and Australia with the aim of engaging in-service secondary school teachers in professional development based on best practices in mathematics with GeoGebra.

Two are the communities involved in the project : the researchers, who projected the tasks and the educational program, and teachers who attended the course. The teachers are also asked to experiment the activity with their classes and to reflect with the other teachers and with the researchers about what happened in class.

Teachers are observed during both the course and the didactical experimentation in class and data are analysed according to the "the Meta-Didactical Transposition" (Arzarello et al., 2014 ; Arzarello et al., 2012 ; Aldon et al., 2013).

The street lamp problem has been projected by the team of researchers in Turin and it was originally addressed to higher secondary school students (14-19 years old). Since we were teaching in lower secondary school (11-13 years old students), we needed to adapt the problem to our context. In particular we paid attention to maintain the "openness" of the problem, the idea of problem solving, but we inserted some more questions to slightly guide the students to understand better the problem.

Theoretical framework and institutional context

Meta-Didactical Transposition

The *Meta-Didactical Transposition* is a new model for framing teacher education projects and its focus is the interaction between the praxeologies of the researchers and the praxeologies of the teachers (in-service or pre-service training) and the dynamic between internal and external components (Aldon et al., 2013 ; Arzarello et al., 2014 ; Arzarello et al., 2012).

It is an adaptation of the Anthropological Theory of the Didactic by Chevallard (1999) to teacher education. Its main theoretical tool is the notion of *praxeology*, which can be described using two levels :

1. the "know how" (*praxis*) : a family of similar *problems* to be studied and the *techniques* available to solve them ;
2. the "knowledge" (*logos*) : the "discourses" that describe, explain and justify the techniques that are used for solving that task. The "knowledge level" can be further decomposed in two components : *Technologies* and *Theories*.

A praxeology consists in a Task, a Technique and a more or less structured Argument that justifies or frames the Technique for that Task (Aldon et al., 2013).

While ATD focuses on the institutional dimension of mathematical knowledge, the MDT model considers the *meta-didactical praxeologies*, which consist of the tasks, techniques and justifying discourses that develop during the process of teacher education.

Two are the communities involved in this project : the community of teachers (who are in training) and the community of teacher-researchers (who designed the task, act as trainers and observe the teachers). Each of these communities has got its own *praxeologies*, the challenge at the end of the project is to create *shared praxeologies*, thanks to the *brokers*.

A *broker* is a person who belongs to more than one community. (e.g. a teacher-researcher belongs to the community of mathematics experts and to the community of school-teachers). Brokers are able to make new connections across communities of practice and facilitate the sharing of knowledge and practices from one community to the other.

In our specific case, Monica belongs to the teachers' community and Elisa belongs to the teacher-researchers' community and acted as a broker during the educational program.

Some of the components of the two communities' praxeologies can change during the educational program and move from external to become internal, regarding the community they refer to.

In our case : let's consider the community of teachers that starts the educational program. Initially, the use of GeoGebra in lower secondary school and the use of open-ended problems are external components for the teachers. However, at the end of the educational program they become internal components in their praxeologies. The researchers' praxeology of designing a task for the teachers could become shared praxeologies when teachers design the task for their students.

National curriculum

In September 2012 the Italian Ministry of education released a new version of the National Curriculum for the first cycle of education (from 3 to 14 years old). The National Curriculum is declined through "Goals for the development of competences" and "Learning Objectives" and explains the expected knowledge and competence at the end of lower secondary school. The National Curriculum is also accompanied by a description of the main ideas of the teaching-learning process and of the different school subjects.

Here you can find some quotations from the National Curriculum of lower secondary school, we recognised as important for the framework of the activity we proposed.

"The resolution of problems is a characteristic of mathematical practice. Problems need to be understood as **real and significant issues**, related to everyday life, and not just as repetitive exercises or questions that are answered simply by recalling a definition or a rule. Gradually, stimulated by the teacher's guidance and the discussion with peers, the student will learn to deal with difficult situations with confidence and determination, representing them in several ways, **conducting** appropriate **explorations**, dedicating the time necessary for precise identification of what is known and what to find, **conjecturing solutions** and results, identifying possible strategies.

Particular attention will be devoted to the development of the ability to **present and discuss** with their peers the solutions and the procedures followed.

The development of an adequate vision of mathematics is of a great importance. This vision does not reduce mathematics to a set of rules to be memorized and applied, but recognizes mathematics as a framework to address significant problems and to explore and perceive relationships and structures that are found and occur in nature and in the creations of men."

Goals for the development of competences

- To explain the procedure followed, also in written form, maintaining control on both the problem-solving process, both on the results.
- To compare different processes and to produce formalizations that allows the student to move from a specific problem to a class of problems.
- To support his own beliefs, giving examples and counterexamples and using appropriate concatenations of statements ; to agree to change his opinion recognizing the logical consequences of a correct argument.

Learning Objectives

- To know the definitions and properties (angles, axes of symmetry, diagonals, . . .) of the main plane figures (triangles, quadrilaterals, regular polygons, circles).

Class contest

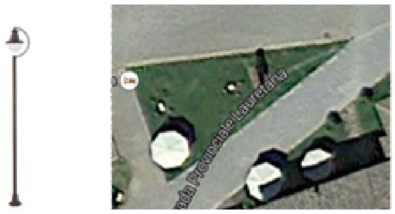
We proposed this activity to 12 years old pupils belonging to two different schools. One class, whose teacher is Monica, comes from "Istituto Don Bosco" in San Benigno Canavese (Turin). It is a 25 student class, including 4 boys with learning disabilities. During the school year they have been showing interest and curiosity in front of Maths problems, especially involving real situations. They are able to work together, in small groups or in couples, helping each other and they are used to discuss the results with the teacher. In January the students started to use GeoGebra as an instrument for exploring the Geometrical content of the curriculum in an active way. They showed, first of all, astonishment and then the strong desire to learn how the software works. Moreover, the students have the possibility to share files and didactical materials on the Moodle platform, performing a real community of practice (Wenger, 1998).

The other class, whose teacher is Elisa, comes from "Scuola Media Holden" in Chieri (Turin). The class is composed by 2 students : a male and a female. They are interested and curious towards the activities proposed during math lessons and they can be placed in an intermediate level of knowledge and competence. They are used to work in groups with a laboratorial methodology and to discuss with the teacher results and ideas. They started to use GeoGebra to explore Geometrical properties (such as angles, perpendicular and parallel lines, . . .) as a support for manipulation of materials (paper folding, paper and pencil, . . .)

The street lamp problem

Part 1

The City Council has decided to build a small triangular pedestrian area planned by the previous administration.
The registered project foresees only one street lamp as illumination for the whole area. Here you are the picture of the pedestrian area.



Can you help the technician who will have to deal with the installation, to find the exact point where the street lamp should be placed?
You can use the picture of the pedestian area and an electric torch to simulate the street lamp.
Now explain how you will proceed to find the best place to put the street lamp.
……

Part 2

Now open the file GeoGebra Lampione.ggb. You will find the pedestrian area to be lit. Try to find, using GeoGebra, the best point.
What did you do? Explain the procedure you followed and why did you choose that point.
……
What are the operational guidance that you could give to the municipal technician to identify the point to put the lamp? What are the relationships of that point with the triangle that defines the pedestrian area?

Part 3

In your opinion does the position of the point depend on the shape of the pedestrian area? What happens if the triangular shape changes? Try to explore the situation with GeoGebra: draw in a new sheet a generic triangle and save the file as Piazza.ggb. Explain what you have discovered.

FIGURE 5.7 – The problem for the student

The street lamp problem

The street lamp problem, as we said before, is an open problem. The starting situation is a meaningful situation for students : the municipal technician has to put a unique street lamp in a triangular pedestrian area, projected by the previous administration. He has to find the best point for the street lamp in order to enlighten all the area without wasting energy. In the picture below you can find the full text of the problem.

Aim of the activity

The main goal of the problem-solving proposed, regarding mathematical contents, is the discovery of the centres of a triangle, focusing on circumcentre. Another fundamental didactical aim is to be able to investigate, to argue and to explain.

The research questions we asked ourselves at the beginning of the teaching experiment can be seen under different points of view.

Students' point of view :

– What is the value added by this activity to the competence of our students'
– Do the students improve their ability of explaining, conjecturing, exploring, . . .'

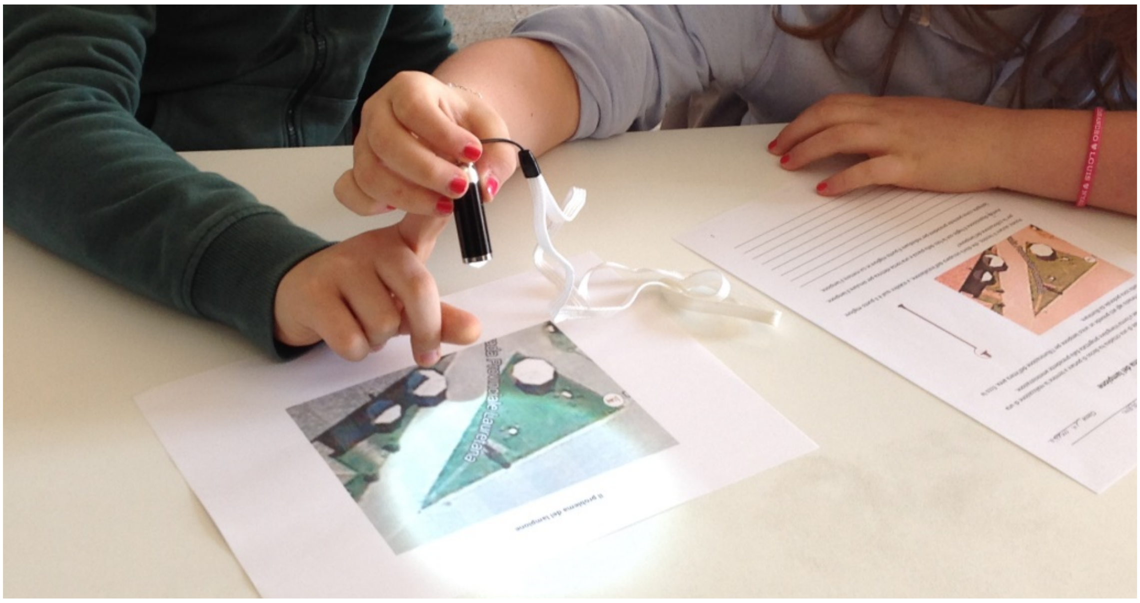


FIGURE 5.8 – Exploration with poor materials

Teachers' point of view :
- What changed in the professionalism of the teacher after this activity ?
- Had the brokering been performed ? Has the activity helped the creation of shared praxeologies'

Description of the activity

The activity was organized in 4 phases : 3 of them were developed by group working and the last one was a collective discussion for the institutionalization of the competences.

The first phase is the analysis of the situation using poor materials. The second phase is the exploration of this problem with GeoGebra while the third one is the exploration of the generic situation with GeoGebra, using the dragging. The last phase is the collective discussion in order to construct together the meanings of the objects involved in the activity.

In the first step, students divided into small groups (4-5 people) began exploring with poor materials : paper and pencil, a torch and the image of the pedestrian area.

Students made their conjectures using the torch to simulate the lamp and draw some of the fundamental elements of the triangle (like perpendicular bisectors, angle bisectors, ...) to find the best point to put the lamp in.

The second step was the static use of GeoGebra. We gave the students a file with the picture of the pedestrian area and asked them to work on it. They reproduce on GeoGebra the same construction made with paper and pencil, but the software helped them to notice that, in some cases, the solution found was not the best one.

The students who chose the barycenter, found some difficulties in fixing the radius. They finally select the farthest point to fix the radius. The students who chose the circumcenter noticed that the circumference passed through all the three vertices. The students who chose the incenter noticed that the circumference was too big.

They used GeoGebra to explore the problem and to find a better solution and sometimes they changed their conjecture.

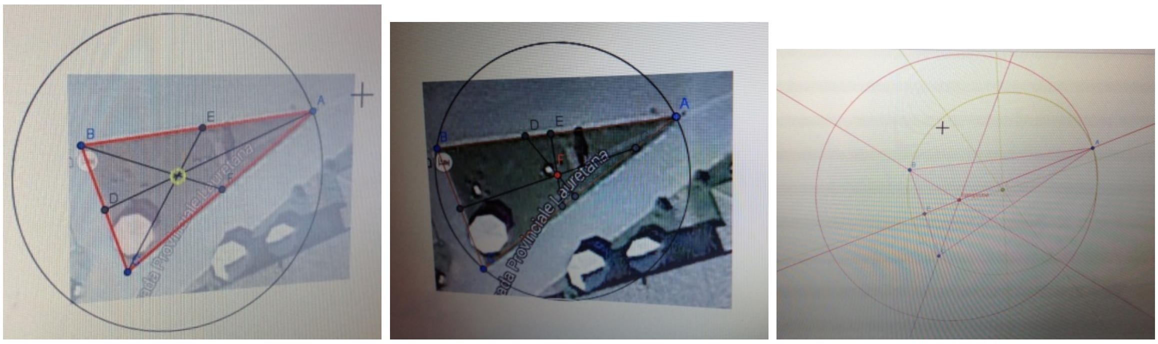


FIGURE 5.9 – Barycenter, circumcenter, incenter versus circumcenter

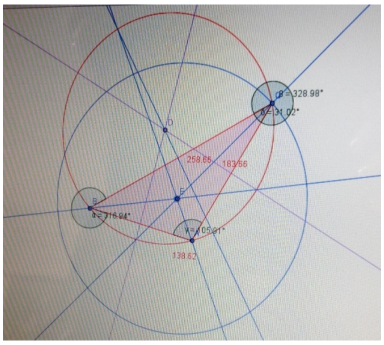


FIGURE 5.10 – Dynamic research of the best point in the case of obtuse angled triangle

The third step involved the dynamic use of GeoGebra, investigating what happens if the shape of the pedestrian area changes. They noticed that the obtuse angled triangle was particular situation : the circumcentre is out of the triangle. Comparing the geometrical solution found with the real situation, students noticed that there were some problems in putting the street lamp outside of the pedestrian area. They discussed about the constraints and the different solutions and in some cases they needed to change their mind.

The last step was the collective discussion. We guided the students to explain their solution in order to convince the other students about their ideas, so they worked on justifying and arguing.

Materials and methodology

The methodology we used is based on the idea of "Mathematical laboratory" (UMI, 2001, 2003) not as a real place, external to the class, but as an approach to mathematics. During and after the laboratory the "Mathematical discussion" (Bartolini Bussi, 1996) is the key point, in fact through

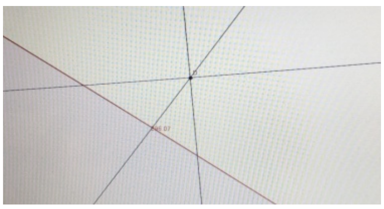


FIGURE 5.11 – Obtuse angled triangle : the circumcenter is outside (zoom tool)

the discussion it is possible to construct meanings and common ideas as a community of practice (Wenger, 1998).

The technology represents another key element of this teaching experiment. We decided to involve technology using a dynamic geometry software (DGS) to explore the problem : GeoGebra. GeoGebra has the power, as others DGS, of being dynamic, so the students can manipulate dynamically the shapes they constructed by dragging them, they can also modify the shape (enlarge, restrict, ...) keeping unchanged the construction protocol (Robutti, 2013).

The observation's methodology

During the activity, we used a logbook to record, day by day, the things done, the materials used and to write observations about our behaviour as well as student's one. Since we gave them a form to work with and to fill in with their considerations, we also took a copy to reflect on our experimentation in teaching methodology. Elisa observed Monica's class during the activity, while Elisa's lessons were videotaped in order to reflect on the didactical practice.

Critical analysis of the activity

There are two different ways of proposing open problems into a class, depending on the moment in which the problem is posed. If the problem comes before the knowledge, then we can speak about discovering and arguing. If the problem comes after the knowledge, we can speak about consolidation and exercises.

In our case, in Monica's classroom, pupils had already studied triangles and triangle centres whereas in Elisa's classroom only triangles and the concepts of perpendicular bisector of a segment, angle bisector, median and altitude had been introduced. Then, in the first situation the aim of the problem-solving was knowledge consolidation, while in the second one was exploration and discovery of geometrical properties.

We are going to critically analyse the activity, starting from two different points of view : the students' one and the teachers' one, answering to the research questions we have asked before.

Since Monica's class was not used to problem solving and laboratory activities, we will focus on the effects on her class and her praxeologies.

The class showed curiosity in front of this kind of activity, really different from the ones they were used to. At the beginning, pupils were wrong-footed because a "right answer", in this kind of work, does not exist. Overcome this difficulty, they argued about the problem, making conjectures and justifying their choices. They seemed deeply involved in thinking and discussing about a real problem and made interesting reflections. Furthermore, students with learning disabilities, that are often bored and distracted during traditional lessons, were pleased to work in groups and to deal with a real situation, showing good intuitive ability.

The discussion led Monica's class to choose the circumcentre for the specific pedestrian area given. When they analysed the generic situation dragging the triangle, they noticed that in the case of an obtuse angled triangle, the circumcentre is outside of the area. So they discussed about the meaning of putting the street lamp outside the pedestrian area, and they chose the barycentre because is always inside and it is possible to choose the height of the lamp according to the circumference needed (passing through the farthest vertex). Elisa's student instead chose the incentre for the case of the obtuse angled triangle.

The students noticed that the solution of an open-ended problem is not unique because it depends on the choices and the constraints made by themselves, for instance the height of the lamp, the possibility of putting the lamp outside the area, the shape of the triangle, the meaning of "best solution" (waste less energy, have a good illumination for the area, put the lamp into the area, ...).

In this kind of activity students feel more involved, because there is not only one expected solution, like in the problems they are used to solve.

The integration between poor material and GeoGebra helped students to construct knowledge, and the dynamic use of GeoGebra gave students space to explore, conjecture and argue. One of the added value of this kind of activity is the mediation of instruments and technology.

At the beginning Monica was sceptical and worried about proposing this activity to her students due to its openness and, furthermore, because the students were very young (12 years old). But she accepted the challenge.

Working in a community of practice, like PLS, has given us the opportunity to share ideas and doubts with other teachers and teacher-researchers, and to transform some of the external components into internal ones. For example : at the beginning the National Curriculum, the use of GeoGebra in middle school classes and the laboratorial methodology were external components for Monica. Thanks to the action of brokering performed by the teacher-researchers, these external components had become internal and the different praxeologies experienced had then become shared ones.

Now, looking at the experience, we feel more confident on proposing laboratorial activities with the use of technology, like Geogebra, to young students, and to use laboratorial methodology in our classes instead of traditional lessons. We noticed that students were able to use their knowledge in a real situation, different from the one in which they have learnt it, improving their competence. And also that students were able to autonomously discover the properties of triangle centres, making choices, arguing and justifying their choices. Finally, they have been able to manage a collective discussion, sharing their ideas and constructing together the meanings.

As teachers we noticed that open-ended problems give the possibility of discussing about various aspects, even different from those designed.

Furthermore, this problem allowed us to introduce some topics that are part of the curriculum of the next years, such as the area of the circle and the area of polygons. All the other topics concerning geometry in the space, physics, ...could be faced in higher secondary schools.

Conclusion

During the activity, students worked in two really different environments : the paper and pencil environment and the technological environment but they used the same mathematics to deal with this problem. The difference is in the methodology used and the opportunities given to students.

Using technology they were able to explore a lot of different situations and understand the problem under different points of view. They could really represent a generic situation ; for instance : when we draw a triangle on paper it is not A generic triangle, but it is THE particular triangle drawn. When we draw instead a triangle with GeoGebra, it is really a generic one. Using the dragging it can change (but maintaining its own properties) and students can appreciate the dynamicity of technologies.

Both the paper and pencil and the technology are important aspects for problem solving but the real potential stands in their integration. Using only paper and pencil or only technology students do not achieve the same results as they use them together. Then the key point is the mediation and the integration of the two environments.

Furthermore, the experience done has been really useful for teachers and students alike. Monica has experienced a new approach and new praxeologies, improving her professionalism as a teacher, while her pupils have been involved with a leading role in the activity : they have taken decisions, discussed, argued and mobilized their competences. Elisa had the opportunity of observing again her didactical practice and to reflect on it.

The interaction between the observer and the teacher underlines important elements the teacher could not notice anyway. Taking part to an international project represents a great opportunity for sharing ideas, methodologies, doubts and for the construction of shared praxeologies, that will be, from now, part of the praxeologies of the teachers involved in the training.

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5.5 Algebraic interactions emerging from a ICT school experience

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Résumé : Cet article concerne la question de recherche suivante : quel type d'interactions apparaît quand un groupe d'éléves de 13-14 ans tentent de résoudre des problèmes algébriques en utilisant les forums électroniques ? En particulier, quel est le rôle des médias pour faciliter le discours mathématique et à aider à travers un enseignement direct ? Ce travail doit être considéré comme un exemple de la compréhension scientifique de base des interactions à travers les processus de co-construction de sens de décision dans des environnements en ligne ou dans des environnements mixtes (Anderson, 2001). L'état des agents de transformation attribués aux TIC vautt la peine d'être pris en compte pour concevoir des interventions spécifiques et pour modifier les modèles pédagogiques, les pratiques en salle de classe, et les contextes scolaires dans les systèmes éducatifs afin de conduire les étudiants vers un importante et satisfaisant apprentissage (Rojano, 2003 : 138).

Abstract : The proposal relates a research proposal (type B) to the subtheme 3. We want to identify the role of forum interactions in a new algebraic experience using ICT By analyzing such a task, we see the role and the implications of forum conversations as a powerful mediator for reflective interactions and global participation. This paper relates to a main following research question : which kind of interactions appear when a group of 13-14 years old Students solve algebraic problems using electronic forums' In particular, which is the role of the media facilitating mathematical discourse and scaffolding by providing direct instruction ? It should be considered as an example for basic scientific understanding of interactions as co-construction processes of sense making in online environments or mixed environments (Anderson, 2001). The condition of transformation agents assigned to the ICT is worth to be taken into account for conceiving deliberate interventions to change the pedagogical models, the practices in the classroom, and the curricular contents in educative systems in order to lead the students towards a significant and satisfactory learning (Rojano, 2003 : 138).

Research framework

Computer-mediated Communication can turn out to be a useful instrument to solve, reflect and discuss problems jointly. The challenge is for teachers to create learning environments that help students to make the transition from basic to high-order skills. In this new scenario, the focus is not on teachers but the process of how the students learn. In such a framework, the use of ICT in first Secondary Schools will entail the gradual disappearance of the limitations of space and time, which will result in a transition toward a usual student-centered model based on cooperative work. On-line forums and blogs have also been recognized as fertile ground for meaning product discussion. Previous researches analyzed forum discourses and also collaborative problem solving activities with future teachers (Bairral & Powell, 2013). But a few researches focus on what kind of behavior and strategies appear when young students solve algebraic problems in an asynchronous way. We know that their online experiences do not generally require them to use dialogue as a way to explore, expand, and drill down into problem solving issues significantly (Jonanssen, 2002).

Teachers have to guide the pedagogical setting towards situations in which relevant aspects are discussed, such as posing questions related to the critical analysis of contexts or the necessity for the generation of new and useful information to promote attention. Participation in our analysis of reflective interactions should be considered as something that improve or restrict mathematics development. We assume some previous research results about the use of electronic forums with geometry problems (Murillo & Marcos, 2011) that the use of the forums of conversation in a digital environment, used to jointly investigate algebraic problem-solving strategies, create favorable conditions, (a) so that the process of problem solving promote reflection and communication of ideas among the students ; (b) for influencing changes affecting the teaching role and the relations that are set out in the classroom ; (c) individual growing and collective development of objects and processes in the topic. Several authors tell us that such e-activities based on group work must be properly structured to avoid the free rider effect, but we decide to use the forums in a completely free way (Johanssen, 2002).

Methodology

The examples presented in this paper, belongs to a part of a wider research in which several problem solving tasks were conducted, analyzed and redesigned through the application of a Design-Based Research methodology (Royo, 2012). For the design of the learning environment (Murillo & Marcos, 2011), we used the Moodle-platform provided by the School. This paper explores two problems and their conversation forums on the virtual environment Moodle, which was new, both for teacher and students at that moment. Oral conversation occurred in the classroom at the same time that they did contributions to the forums. In addition to using computers, students had paper and other material written or manipulative aids as instruments of work to look for strategies for resolution of the problems. The collected data for our study is constituted by registered dialogues on the forum, and also audiovisual records of some moments of the session ; direct records of the diaries of the students in the Moodle platform ; record direct from the journal of the teacher in the Moodle platform.

We have analyzed the dialogues by ethnographic methods. We think that it should lead the mathematical reasoning through observation of individual cases, guess, check and argumentation, since thus prepares the task of orientation of the process of generalization, one of the main ways of introduction of the algebra. To follow our aims, student's interactions had been analyzed by using educational profile using e-accessibility and e-connectivity. By using content analysis we identify algebraic contributions in a specific task, and quantifying the use of problem solving strategies in terms of applying and answering, grounding, or interpreting other's contributions. Some easy classical situations were used. It's the case of number of possible segments using n dots, or magic ball problem in which you have related signs to numbers 1 to 99, and ask for subtracting a chosen number 43 to the inverse 34, and find the sign. The teacher didn't see the result, but magic ball predict the sign.

Results and discussion

The activities lead to the development of generalization and symbolism processes. The value of technological tools such as virtual learning environments is not to replace the role of the teacher, but enhance the distributed teaching presence, creating a context that promotes the understanding and development of growing significant algebraic knowledge. Content categories and interactional analysis were helpful to analyze if the interactions are focused on certain aspects of problem solving activity. Such tools provide the possibility to understand how the interactions relate the educational profile with algebraic content issues and strategies. Some students proposed the use of known manipulative as Student 9 : "We can take the geoboard and go testing with dots and segments". Some other students sought convincing verbal explanations to the formula.

After categorizing and observing the results, we found that in almost all the tasks, electronic forums enable individual and collective construction of objects and processes in the learning of algebra (Royo, 2012), and improve generalization attitudes, similar to face-to-face conversations. Interlocutive interactions (based on habitus, in which no new math meanings are conceived) influence the development of mathematical ideas and reasoning in diverse ways. We can see, during the dialogues, how the use of multiple representations emerging as the use of polycubes to encourage symbolic representations by oral discussion about them. Let's say Student 13 comment : ”5 dots and $n = 4$: $1 + 2 + 3 + 4 = (1 + 4) + (2 + 3) = 5 + 5 = 10$, and that's equal to : $(1 + 4) \times 4/2 = 5 \times 2 = 10$ $\frac{(n+1)n}{2}$“ after some discussion about manipulative aids. In the magic ball problem forum, Student 6 & 9 says : ”It always appears a multiple of 9, and all these numbers are related to the same symbol. Thus, it appears the sign you have prepared before“. Some other students giving particular examples, or indicate a simple agreement, not enough to be considered as a mathematical proof. In such cases, the teacher feels the need to appointing “do you feel being sure that always the result will be a multiple of nine ? Please, tell us when you have a convinced argument and write it in a wiki-space“

It was observed that only after eleven particularization contributions, Student 13 changes the representation, and says that ”a number xy is written as $10x + y$ and the contrary is yx, then the subtraction is $9x$“. Even being a wrong symbolic statement, gives opportunities for future comments in a generalized way, using letters. Many other examples are given, and much time is needed to find an extended symbolic written explanation ”it is always nine times the difference among the two digits“ as a verbal way of generalizing. Ten messages after a new symbolic expression was used to say $9(x-y)$ as an expression of a convincing generalization. With such interventions, we see the difficulties relating symbolic use of patterns observed by the students, and described in their comments. In figure 5.12 we can see the interaction system during the two exemplary problems (Royo, 2012) in which some students appear as a node of meaningful interactions (bigger circles) and we can see that different nodes appearing in such different problems. Colors are used to indicate the type of provocation (math, algebraic, simple agreement...).

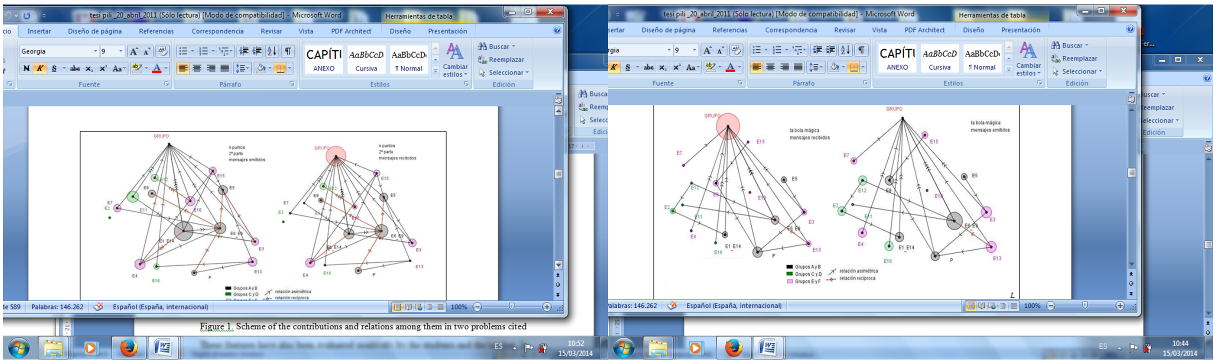

FIGURE 5.12 – Relations among contributions in a n-dot and magic ball problems.

Qualitative content analysis about the sentences not fully described in this paper, shows that generalization methods and symbolic language are close to which it's regularly presented in face-to-face classrooms, but they appear more quickly. It also spontaneously emerges exchanging the representations not always usual in face-to-face classrooms. Interactional data corroborate the findings of authors who point out that these instruments allow asynchronous math reflections which can facilitate collective learning (Murillo, & Marcos, 2011). We also found that higher participative profile students, have more math interventions (grey and green nodes in figure 5.12, explained in detail in Royo, 2012). Nevertheless, the peer-interactions introduced a new algebraic experience in which personal slow inductive processes should be legitimated instead of introducing quick top-down proof analysis, syntax centered.

Conclusion and perspectives

According to global data for all the problems during the broader study (Royo, 2012), the number of contributions with a focus on the algebraic content (grey and blue) is significantly less than the contributions with a focus on the relationship with other participants. However, the ratio increases significantly time after time and depending on the problem that it was discussed in the Forum. We found enough evidences to tell that even the interlocutions observed support the reflective development of the participants' mathematical ideas and reasoning (Bairral & Powell 2013) promoting multiple unexpected representations as manipulative (not related to web environment, and usually presented in the task statements). In such environment, Moodle act only as an artifact with manipulative aids and language (Hitt, 2013), to identify that structural indicators (as number of interventions or nodes) are not enough to see algebraic evolution during the activity. In fact, in our experience we also see that tables are needed to enlarge a generation of the rule, manipulative aids are fundamental to see the rule as being general, and the forum gives the opportunity to re-write the sentences assuming more precise language for generalizing and proving.

Electronic forums act as agents of change that affected the teaching role and in the relationships and interactions established in the classroom. Asynchronous debate gives opportunities to emerge and solve personal difficulties about generalization without teacher intervention. Thus a need for analyzing the new role of the teacher as an mathematics web animator who manage not only artifacts but positive interactions.

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5.6 Proving processes in a Dynamic Geometry Environment : A case study

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Abstract : Knowing that new technologies act as conceptual reorganizers of knowledge, investigating their potential with respect to proof processes could provide insights and useful information toward understanding whether and how these tools can support proving activities. This paper presents the results of an analysis of students' cognitive processes when working in a Dynamic Geometry Environment (DGE) and the obstacles that they face while they elaborate geometric proofs. We observed six pairs of ten-graders (15-17 years old) in a private school in Lebanon, while working on geometrical open proof problems using dynamic geometry software, namely GeoGebra. The aim of the study was to analyze the interactions between students and the DGE with special attention to the different dragging modalities employed, the types of conjectures formulated and justifications produced, and the ways the DGE effects students' conjecturing, argumenting and proving processes. Within the limited scope of this paper, we will describe and analyze the work of one student pair.

Introduction

One common way that teachers use to introduce the notion of proof is to raise students' doubts about mathematical properties and then present proof mainly as means for verification and conviction. De Villiers (2004) argues against this approach that places conviction as a result of proof saying that conviction often precedes proof and should be utilized to motivate its finding. Therefore, proof is not only a question of *making sure*, but also a question of *explaining why*.

The learning of proof and proving in school mathematics should start at an early age and grow in importance and rigor through grade levels. Teachers play a major role in successfully implementing this process. Their views about the importance and role of proofs, their teaching methods used with students in classrooms, their interpretation and implementation of curricular tasks that have the potential to offer students opportunities to engage in proving, their diagnosis of students' difficulties in proving and their design of instructional interventions to help overcome these difficulties, will determine if and how proof will be integrated in school mathematics (Hanna & de Villiers, 2008).

According to the framework developed by Marrades and Gutierrez (2000), there are two main types of justifications, empirical and deductive, and 16 sub-types. In the Lebanese curriculum students are mainly taught and required, to elaborate deductive justifications by structural thought experiment. Justifications are deductive when students validate their conjectures in a general way by the decontextualization of the arguments employed and the use of logical deductions. Thought experiment, which is a type of deductive justifications, is when a specific example is used to help organize the justification. There are two types of thought experiment : transformative and structural. Structural justifications are sequences of logical deductions derived from the given of the problem and definitions, axioms, and theorems.

Since a variety of new technologies (CAS, DGE) are made widely accessible to the school communities, it is therefore necessary to consider their potentialities as related to the teaching and learning of proof.

Dynamic Geometry Environment (DGE) consists of a virtual environment where accurate construction of geometric figures can be carried out. The key characteristic of a DGE is that free elements of the construction can be moved and, as they are moved, all other elements self-adjust automatically while preserving all dependent relationships and construction constraints. DGE provides many tools of chief importance with respect to the work in geometry. They allow the students to execute complex constructions and test them. A variety of basic construction tools are available in DGE like *perpendicular lines, bisectors, polygons, circles, conics, reflections* and the like. The Dragging tool allows the student to manipulate objects on the screen and get real-time visual feedback. The *Trace tool* allows the student to see the trace of a point or object when dragged. The Measuring tool measures distances, length, angles and perimeters. The Macro-construction tool permits the definition of new tools according to the needs of the student. The following sections will describe each of the previously mentioned tools.

The dragging tool is of chief importance in DGE as it helps students perceive new functions of proof and overcome the difficulties faced when dealing with proof in a paper-and-pencil environment. The main function of dragging relies in the feedback it gives to students' activity and in the validation power it provides. If the constructed figure keeps its desired properties under dragging then it was properly constructed. Therefore students can, in such environments, verify their constructions through dragging instead of a theoretical verification (Mariotti, 2000). This powerful feature helps the students gain greater conviction of the truth value of theorems, and identify counterexamples and incorrect construction through the feedback it provides. Since in the Lebanese curriculum students are mainly taught to elaborate deductive justifications by structural thought experiment, we expect them to do so in DGE as well.

Olivero (2002) classified the dragging modalities according to the different aims that students want to achieve . The modalities are :

- Non dragging : Not moving anything on the screen.
- Dragging test : Moving dragable or semi-dragable points in order to see whether the figure keeps the intended properties.
- Wandering dragging : Moving the basic points on the screen randomly in order to discover configurations or regularities in the figures.
- Bound dragging : Moving a semi-dragable point, which is constrained to move on a constructed object (e.g. a point on a circle).
- Guided dragging : Dragging the basic points of a figure in order to give it a particular shape.
- Lieu muet dragging : Moving a basic point so that the figure keeps a discovered property ; that means you are following a hidden path (lieu muet), even without being aware of this.
- Line dragging : Drawing new points on the ones that keep the regularity of the figure.
- Linked dragging : Linking a point to an object and moving it onto that object.

Different types of dragging modalities are used to complete different tasks in DGS, such as : wandering dragging for exploring constructions by moving points on the screen ; lieu muet dragging for discovering the invariant properties of the figure ; line dragging for generating conjectures and dragging test for supporting the production of conjectures.

Restrepo (2008) developed a parallel classification that categorizes dragging modalities by the mathematical purpose behind its use. The framework is translated for this article.

- Aimless dragging : dragging without a prior mathematical purpose.
- Dragging to adjust : it is used by the student who doesn't possess sufficient knowledge to construct a robust figure and who assumes that such strategy is enough to obtain a correct figure.
- Soft dragging or guided dragging : dragging the base points of the construction to give it a momentarily shape or certain properties.
- Exploratory dragging : it consists of three subtypes namely, dragging to identify the invariants of the figure, dragging to observe the variations during movement, and dragging to identify the trajectory of a point.

– Dragging to validate or invalidate : it is divided to three sub-types namely, dragging to validate a conjecture or property, dragging to validate a construction, and dragging to invalidate a construction.

Methodology

The difficulties that students face when approaching proof are well documented in the literature, but since proof is an important aspect of mathematics and occupies a central place in the curriculum, students should be given the opportunity to truly and fully engage with this mathematical experience. Therefore it is crucially important to investigate ways of supporting students in their approach to proving. Knowing that new technologies act as conceptual organizers and change how knowledge is constructed, investigating their potential with respect to proof may provide insights and useful information toward explaining whether and how these tools can support proving activities.

The research consisted of a series of observations which took place in secondary school classrooms (15-17 year old students). The context is that of open geometrical proof problems within a dynamic geometry environment, namely Geogebra. The main purpose of the study was to investigate the students' mental processes involved in the construction of conjectures and proofs in geometry, when students are working on open proof problems and interacting with a dynamic geometry environment. Special attention was given to the interplay between the spatio-graphical field (including Geogebra objects, paper drawings, etc) and the theoretical field (including geometrical properties, theorems and definitions) (Laborde, 1998). In particular, the study focuses on two kinds of transformations that the proving process involves : transformations on objects and transformations on statements. The transformations on objects involve manipulations of objects, whether on paper or in the Geogebra environment, for example via dragging. The transformations on statements refer to the shifts from facts and experiences in a precise space and time to de-contextualized and de-timed logical statements of the form 'if . . . then'.

Given this background, the particular aims of the study are :

1. To investigate the process of conjecture generation and proof writing in solving open proof problems within a dynamic geometry environment.
2. To investigate students' interactions with the dragging feature of the dynamic geometry environment and the role of those interactions in the development of conjectures.

The present study was conducted in a private school in Lebanon. It falls under a more general school-wide project aiming at the integration of technology in mathematics classrooms. Given that students feel the necessity of exchanging ideas when working on a problem, and in order to externalize their thought processes, students were working in pairs to solve open proof geometry problems using GeoGebra.

The chosen problem was adopted from Olivero (2002). The aims investigated in the present study align with those of Olivero's study, which gives us a higher level of confidence that the problem will allow us to observe the desired processes. Also, the chosen problem lies in a conceptual domain familiar to the grade-10 Lebanese students. The nature of the problems is not changed by the DGE which only acts as a visual amplifier facilitating the mathematical task. This type of problem acts as a window on students' ideas and understandings (Laborde, Kynigos, Hollebrands, & Strasser, 2006). The problem was especially chosen to uncover the cognitive processes mobilized in situations involving dragging, conjecture generation and proof writing within a DGE.

Data was collected using video-recording and collection of materials, namely any paper trace generated by students, together with their GeoGebra files. The analysis of the case studies was based

Vicky	[reformulates the statement of the problem, saying it to Eric] Manipulate $ABCD$ and observe how $HKLM$ varies as a function of $ABCD$.
Eric	[he drags $ABCD$ to make it a rectangle (Figure 5.13)] If it [$ABCD$] was a rectangle, then it [$HKLM$] is a square.
Vicky	And if it [$ABCD$] was a square ...
Eric	If it [$ABCD$] was a square [he drags $ABCD$ to make it a square (Figure 5.13)] if it was a square [$ABCD$] they [points H, K, L and M] coincide. And if it [$ABCD$] was a parallelogram [he drags $ABCD$ to make it a parallelogram (Figure 5.14)] if it [$ABCD$] was a parallelogram then they [points H, K, L and M] are a rectangle.
Vicky	But we need to say why.

Table 5.1 – Excerpt from the transcript of Eric and Vicky

on the categorization of Dragging Modalities made by Olivero (2002) and the framework for Types of Justifications developed by Marrades and Gutierrez (2000).

Results

In this paper, we will focus on the work of one pair of 10-graders, namely Eric and Vicky. They are high achievers in mathematics as reported by their professor. They had already used GeoGebra few times before this session, to work on construction and exploration problems. The problem given to the students is the "Angle bisectors of a quadrilateral" problem (Olivero, 2002) :

1. Let $ABCD$ be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, K, L, M of pairs of consecutive bisectors.
2. Drag $ABCD$, considering different configurations and explore how $HKLM$ changes in relation to $ABCD$.
3. Write down your conjectures and prove them.

The work of Eric and Vicky was divided into two phases : a conjecturing phase and a proving phase.

The Conjecturing Phase

During the conjecturing phase, Eric used Guided Dragging, which is dragging the basic points of a figure in order to give it a particular shape (Olivero, 2002). The students transformed $ABCD$ into different geometrical shapes and both Eric and Vicky observed how $HKLM$ varied as a result of the changes effected to $ABCD$. The following is an excerpt from the video transcript, supported by students' Geogebra files. The excerpt presents a look onto the students' elaboration of conjectures (we explain in brackets what the students are doing or which object they are referring to) :

Thus in this phase Eric and Vicky developed three conjectures for the cases of ABCD rectangle, square and parallelogram. When Vicky pointed out that they need to prove these conjectures, the conjecturing phase was interrupted and they started working on the proofs

The Proving Phase

In the following section, we present the different cases investigated by the observed pair and their proving attempts. We also connect the proving attempts to the framework developed by marrades and Gutierrez (2000)

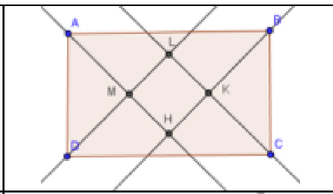
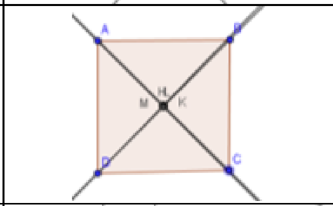
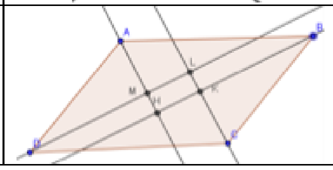
Figure 1		Conjecture 1: If ABCD is a rectangle then HKLM is a square.
Figure 2		Conjecture 2: If ABCD is a square then H, K, L, and M coincide.
Figure 3		Conjecture 3: If ABCD is a parallelogram then HKLM is a rectangle.

FIGURE 5.13 – Conjectures developed in the conjecturing phase

Case 1 : $ABCD$ is a rectangle

Eric and Vicky started proving their first conjecture : "If $ABCD$ is a rectangle then $HKLM$ is a square (Figure 5.13)." Using angle equalities and relationships (45°-90°-45°), they were able to show that $HKLM$ is a rectangle, by showing that it has 90° angles. Then to further show that $HKLM$ is a square, they tried to use congruent triangles but failed to do so.

According to Marrades and Gutierrez (2000) proving that $HKLM$ is a rectangle using angle equalities and relationships is a deductive justification by structural thought experiment. And failing to show that $HKLM$ is a square using congruent triangles is a failed deductive justification.

Case 2 : $ABCD$ is a square

Eric and Vicky went on to proving their second conjecture : "If $ABCD$ is a square then H, K, L and M coincide" (Figure 5.13). They constructed a deductive proof by structural thought experiment (Marrades & Gutierrez, 2000) saying that the angle bisectors in a square are also its diagonals, which intersect. so the points H, K, L and M coincide at a single point.

It is important to note, in this case, that the premise of the students' conjecture could be made less specific, as the points H, K, L and M would coincide if $ABCD$ was a rhombus, which is a more general case than the case of the square. The conjecture could then be refined to become : "If $ABCD$ is a rhombus then H, K, L and M coincide".

Case 3 : $ABCD$ is a parallelogram

When Vicky and Eric wanted to explore and prove the case of $ABCD$ being a parallelogram, Eric dragged A, B, C and D to form a parallelogram. The shape so happened to be a rhombus, in which case $HKLM$ coincided (Figure 5.14). However, Eric and Vicky thought that $ABCD$ was simply a parallelogram and elaborated a new conjecture, different from Conjecture 3 that has the same premise, thus emerged Conjecture 4 : "If $ABCD$ is a parallelogram then H, K, L, and M coincide". However, Eric and Vicky didn't notice that this new conjecture contradicts the Conjecture 3, previously elaborated in the conjecturing phase : "If $ABCD$ is a parallelogram then $HKLM$ is

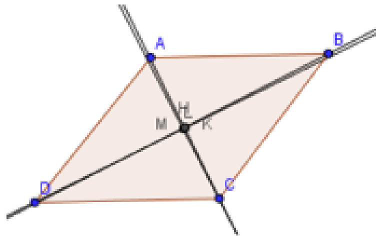


FIGURE 5.14 – $ABCD$ is a parallelogram

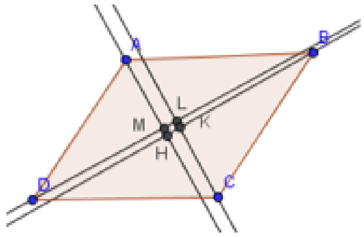


TABLE 5.2 – $ABCD$ is a rhombus

a rectangle". Since the conjecture was incorrect, it led to a failed justification, as Eric and Vicky justified their conjecture by saying that in a parallelogram the bisectors which are also the diagonals intersect at one point.

Case 4 : $ABCD$ is a rhombus

Finally Eric and Vicky dragged points A, B, C and D with the intention to form a rhombus. But the figure was not precisely a rhombus (Figure 5.2) since the sides were not exactly congruent. Thus when they saw that points H,K, L, and M formed what they thought to be a small square, instead of intersecting at one point, they generated another wrong conjecture, Conjecture 5 : "If $ABCD$ is a rhombus then $HKLM$ is a square". They argued that in a rhombus the diagonals, which are also the bisectors, are perpendicular, so the angles are right, making it a rectangle, which is subsequently a failed proof.

Our categorization of failed justifications

According to the framework of Marrades and Gutierrez (2000) there were two types of justifications elaborated by the students : deductive justifications by structural thought experiment and failed deductive justifications.

However, we noticed that the failed justifications cannot be combined into one category since they are of different nature and do not all originate from the same step in the proof process. Thus, we distinguish three types of failed justifications :

1. Justifications that failed at the theoretical level : In this type, students succeed in constructing a correct figure and in developing an adequate conjecture but fail in finding a theoretical support for the conjecture. For example, in the first case ($ABCD$ is a rectangle), based on the figure properly constructed, students knew they had to show that HKLM is a square but failed in appropriately using congruent triangles to do so.

2. Justifications that failed at the conjecturing level : This type of failed justifications represents the cases where the figure is correct but the subsequent conjecture simply does not correspond to the figure at hand. Obviously, a failed conjecture leads to a failed justification. For example, in the third case ($ABCD$ is a parallelogram), the figure was a rhombus properly constructed but the conjecture was treating the case of the parallelogram resulting in a mismatch.
3. Justifications that failed at the graphical level : This last type represents the cases where the figure is inaccurate leading to a failed conjecture and a failed justification. For example, in the fourth case ($ABCD$ is a rhombus), the constructed rhombus was inaccurate thus creating the illusion of the square (actually rectangle) $HKLM$. Therefore the conjecture and justification were incorrect.

Clearly each type of failed justification relates to a different kind of difficulty : Justifications that failed at the theoretical level reveal a problem in the comprehension and application of mathematical properties and theorems and thus belong to the theoretical field. Justifications that failed at the graphical level show that students still struggle with the use of the different instruments available in the DGE and thus belong to the spatio-graphical field. Justifications that failed at the conjecturing level reveal a difficulty in the switch between the spatio-graphical field and theoretical field as students fail in translating the figure at hand into a mathematical conjecture. The last two types could have been avoided if students had tested their conjectures in more than one case before trying to prove them. Although they are aware of the existence of tools that they may use to verify some properties of the figures, students fail to use those tools and rely on perception to determine the type of shape they end up getting during their dynamic exploration.

Discussion

In general terms, we can see that the new experiences encountered by students in the spatio-graphical field overpowered the longstanding theorems and properties established in the theoretical field since grade 7. For example, in case 3 (ABCD is a parallelogram) Eric and Vicky assumed the figure to be a parallelogram and they observed that the angle bisectors intersect at one point. The perceptual element overpowered their geometrical prior knowledge so that, instead of questioning the correctness of the figure, they bent a well-known property in order to correspond to what they saw on screen. In brief, they tend to over trust the observed shapes and properties that they perceptively recognize at first sight, and the first look at the figure seems to exclude the mathematical look at this figure.

In a paper-and-pencil environment, the students participating in this study are trained to reflect abstractly and concretely (on the paper) simultaneously as they use colors to mark specific objects and relations on the figure and they jot down their ideas to single out what is interesting and needed. However, when working in DGE, Eric and Vicky gave up the paper and pencil completely and relied purely on visual perception and abstract thinking which was difficult for them to control and organize. There was a deep schism between DGE and the paper-and-pencil environment, the use of one excluded the other. Eric and Vicky perceived DGE to be a mere construction tool, much like the ruler and compass, as evidenced by the limited use of construction tools. They were not open to the possibility of exploiting the potentialities of this new environment in the proving process. Despite the fact that they are high achievers in mathematics they failed in transferring the skills acquired in the paper-and-pencil environment to the DGE in order to structure their thoughts and organize their work method to successfully solve the problem.

Following the classification of dragging modalities developed by Olivero (2002) we note the use of only one dragging modality which is Guided Dragging, which results in soft constructions (Healy, 2000). This strategy is implied by the nature of the problem, especially in the conjecturing phase : having to explore different types of quadrilaterals, the obvious strategy is dragging ABCD to obtain different soft quadrilaterals. However, after developing the conjectures, constructing robust figures to verify these conjectures could have saved the students many of the mistakes seen above. Eric and Vicky viewed dragging as a graphical tool used to modify the shape of drawings. They focused on the movement properties of the drawing but did not use dragging, neither to test conjectures nor to explore the relational properties of the figures. The dragging tool facilitated the conjecturing phase but was completely disregarded in the proving phase, as students reflected on a static figure similar to the paper-and-pencil environment. The only type of constructions used is that of soft constructions.

Using the classification of Restrepo (2008) we can see that students relied mainly on Dragging to Adjust which is the strategy used when students do not possess the required knowledge to construct robust figures and assume that soft dragging is sufficient to obtain the correct figure. They were not able to develop the instrumented action schemes needed to skillfully use Dragging to Validate a Conjecture or Property.

In conclusion, Eric and Vicky were not able to detach themselves from the drawing in order to access the figure, as distinguished by Laborde and Capponi (1994). They did not distinguish the properties of the drawing which correspond to the figure from the ones that are only perceived spatial properties and cannot be used in the conjecture and proof. They used the dynamic figure to identify the property that has to be proven rather than as a tool to explore the validity of the property.

These students, who are used to being told what to prove by the teacher and/or the problem statement, used the DGE just to replace this missing part, when faced with a problem that left hidden the conclusion to be proved. As soon as they adjusted and perceived a certain figure they considered it to be a source of authority, like the teacher or the problem statement. Since whatever the teacher says is considered to be true and students do not question what the teacher says, they transferred this thinking to DGE by not questioning any figure or checking its truth value.

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5.7 Primary graphs

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Résumé : Cet article décrit une activité réalisée dans les écoles primaires pour motiver les jeunes étudiants de profiter de mathématiques et la voir d'une manière créative. Ceci est possible par l'introduction de la théorie des graphes, en raison de la possibilité de dessiner et de jouer avec plusieurs graphes. La proposition est fondée sur un véritable laboratoire de mathématiques connu en troisième et quatrième années de l'enseignement primaire.

Abstract : This paper describes an activity carried out in primary schools to motivate young students to enjoy mathematics and see it in a creative way. This is possible by introducing graph theory, because of the possibility to draw and to play with several graphs. The proposal is based on a mathematics laboratory experienced in third and fourth classes of primary schools.

Introduction

Everyone should enjoy learning, because if you do something for fun, rather than for duty, then you can get the maximum output with the minimum effort. Even more this is true for children.

More and more students have a negative attitude towards mathematics (Di Martino, 2007), originated from unfit teaching methodologies asking young students to learn without manipulating, seeing with their own eyes, connect mathematics concepts with reality (Mammana & Milone, 2009a, Mammana & Milone 2009b)

With this respect we planned and experimented a math classroom activity for primary school children, on the concept of graph. Some graph theory topics, in fact, are sufficiently simple to be proposed at primary school level : the first aim of the activity is to make children enjoy math, but the inner goal is to make children starting to model real situations with the aid of graphs. The object "graph" is good, in fact, not only because you can see it, draw it and play with it, but also because it is quite useful to describe situations of real life by schematising them.

Not by chance there is an increasing attention toward graph theory in several international projects (see, for instance, http ://math.illinoisstate.edu/reu/).

The approach of this activity is the one described in (Aleo et al., 2009), based on mathematics laboratories (Chiappini, 2007), but adapted to 8-9 years old students and enriched with a huge use of technology (dynamic software to handle graphs, online games, multimedia interactive whiteboard), because addressed to native digital, very good in learning with technologies.

The teaching strategy is based on horizontal teaching (Ferrarello, Mammana & Pennisi, 2014) i.e. a way of teaching/learning in which the teacher enters in the intersection between teacher's knowledge and students' knowledge and enlarge this intersection. Horizontal teaching forces the teacher to go into students' real life, to understand their needs, to look at reality with their own eyes, so it requires an extra effort for the teacher, but it is effective and sometimes it enlarges not only students' knowledge, but also teacher's knowledge.

Here, for "reality" we do not mean only things that exist and you can touch, but also situations and characters that are familiar to children, so to motivate their study, for example cartoons characters, personal relationships, ...

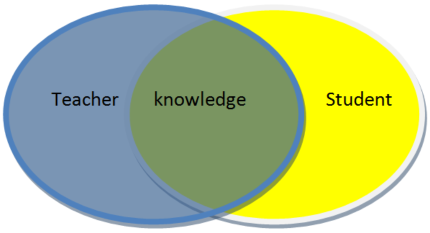

FIGURE 5.15 – Horizontal teaching

Technologies

Technology is a precious tool for such an activity. Nowadays 8-9 years old children were born with technology, which takes part of their real and today life.

On the other hand the use of technology is quite required in teaching : just think to 2.0 classes, or all the existing software and apps addressed to math teaching and learning.

The activity was carried out in a class with a Multimedia Interactive Whiteboard (together with classical blackboard and coloured chalks) : children manipulated graphs with their own hands, dragging nodes and deforming edges both with a dynamic graph editor and on line games, namely :

- *Yed Graph Editor* (http ://www.yworks.com/en/products_yed_about.html), a dynamic software to draw and explore graphs. This is useful because you can easily draw nodes of many shapes, representing several objects (people, geometric shapes, and every picture you want to import) and because you can easily explore graphs by dragging nodes or by adjusting edges.
- *Icosien* (http ://www.freewebarcade.com/game/icosien/) an online game to find Eulerian, semieulerian and Hamiltonian paths in given graphs (Fig 5.16, Fig 5.17), by wrapping the string around the nails to create the given shape in each level.

(http ://www.kongregate.com/games/ewmstaley/strand) to draw planar graphs with nodes of given degrees. It is useful also to reason about degrees.

Activity and topics

The activity has been carried out twice, last academic year (2012/2013) and this academic year (2013/2014) and it was leaded by the author of this paper (that is not a teacher in a primary school) tutored by an assistant, who helped in practical duties.

Both times the activity consisted in 12 meetings, once a week, in the afternoon. Children came voluntarily. The content for the activity are summarized in the following table together with some of their related activities.

Theoretical references to topics can be found in (Higgins, 2007 ; Wilson, 1996)

Here we describe some of the activities that have been carried out through several intriguing examples arising from "real" problems. We briefly present the following examples, giving motivations to use them.

Introduction to graph theory

The Königsberg bridges problem.

The matching of Disney princesses with their boyfriends.

A draw with Disney princesses and boyfriends was given to students, and to each princess was

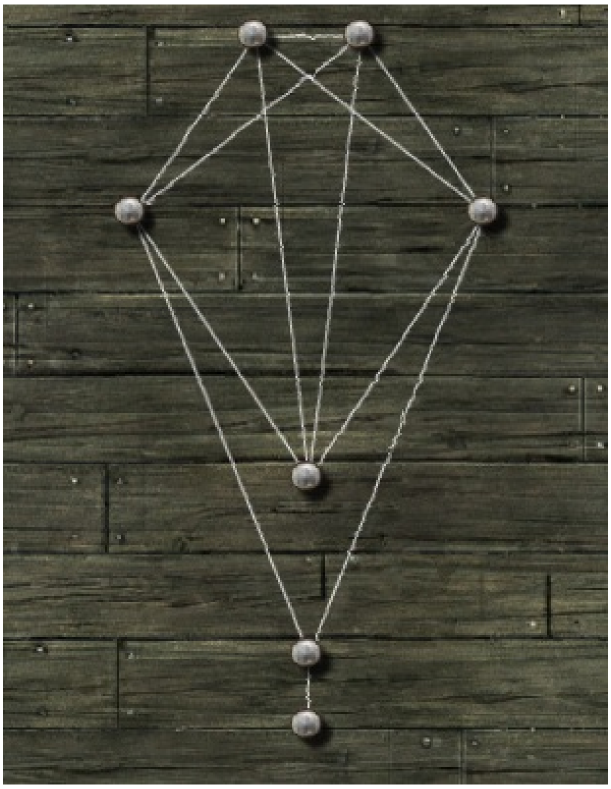

Figure 5.16 – Semieulerian graph

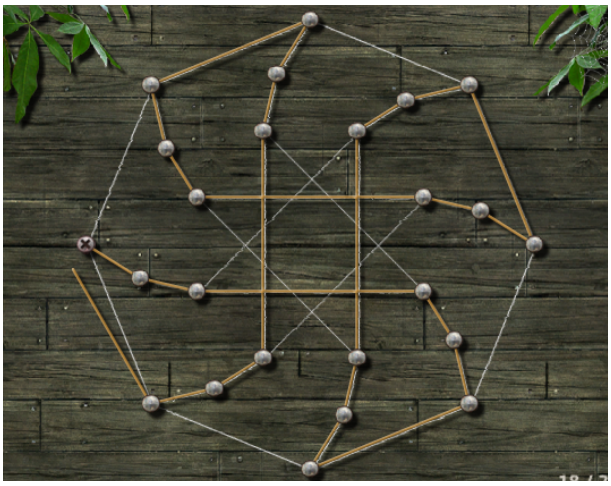

Figure 5.17 – Hamiltonian graph

assigned a boyfriend. But all the princess were matched to wrong boyfriends. Students were asked to delete wrong connections and draw new edges so to get the right matchings.

This graph is useful to introduce degree of a node (how many boyfriends a princess can have ?) and isolated nodes (has Ursula any boyfriend ?)

Moreover, one can use such a graph afterwards for the explanation of the mathematical notion of function : if we delete Ursula and require every princess to have only one boyfriend, than we get a function from the set of princesses to the set of boys (a non surjective function actually, because

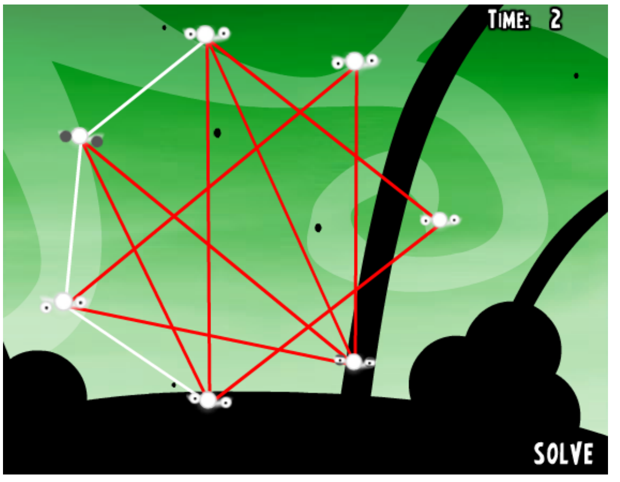

FIGURE 5.18 – Planar graph in Fly Tangle

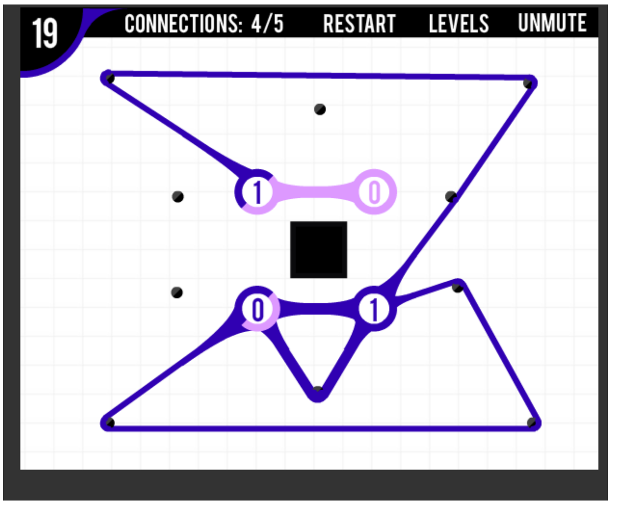

FIGURE 5.19 – Strand online game

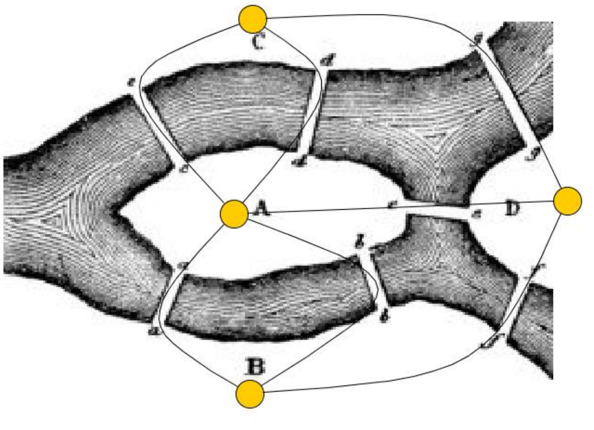

FIGURE 5.20 – Konigsberg seven bridge puzzle

Topics	**Activities**
Introduction of graphs and basic definitions : graphs to solve real problems, node, edges, degree of nodes, paths, cycles.	The Königsberg bridges problem ; the matching of Disney princesses with their boyfriends ; the genealogic tree of Dragon Ball cartoon ;
Eulerian and semieulerian graphs : graphs whose edges you can visit just once.	Graphs you could draw without lifting the pencil from the paper and drawing each edge only once ; words or sentences you can discover in a graph whose nodes are letters.
Hamiltonian graphs : graphs whose nodes you can visit just once.	the problem to sit around a table with friends both on your right and on your left ; Violetta's tour.
Planar graphs : graphs that can be drawn on the plane in such a way that its edges intersect only at their endpoints.	Three cottages problem.
Graph colouring : colour nodes of a graph in such a way adjacent nodes have different colours.	Maps you can colour by using the least numbers of colours such that adjacent regions have different colours.

TABLE 5.3 – Table 1 –Topics and activities

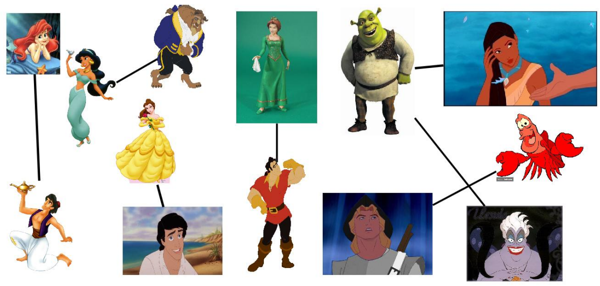

FIGURE 5.21 – Wrong matching of Disney princesses

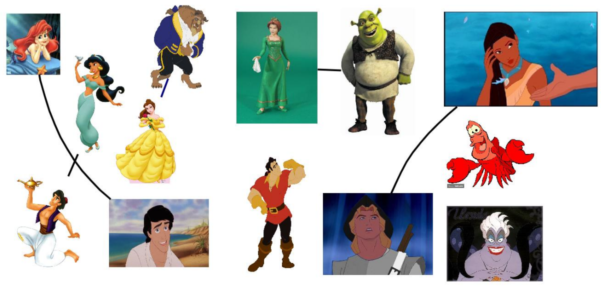

FIGURE 5.22 – Right matching of Disney princesses

of the presence of Gaston and Sebastian). Many other examples of graphs can be used in order to introduce the notions of function and relation in general.

Children were asked to properly construct the genealogic tree of Goshin, whose family is very complicated.

This last graph is useful to introduce trees, because it is immediate to notice that the graph is connected and you cannot have a "cycle", i.e. no character could be forefather of himself. After this

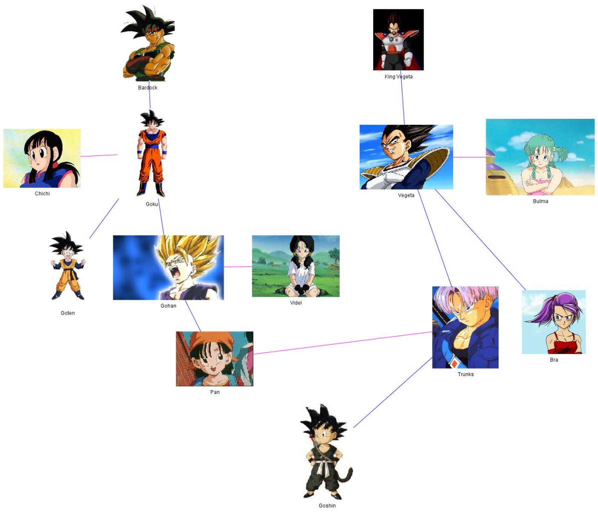

FIGURE 5.23 – Genealogic tree of Goshin

children were asked to draw their own genealogic tree.

Eulerian and semieulerian graphs

Graphs you could draw without leaving the sheet with your pen and without drawing any edge twice . We called these graphs "walkable".

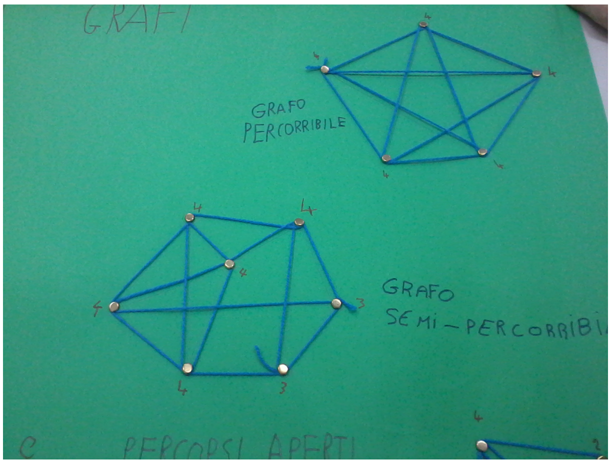

FIGURE 5.24 –
Graphs realized with real string

Finding words or sentences in a graph made of letters, like the following.

The graph in Fig 5.25 was used to practice paths, and especially semieulerian paths (Can you read the sentence hidden in the graph ?) Starting from a node of odd degree (G in this case) you get the sentence "Grafo percorribile" (Italian for "walkable graph"), finishing in the other node of odd degree (E).

This graph was also used to introduce loops, i.e. R can be repeated, (because in the word "percoRRibile" we have a double R) and multiple edges (because in the word "percorrIBIle" there is a sequence of I, B and I again).

Hamiltonian graphs

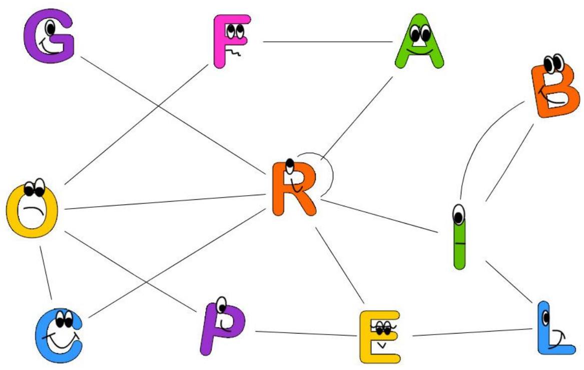

FIGURE 5.25 – Open walkable sentence

Besides classical examples, like a proper accommodation of persons around a table so that every person has a friend both at his right and at his left, we used the following Violetta’s tour

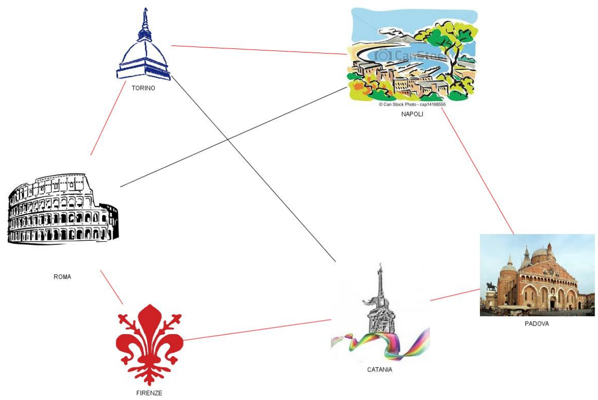

FIGURE 5.26 – Possible Hamiltonian tour

Also in this case children explored several Hamiltonian graphs by means of online game Icosien.

As for planar graphs and graph colouring we used the well-known three cottages problem and the maps colouring problem, respectively.

An instructional unit : eulerian and semieulerian graphs

In this section we shall describe the path through a whole topic : eulerian and semieulerian graphs. We called these graphs closed walkable and open walkable graphs, respectively, since “eulerian” for children is nonsense, while “walkable” recall them the activity of walking through the edges of the graph. The instructional unit proceeded with the following phases :

Pencil and paper games : many figures were presented to the pupils, as in picture Fig 5.27, asking to draw the graphs without lifting the pencil from the paper and passing just once through every edge.

Students were very excited about this game and they often thought it was always possible, especially after several successes, as for the graph e in Fig. 5.27. They thought themselves to be unable to do it, rather than it was impossible. And when we claimed that it was impossible to draw that graph, they didn’t believe it, keeping on trying.

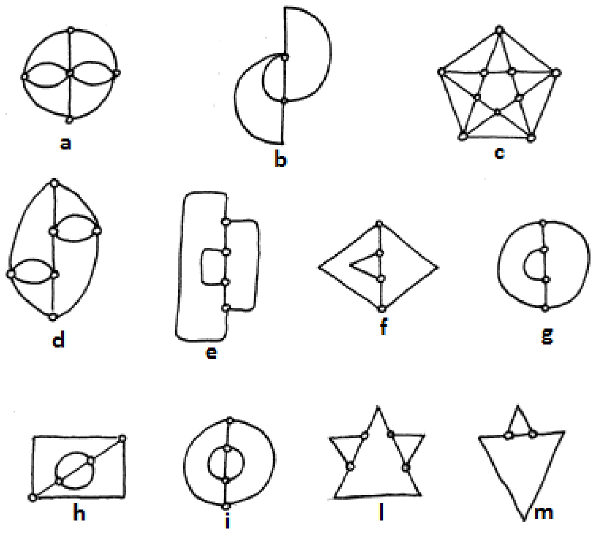

FIGURE 5.27 – Graphs to be decided about eulerianity

Practice with sentences : after that, we practised with words and sentences, as said in the previous section. Children played with anagrams and graphs related to possible configurations of letters, as in the popular game ruzzle (where letters can be used consecutively if they are neighbours in vertical, horizontal or diagonal direction), i.e. two letters are connected if you can use them consecutively in a word. For instance, looking at Fig 12, to form the word "sea" you follow the path 's', 'e', 'a' and you can do that because 's' and 'e' are connected and 'e' and 'a' are connected, while you cannot form the word "tea", because 't' and 'e' are not connected.

*Argumentation by chalks :*The on-line game could not be used to argument about odd degrees in semieulerian graphs, otherwise children would have been too focused on the game itself. We used blackboard and coloured chalks. In the graph in figure 5.30 we used oriented edges to trace one of the paths found by pupils, i.e. (1,2), (2,3), (3,4), (4,2), (2,5), (5,4), (4,1), (1,5) ; then we coloured every source with a green chalk and every sink with a red chalk. Then we focused on two nodes : the starting source, 1 in the figure, and the ending sink, 5 in the figure, and we counted the green and red edges. In 1 we have two green edges and a red one, because we first go out through edge (1,2), then we get in through edge (4,1) and at last we go out through (1,5). So we have an odd number of edges (two out and one in). Similarly, we examined node 5 : we have one green edge going out and two red edges getting in, one of which we used to complete the path.

In the other nodes we go out and in the same number of times, because they are passing nodes.

In such a way children understood the motivation for the two odd-degree nodes for semieulerian graphs. For the eulerian graph we just noticed that the starting point coincides with the final point, so we have the same number of outgoing out and incoming edges.

The whole phase of argumentation was led by the teacher, stimulating pupils with questions, encouraging them to express their thoughts, making them reflect on their own actions and claims. And finally exulting for their good insights and reasonings.

In order to make every child an active part of the activity, the class was divided into three teams, according to the preferences of the pupils themselves : drawers, writers and thinkers. The thinkers invented the graphs to be used, the drawers draw them down and the writers wrote the title and some captions. As for the wrapping of the string, all the children contributed, because for any member

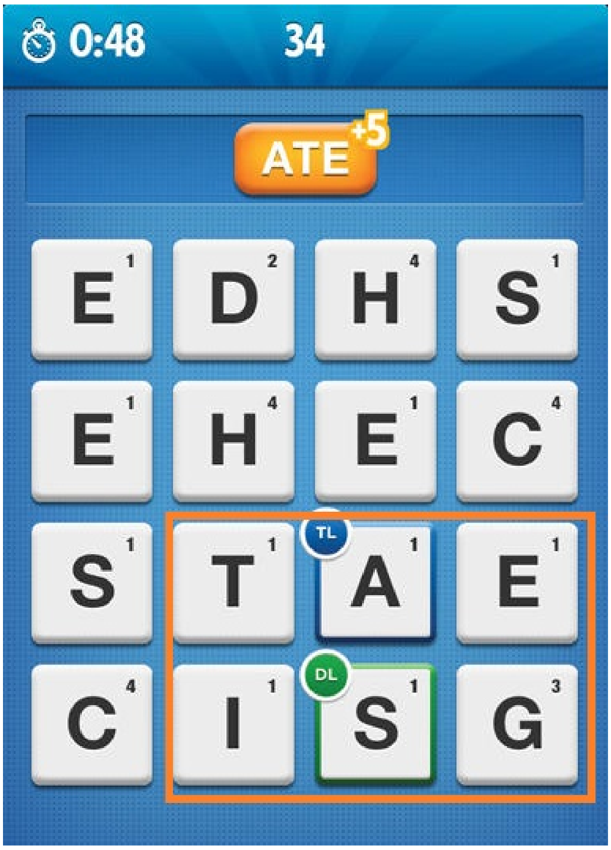

FIGURE 5.28 – display of game ruzzle

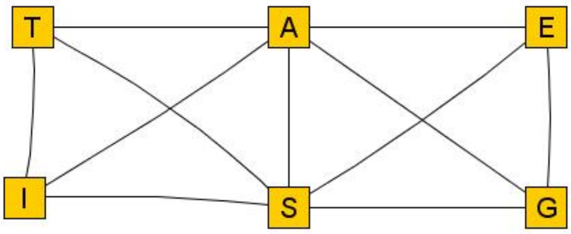

FIGURE 5.29 – graph related to the red rectangle in fig 5.28

of the three teams it was very important to join mind and body, as well as putting into practical actions the concepts they had studied .

Final "fighting" with parents :parents were invited to the last meeting for a Children vs Parents contest based on line games (Icosien and Strand) ; the victory of Children was overwhelming ! In fact children not only practised but also knew the tricks. Children were very proud to win against their own parents and to know things that mothers and dads did not.

Results and conclusions

Students of both schools were enthusiastic to see mathematics as an enjoyable subject, without numbers, calculations, systematic operations, but rich of princesses, relatives, football player, . . .,

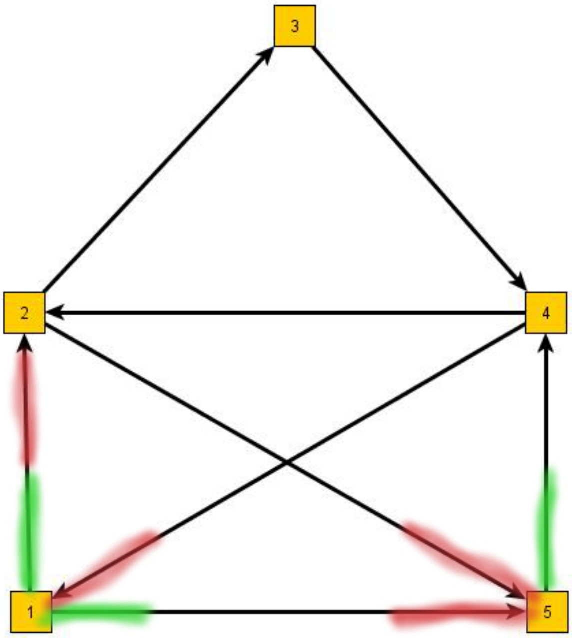

FIGURE 5.30 – graph used to argument

FIGURE 5.31 – Poster realized by pupils

and real situations.

Pupils finally saw mathematics as a game. Actually mathematics is a game indeed : you have to reach a target staying into the rules.

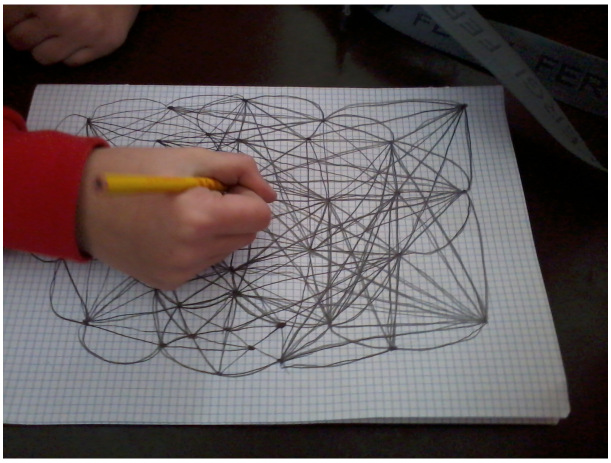

FIGURE 5.32 – Lilia drawing an eulerian graph of her

Referring to the activity of pencil and paper games on the previous section, Lorenzo (8 years old !) after the teacher said that the graph f in Fig.5.27 was not walkable said : "Then graph **g** is not walkable too ; they are the same !" So, that child was able to see the model underneath the drawings, without any particular explanation of the teacher about what is a mathematical model.

Some teacher of the students have been interviewed : they noticed that children during the activity had an increase of their logical skills and, especially, they were more active in class, showing to be able to pose questions instead of passive listening to lectures. So pupils "learnt to learn" in a more active way, not only in the graphs' activity : they brought this new skill in class applying it in the whole learning process.

The more difficult activity was the argumentation. Students were much more interested in playing and learning new "tricks" rather than in reasoning about the explanation of why the trick worked. But engaging them in the discussion, inviting them to participate actively, instead of just listening helped them in concentrate on the topic.

In the second course, my class tutor took notes every lecture and then she organized meetings with other primary math teachers to share materials and ideas, for possible future works. This work is very useful, because primary teachers (but also higher school teachers) do not know graph theory, that is a precious tool to be used in class to model problems from real life, to represent relations, to mathematize situations. And teachers who took part to these graphs' meetings had the opportunity to confront each other on questions about teaching/learning of mathematics.

In the end we report some of the comments written by children : "I think that graphs are more funny than games", "I think that this laboratory on graphs helped me to reason more quickly", "Mathematics is beautiful, intriguing and is of help".

ACKNOWLEDGEMENTS

The author is very grateful to class tutors in primary schools Giovanna Cuccu and Giuseppina Arnao for practical support with students.

The author also wants to thank Flavia Mammana for useful comments, and, last but not least, many thanks to all the pupils involved in the activities for their enthusiasm and fondness.

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5.8 La calculatrice comme milieu expérimental

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Résumé : Dans ce texte, à travers l’étude de tâches mathématiques intégrant l’utilisation de la calculatrice, nous nous illustrons de quelle façon cette utilisation peut prolonger l’expérience mathématique de l’élève, dans le même sens que l’on dit que l’outil prolonge la main. Nous nous appuyons sur quelques activités prototypiques, destinées à des élèves de différents âges. Nous concluons par une réflexion sur les conditions de l’intégration de ce type d’activité dans l’enseignement en nous fondons sur les notions de milieu et de praxéologie.

Abstract : In this paper, through the study of mathematical tasks incorporating the use of the calculator, we illustrate how this use may extend the mathematical experience of the student, in the same sense that it is said that tool extends hand. We rely this on some prototypical activities for pupils of different ages. We conclude with a reflection on the conditions of the integration of this type of activity in teaching us basing on the concepts of « milieu »and praxeology.

Quelques activités pour produire des expériences à tout âge

Diviseurs sans multiplication

Dans une expérimentation sur le long terme, avec des élèves de 6-7 ans, nous avons travaillé le type de tâches « cibles »suivant (encadré 1) :

> À l’aide de la calculatrice, chercher les nombres qui permettent, avec des additions répétées, d’atteindre des valeurs cible entières. Les nombres cherchés correspondent aux diviseurs de la cible. Par exemple, la cible 48 peut être atteinte en additionnant de manière répétée le 1, 2, 3, 4, 6, 8, 12, 24 et le 48.

Encadré 1. La situation « des cibles »

Ce travail s’est poursuivi pendant quelques mois, à raison d’une séance par semaine, dans un dispositif alternant recherche individuelle et mise en commun au tableau noir aboutissant à des tables de ce type (encadré 2).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
18	1	2	3			6												18							
19	1																		19						
20	1	1		4	5					10										20					
21	1		3				7														21				
22	1	2									11											22			
23	1																						23		
24	1	2	3	4		6		8				12												24	
25	1				5																				25

Table 5.4 – Encadré 2. Exemple de table remplie (correctement) pour les cibles de 18 à 25

Ce travail, décrit dans Del Notaro et Floris (2011) a permis l’étude de propriétés des nombres entiers telles que parité, multiples et diviseurs, nombres premiers. L’utilisation de la calculatrice y a joué un rôle important, permettant de définir les propriétés comme des actions possibles ou non avec la calculatrice. Si les phénomènes habituels de recherche frénétique (Meissner, 2005) dit de « pêche »par Artigue, 1997) ont eu lieu, en remplacement parfois de l’utilisation de connaissances institutionnalisées, sur le moyen terme une évolution de ces connaissances a été observées, en particulier avec une première approche de la multiplication.

Diviseurs avec division : racine carrée

Un travail sur les diviseurs a également pu être proposé à l'école secondaire par des étudiants stagiaires, en exploitant la touche « table »de la calculatrice TI30XS Multiview, en dotation à Genève menant à la recherche de racines carrées par l'observation du renversement, la valeur de x dépassant celle de y.

x	$y = \frac{60}{x}$
1	60
2	30
3	20
4	15
5	12
6	10
7	8,57...
8	7,5
9	6,66...
10	6
⋮	⋮

Table 5.5 – Encadré 3. Utilisation de la touche Table

Propriétés arithmétiques

Les activités suivantes sont à travailler avec des élèves de la fin de l'école primaire ou du début du secondaire (11-12 ans). Elles sont été proposées à des enseignants du secondaire en formation et se sont révélées non immédiates difficiles pour certains d'entre eux.

Les tâches suivantes permettent de travailler la numérotation de position.

– Faire afficher le nombre 89254. Sans effacer ce nombre, à l'aide d'opérations mathématiques, faire afficher le nombre 89454.
– Faire afficher le nombre 89254. Sans effacer ce nombre, à l'aide d'opérations mathématiques, faire afficher le nombre 892054.
– Faire afficher le nombre 4.56. Sans effacer ce nombre, à l'aide d'opérations mathématiques, faire afficher le nombre 4.056.

Les tâches suivantes permettent de travailler la distributivité

– Sans utiliser la touche de multiplication et en un minimum d'opérations sur la calculatrice, calculer les produits suivants : 387×204 et 387×199.
– Déterminer tous les chiffres d'une plus grande puissance de 7 possible.

Simplification de fractions

Nous nous intéressons aux procédures de simplification de 'grandes' fractions lorsqu'une calculatrice est à disposition. Pour des numérateurs et dénominateurs au-delà de 100, les élèves ne peuvent généralement pas faire appel à un répertoire mémorisé de multiples des entiers pour la recherche d'un diviseur commun et doivent donc exploiter d'autres techniques pour effectuer ce type de tâche. Dans le cadre d'une recherche (Weiss & Floris, 2008), une série de fractions à simplifier ont été proposées à différents types d'élèves, avec autorisation d'utiliser la calculatrice (encadré 4).

a) $\frac{2500}{7500}$	*b)* $\frac{72}{108}$
c) $\frac{241}{150}$	*b)* $\frac{176}{165}$
d) $\frac{256}{243}$	*b)* $\frac{749}{7000}$
g) $\frac{187}{340}$	*b)* $\frac{110}{264}$

Table 5.6 – Encadré 4. Les tâches proposées

Il est apparu que de nombreux élèves ont considéré comme irréductibles des fractions comme 187/340, car ils ne poursuivent pas leurs recherches de diviseurs communs au delà de 10. Ces élèves, soumis au contrat didactique habituel dans lequel on se contente de leur demander de simplifier les fractions par 2, 3, 5, 7 ou 10, voire 11. Les tâches proposées, à la frontière de ce contrat, les conduisent à une extension de leur réalité mathématique et permettre qu'ils apprennent quelque chose qui leur avait échappé jusque là : l'existence d'une méthode permettant de rendre toute fraction irréductible. Et nous songeons ici à la décomposition des numérateurs et dénominateurs en facteurs premiers, plus transparente que l'utilisation du PGCD (encadré 5) :

$$\frac{637}{1183} = \frac{7^2 \times 13}{7 \times 13^2} = \frac{7 \times 7 \times 13}{7 \times 13 \times 13} = \frac{7}{7} \times \frac{7}{13} \times \frac{13}{13} = \frac{7}{13}$$

Encadré 5 Simplification par décompositions

Nombres non décimaux

Considérons l'activité suivante :

1. Déterminer l'écriture décimale de 3/7 avec la calculatrice. Copier sur la 3ème ligne de la calculatrice (TI-30XSMultiView) le résultat obtenu et multiplier ce résultat par 7. Que conclure ?
2. Faire à nouveau 3/7, puis multiplier immédiatement par 7 (touche Ans puis 7). Que conclure cette fois ?
3. L'écriture décimale de 3/7 est-elle périodique ? Si oui, déterminer cette période. Peut-on le faire à l'aide de la calculatrice ?

L'activité qui précède permet de mettre en évidence la gestion des arrondis par la calculatrice. Elle nécessite aussi une réflexion numérique. De même que pour l'exercice de grande multiplication proposé ci-dessus, elle suggère un travail de négociation entre les réponses de la calculatrice et celles que l'on peut obtenir en utilisant les algorithmes habituels (papier/crayon). Elle conduit à approfondir les connaissances sur le fonctionnement de la calculatrice, avec les décimales supplémentaires traitées mais non affichées. Une activité analogue peut être faite, en partant de la racine carrée de 2.

Suites numériques, limites

Dans l'enseignement de l'analyse pré-universitaire, la notion de limite peut se fonder sur une réalité numérique, mais celle-ci est entièrement à construire. L'étude de suites fournit des expériences pour les élèves. Outre les suites arithmétiques et géométriques habituelles, l'examen de suites données par Un = formule mathématique et Un = formule mathématique permettent d'affiner la définition et de mettre en défaut certaines conceptions : la première suite permet de justifier l'étude de grandes valeurs et la second met en défaut l'idée qu'une suite n'atteint pas sa limite.

Balises théoriques : Milieu, praxéologie

Ce qui caractérise et rassemble les tâches ou activités présentées, c'est une utilisation de la calculatrice comme outil d'apprentissage des mathématiques. Peut-on alors considérer la calculatrice comme un milieu d'apprentissage ? Pour Brousseau (1986), un milieu d'apprentissage est constitué de différents éléments qui vont rendre possible l'apprentissage visé par l'enseignant dont, en particulier, des résultats d'actions de l'élève tels que des calculs, des dessins ou des manipulations.

C'est dans ce milieu d'apprentissage que travaille le lien entre ce qui a été proposé par le maître et ce que l'élève a réalisé. Les résultats d'actions effectuées à la calculatrice peuvent faire partie de ce milieu. En elle-même, la calculatrice ne constitue pas un milieu, pas plus qu'un calcul isolé effectué avec crayon et papier. Un lien doit être construit et on a donc également pu parler d'instrumentation didactique, c'est-à-dire d'une prise en charge officielle en classe de l'utilisation de la calculatrice. Prenons l'exemple de l'opération 9999999999+1 qui fournit sur la calculatrice TI34II la réponse 1.E1010. Sans un travail préalable d'enseignement, cet élément ne fait pas partie des éléments objectifs du milieu d'apprentissage. L'enseignant, qui doit avoir réponse à tout ce qu'il a contribué à provoquer -c'est le contrat didactique- ne pourra que donner une réponse évasive à la question d'un élève « trop »curieux. Une instrumentation didactique qui prendrait en compte cette réponse serait relativement complexe. Elle inclurait un travail sur le nombre de chiffres d'un nombre entier, travail menant à des petits théorèmes, tels que « en effectuant une addition, le nombre de chiffres n'augmente pas ou augmente de 1 ». Ainsi qu'un travail sur le nombre des chiffres affichés par la calculatrice. C'est bien entendu avec la multiplication que ce travail se révélerait le plus intéressant : pour quel type de multiplication le nombre de chiffres du résultat correspond-il à la somme du nombre de chiffres des multiplicandes ?

Un travail de ce type, s'il reste isolé, n'a qu'un intérêt anecdotique. L'étude ne peut prendre son sens que dans le cadre d'une perspective à moyen terme comprenant un travail technique légitimé par les mathématiques, avec des fondements théoriques, des règles et des propriétés. Dans ce cas, ces fondements correspondent à l'écriture positionnelle des nombres en base dix et à toutes les propriétés mathématiques sur lesquelles elle s'appuie, particulièrement celles de la structure d'anneau. C'était bien le projet des plans d'études des années 1970, avec le travail sur les différentes bases, et peut-être a-t-on jeté le bébé avec l'eau du bain : une organisation mathématique ne se réimplante pas du jour au lendemain dans un cursus (ne pas comprendre ici que nous préconisons le retour en classe du calcul en bases différentes de dix, ni l'introduction de l'étude des anneaux !).

On peut exprimer les choses d'une autre façon, en nous référant à la notion de praxéologie (Chevallard, 1999). Un milieu d'apprentissage ne peut fonctionner durablement sans la présence d'un vocabulaire, décrivant des actions et des propriétés relativement aux résultats de ces actions. Ces propriétés permettent de contrôler les actions, de prévoir des résultats : des praxéologies, que nous appelons ici aussi mini-cultures. Dans le cas des cibles, les élèves utilisent des règles telles que « en ajoutant plusieurs fois le même nombre pair, on obtient toujours un nombre pair », qui leur permettent d'éviter certaines recherches en éliminant des nombres impairs pour les cibles paires. Un autre exemple de règle pour cette même activité est le fait que les diviseurs forment des paires, de sorte qu'une fois l'un des diviseurs trouvé on peut en déduire l'autre à partir de l'action effectuée : le nombre d'addition effectuée avec le premier diviseur. Dans ce cas, la mini-culture se constitue comme l'ensemble des règles et du vocabulaire permettant de contrôler la situation et se travaille en tant que telle, par des institutionnalisations consignées sur une affiche ou le cahier de l'élève. Autre exemple, la simplification des fractions. La diversité des fractions proposées conduit les élèves à utiliser d'autres techniques que la technique élémentaire de recherche de diviseurs communs élémentaires, qui faisait penser à certains d'entre eux qu'il suffit de se limiter à recherche des diviseurs inférieurs à 10. Dans le cadre de dispositifs adéquats, ces nouvelles techniques peuvent permettre l'explicitation de nouvelles propriétés pour la simplification des fractions.

Dans l'exemple des suites numériques, les tâches proposées conduisent à une exploration des différentes situations possibles, citons le fait qu'une suite croissante ne diverge pas forcément ou qu'une suite bornée ne converge pas forcément, cette exploration permettant de conduire aux théorèmes standards sur la convergence des suites.

Nous insistons sur la notion de résultats matériels, qui peuvent être des objets ou des traces sur un tableau noir, sur du papier, sur l'écran d'une calculatrice. Ce milieu matériel est indispensable en vue du rappel des actions effectuées et pour l'élaboration de conjecture. L'étude du déroulement de la « course à vingt »(Brousseau, 1986) met en évidence ce rôle du milieu. A cet égard, l'utilisation de calculatrices conservant l'affichage et la mémoire des opérations effectuées est très importante, ainsi que la possibilité d'en présenter le résultat à l'ensemble de la classe en utilisant une calculatrice adaptée (transparente pour rétroprojecteur ou en reliant la calculatrice à une tablette de rétroprojection). Les travaux de Guin & Trouche (2002), de Artaud (2003) et de Kieran & Guzman (2003) en particulier, montrent l'importance de l'exploitation didactique des éléments du milieu.

Conclusion

Les expériences des élèves permettent de constituer des milieux pour les apprentissages mathématiques. Ils constituent la réalité dans laquelle ils anticipent leurs actions, et agissent. La technologie, même la plus « simple »vient complexifier cette réalité, ajoutant des rétroactions spécifiques, parfois utiles et valides mais parfois aussi surprenantes. C'est par une genèse instrumentale (Rabardel, 1995) que l'outil s'intègre peu à peu. Souvent, cet aspect n'est pas pris en compte dans les plans d'études, ou alors de façon minimale. C'est ainsi le cas du nouveau Plan d'Etude Romand (CIIP, 2010). Il revient donc aux enseignants de piloter l'intégration de l'outil. Nous avons présenté ici quelques pistes montrant comment c'est possible. Ces exemples mettent aussi en évidence en quoi le rapport aux mathématiques peut évoluer avec la technologie, même si bien sûr les mathématiques restent elles-mêmes.

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5.9 La pensée arithmético-algébrique dans la transition primaire-secondaire et le rôle des représentations spontanées et institutionnelles

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Résumé : Durant la CIEAEM65 ont été présentés les premiers résultats de notre recherche portant sur la pensée arithmético-algébrique. Cette année, nous souhaitons contribuer en rapportant l'analyse des résultats de la 2e partie de cette recherche. En effet, une expérimentation a pris place suite à la première autour du même thème, une pensée arithmético-algébrique comme prélude à la pensée algébrique à l'école secondaire. Dans le processus de construction d'une pensée arithmético-algébrique, nous avons analysé les productions spontanées des élèves face à une activité liée aux nombres polygonaux dans une approche qui suit un chemin naturel vers l'algèbre plutôt qu'une approche par résolution d'équations. Nous avons pu constater que ces représentations spontanées peuvent évoluer, dans un environnement à la fois technologique d'apprentissage en collaboration et de débat et d'auto-réflexion (ACODESA).

Abstract : In CIEAEM65 we present the first results of our investigation on the arithmetic-algebraic thinking and now we will continue with the analysis of the results of the second part of our research. In this paper we present the results of the continuation of an experiment related to an arithmetic-algebraic thinking as a prelude to algebraic thinking in secondary school. In the process of building an arithmetic-algebraic thinking, we analyze the spontaneous students' productions when solving an activity related to polygonal numbers. We show that spontaneous students' representations may evolve in technologically collaborative learning environment, debate and self - reflection (ACODESA). In our approach to algebra, we chose a natural path instead of an approach by solving equations.

Introduction

Dans le passé était présente une tendance à caractériser, d'un côté, la pensée arithmétique (p.e. voir Verschaffel et De Corte, 1996), et d'un autre côté, la pensée algébrique (p.e. Kaput, 2000, 2008 ; Kieran, 2004, 2007). Ainsi, les chercheurs se sont penchés sur les difficultés d'apprentissage des mathématiques en essayant de comprendre les obstacles liés à la transition entre l'arithmétique et l'algèbre. Une de ces approches était centrée autour de l'obstacle épistémologique lié à la résolution d'équations (p.e Filloy & Rojano, 1989). Les chercheurs se sont ainsi attardés à trouver un chemin facilitant le processus de transition des élèves de l'arithmétique vers l'algèbre (voir tableau I).

Caractérisation d'une pensée arithmétique	Transition vers l'algèbre	Caractérisation d'une pensée algébrique
Verschaffel-De Corte (1996) : 1. concepts numériques et sens du nombre, 2. sens d'opérations arithmétiques, 3. la maîtrise des faits arithmétiques de base, 4. calcul mental et écrit, et 5. problèmes de mots en appliquant des connaissances numériques et compétences arithmétiques.	Obstacles à surpasser (p.e. voir Filloy et Rojano, 1989)	Modèle GTG de Kieran (2004/2007) : G) L'activité générationnelle...formation des expressions et des équations qui sont les objets de l'algèbre... T) L'activité de transformation... factorisation, l'expansion, substitution d'une expression dans une autre,...résolution d'équations et des inéquations,... G) L'activité globale/méta...l'algèbre est utilisée comme un outil, mais qui ne sont pas exclusifs à l'algèbre [...] résolution de problèmes, la modélisation,...

Table 5.7 – La transition de l'arithmétique vers l'algèbre.

Une caractérisation des composantes de la pensée algébrique a donné lieu à une distinction entre deux chemins (Kaput, Idem ; Blanton & Kaput, 2011 ; Artigue, 2012) :

a un chemin naturel de l'arithmétique vers l'algèbre,

b un chemin symbolique direct dans la construction de la pensée algébrique.

Le paradigme « Early algebra »(aux États-Unies) est précisément né de cette idée que l'on peut suivre « un chemin naturel de l'arithmétique vers l'algèbre ». De notre point de vue, les chercheurs liés à ce paradigme ont choisi deux voies distinctes. Dans une de ces voies, on peut interpréter que certains chercheurs souhaitent construire une espèce d'« autoroute vers l'algèbre »(disons que, p.e., Carraher, Schliemann & Brizuela, 2000, 2006 vont dans cette direction avec leurs idées sur l'approche arithmétique des fonctions). Pour d'autres chercheurs, un besoin de mieux comprendre ce paradigme afin d'éclairer le problème de la transition vers l'algèbre et les conséquences de ce mouvement se fait sentir (Radford, 2003, 2011). Nous pouvons résumer en disant que les chercheurs, dans le passé, étaient centrés sur les difficultés autour de la transition de l'arithmétique à l'algèbre, le mouvement « Early algebra »ayant donné lieu à un nouveau paradigme qui a divisé les chercheurs en deux.

Cadre théorique

Nous sommes intéressés à la construction des connaissances d'un point de vue social. De plus, nous pensons que la technologie peut aider à la construction de cette connaissance. Nous penchons pour une théorie post Vygotskienne liée à la théorie de l'action selon le cadre théorique d'Engeström (1999) et qui a évoluée vers le cadre théorique de Leontev comme le souligne Nardi (1997) autour de la théorie de l'activité dans sa 5e génération :

L'objet de la théorie de l'activité est de comprendre l'unité de la conscience et de l'activité. La théorie de l'activité intègre les notions d'intentionnalité, l'historicité, la médiation, la coopération et le développement dans la construction de la conscience. ... [celle-ci] est dans la pratique quotidienne : vous êtes ce que vous faites. Et ce que vous faites est fermement et intimement intégré dans la matrice sociale dans laquelle chaque personne est une partie organique. Cette matrice sociale est composée de personnes et artefacts. Les artefacts peuvent être des outils physiques ou systèmes de signes...(P. 4)

Étant donné que nous sommes intéressés à l'utilisation de la technologie et aux processus arithmético-algébriques, notre choix s'est porté sur le contenu mathématique lié aux nombres polygonaux. De plus, étant donné que les nombres polygonaux ont été utilisés pour analyser des processus algébriques dans un milieu technologique, nous avons décidé de revisiter les travaux des chercheurs Healy & Sutherland (1990) qui ont travaillé avec Excel et les travaux de Hitt (1994) qui a utilisé Excel et LOGO.

L'expérimentation menée par Healy & Sutherland (Idem) s'est attardée aux nombres triangulaires. Les élèves de première année du secondaire ont construit dans l'environnement Excel l'expression "trig. n = na before + position". Nous nommons ce type de représentations des « représentations spontanées », elles proviennent des « représentations fonctionnelles »qui se sont formés chez les individus à travers l'activité mathématique dans laquelle ils sont immergés. Ce type de représentation est éloignée de la représentation institutionnelle qui se traduit par $T_n = \frac{n(n+1)}{2}$.

Notre question de recherche repose sur comment, après une construction personnelle qui s'appuie sur la production de représentations spontanées, nous pouvons faire évoluer ces représentations vers une représentation institutionnelle liée aux nombres triangulaires. Nous pensons que cette évolution serait favorisée dans un milieu de construction sociale des connaissances.

Pour notre expérimentation, les activités élaborées par Healy & Sutherland et par Hitt ont été reprises et réaménagées afin de créer des situations d'apprentissage dans le milieu Excel et en utilisant un applet appelé POLY. Notre intention était de promouvoir la production de représentations spontanées (papier crayon) et la visualisation dans un milieu socioculturel, de faire évoluer ces représentations vers les représentations institutionnelles (voir Hitt, Saboya et Cortés, 2013).

Notre hypothèse d'investigation

Nous pensons qu'il est important de construire une pensée arithmético-algébrique nécessaire au passage de l'arithmétique à l'algèbre et vice versa. Ainsi, nous voulons construire chez l'élève une pensée arithmético-algébrique dans un environnement d'apprentissage collaboratif, de débat et d'autoréflexion (ACODESA) dans une approche socioculturelle.

Méthodologie

Nous avons choisi deux populations de différents niveaux d'études pour étudier notre hypothèse d'investigation.

a 13 élèves de première année (12-13 ans) d'une école secondaire au Québec,

b 14 élèves de troisième année d'études (15-16 ans) dans une école secondaire du Mexique.

Dans les deux expérimentations menées, les 5 étapes de la méthode ACODESA nous ont guidées.

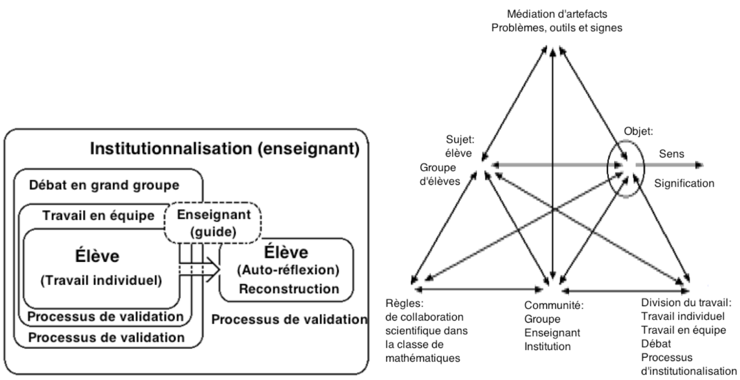

FIGURE 5.33 – ACODESA et la théorie de l'activité en suivant le modèle d'Engeström (1999).

Dans ce document, nous allons analyser les données des 13 élèves du Québec. Dans Hitt, Saboya et Cortés (Idem) nous avons présenté la première expérimentation menée. Un mois et demi cette expérimentation, nous sommes retournés voir les élèves pour leur demander de refaire la même activité sur les nombres triangulaires. Nous avons posé un défi à un de ces élèves qui s'était particulièrement démarqué dans la première expérimentation, une étude autour des nombres pentagonaux.

Qu'est-ce que les élèves ont retenu après un mois et demi ? Comme nous n'avons pas beaucoup d'espace dans ce document, nous présentons brièvement sous forme de schéma les représentations spontanées des élèves et les résultats obtenus après analyse des productions des élèves ayant réalisé cette activité après 45 jours.

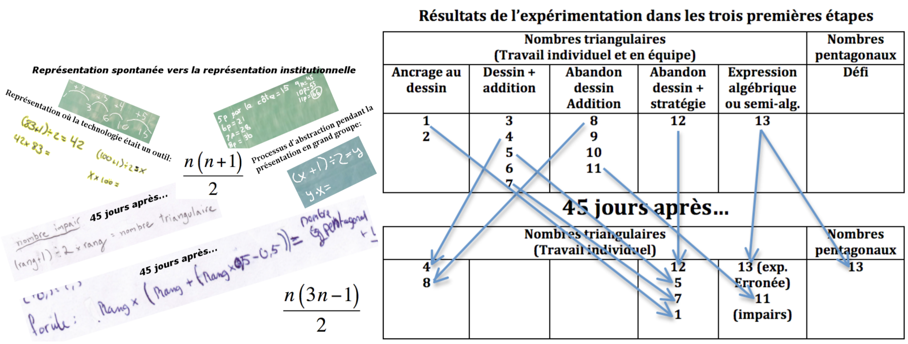

FIGURE 5.34 – Résumé des résultats de la 2e partie de l'expérimentation.

Conclusions

Tel que nous l'avons précisé, nous avons privilégié une approche naturelle pour l'introduction de l'algèbre selon Kaput (1998, 2000), Artigue (2012) et Blanton & Kaput (2011). Les élèves se sont engagés dans les tâches d'abord dans un environnement papier crayon, après avec Excel et l'applet POLY. Nous pouvons remarquer que dans la première expérimentation différentes représentations spontanées ont émergées et nous avons pu assister lors de la deuxième expérimentation à une évolution de ces représentations.

En fait, 45 jours après la 1ere étape de la expérimentation, le garçon qui avait trouvé une expression algébrique pour calculer n'importe quel nombre triangulaire, avait oublié « sa formule »et il a donné une réponse erronée pour le 11e nombre triangulaire. Il n'a pas vérifié sa formule avec les exemples données (les premiers 4 nombres triangulaires). Par contre, ce que lui n'a pas oublié c'est le processus de construction pour proposer une expression pour calculer n'importe quel nombre pentagonal. Selon le tableau, il a eu aussi d'autres élèves qui on fait une reconstruction de la méthode pour calculer les nombres triangulaires. Le travail en collaboration et la technologie ont joué un rôle important dans cette évolution. Un fait ce que nous pouvons remarquer, est que les élèves, dans le processus de résolution, ont produit des représentations qui sont loin des représentations institutionnelles, mais ils ont résolu les tâches, et de ce point de vue, le passage aux représentation institutionnelles peut se faire sans « friction »avec les élèves.

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5.10 The activity of programming on the continued education of the mathematic teacher

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Résumé :Cet article vise à présenter deux exemples d'activités sur la programmation et les concepts mathématiques qui faisaient partie de la recherche sur la formation continue des enseignants de mathématiques à l'usage des technologies numériques, développé dans le projet Observatoire de l'Enseignement , de UNIBAN , financé par la CAPES . La méthodologie utilisée dans les études a été développeé à partir du point de vue de l'analyse qualitative sur des actions de formation à destination des enseignants de mathématiques qui travaillent dans Basique Education. Dans le premier exemple, la programmation de la situation est fait avec le logiciel GeoGebra et la notion de symétrie et dans le second, la programmation avec le langage Scratch pour la création d'un micro monde pour explorer l'idée de se concentrer dans la generalization. Les recherches ont été développées , en se concentrant actions de formation sur la base de l'approche constructiviste, qui nous ont permis de constater qu'il existe une possibilité pour les enseignants de l'éducation de base de reconstruire leurs connaissances professionnelles dans la perspective d'intégration de TPACK –Technologie, pédagogie et la connaissance du contenu.

Abstract :This article aims to present two examples involving the activity of programming and mathematical concepts that were part of research about the continued education of Mathematics teachers for the use of digital technologies, developed in the Education Observatory Project, from UNIBAN, funded by CAPES. The methodology used in the studies was developed from the perspective of qualitative analysis involving training actions aimed at mathematics teachers who work in Basic Education.In the first example, the situation involved programming using the Geogebra software and the concept of Symmetry and in the second, the programming with the Scratch language for the creation of micro world exploring the idea of generalization.The focus in that such research were developed, focusing formative actions based on the constructionist approach, allowed us to observe that there is a possibility for teachers of Basic Educationto rebuild their professional knowledge in the integrative perspective of TPACK - Technology, Pedagogy and Content Knowledge.

Introduction

Education in the digital culture has been an instigator theme for many researchers from different areas of knowledge, who seek to understand the impacts of the presence of digital technologies in the daily lives of students. Such technologies demand and imprint the creation of new forms of communication, information search and representation of knowledge.

Technologies, especially computers, reached the Brazilian public schools as a policy of the federal government in the late 80s. From that time until the present day multiple deployment projects, accompanied by training of teachers for the pedagogical use of technologies have been developed meeting the needs of the education system and the new specifics of technological advances.

It is interesting, however, to remind that in the beginning of this process, schools did not have many choices of computing resources : one possibility was to use programs of the Computer-Aided Instruction type, based on the behaviorist conception of learning programs and other, Programming Language logo designed based on constructivist principles of learning, and another, the Logo programming language, designed based on constructivist principles.At that time, the most comprehensive option was taken by the programming language, due to the strong influence of the ideas of Papert and his Brazilian disciples, who lived at the peak time of the Logo programming language in Massachusetts Institute of Technology (MIT).

Papert was one of the pioneers in the history of Computing in Education, who along with his colleagues created the Logo programming language (derived from Lisp) geared towards the use in the educational context of Basic Education.He sought to understand the relationship between man, technology and the nature of learning. To guide the use of the programming activity in the processes of teaching and learning, Papert (1991 ; 1985) developed a pedagogical approach called Constructionism.Such approach emphasizes the creation of learning environments that allow different individuals to engage in reflective activities, those that promote both "learn-with" like "learn about thinking." Is the idea of hands-on and head-in.This means that the subject can learn by doing, by building something that is meaningful to him and that allows an emotional and cognitive engagement with the production process. These principles emphasize the autonomy, curiosity and authorship of the learner from the perspective of Freire (1996).

Such powerful ideas, however, did not become widely effective in the context of basic education for various reasons mentioned in several studies. Among which we highlight the gap that has developed between the rapid advancement of technologies and the process of deployment of computers in schools. In addition, teacher education, which has a fundamental character, became increasingly more complex, because it requires new reconstructions of knowledge involving the integration of different areas : Technological, Pedagogical, and Mathematics. This approach to integrating technology into specific content and teaching is represented by TPACK (Technological Pedagogical Content Knowledge) from Mishra and Koehler (2006), who extended Shulman's idea of Pedagogical Content Knowledge, considered necessary for teacher actions in the classroom (Shulman, 1986).

This model TPACK consists on the intersection of three different types of knowledge : Content (CK), Pedagogical (PK) and Technological (TK). However, this model involves other intermediary interrelations among knowledge types. Therefore, the interrelation between Pedagogical Knowledge (PK) and (CK) Content knowledge originates the Pedagogical Content Knowledge (PCK) that refers to knowing to teach a certain curricular content. The Technological Knowledge (TK) and Content Knowledge (CK) originate the Technological Content Knowledge (TCK) that refers to being able to select the technological resources that leverage the learning ability of a certain curricular content. The interrelation of Technological Knowledge (TK) and Pedagogical Knowledge (PK) generate the Technological Pedagogical Knowledge (TPK) that refers to the understanding of the implications of the use of these technological resources in the teaching and learning process. Finally, the integrating point of these interrelations is the Technological Pedagogical Content Knowledge (TPACK).

This structure of the TPACK has aided in the understanding of the teacher's pedagogical appropriation of the technological process, considering new opportunities for teaching and learning were brought by the Internet, especially Web 2.0 and various software that were created, such as the math :Cabri, Geogebra, Winplot, among others. From the point of view of hardware, there is also a growing movement of availability of mobile devices (tablet, mobile, netbook, etc.) that requires a new management in the classroom and a differentiated pedagogical practice from the usual one, already consolidated in educational culture.

Advances can expand and diversify the forms of teaching and learning, pushing the limits of time and classroom space, allowing the student to develop the investigative thinking in the search for new information and insights, as well as the synchronism in the forms of communication that can promote collaborative learning and sharing of discoveries, trials, reflections and different ways to represent knowledge.

It is interesting that in this current scenario permeated, more and more, by a great diversity of digital technologies, the programming activity begins to resurface by the recognition of its educational potential.

In this sense, this article aims to present two examples involving the programming activity and mathematical concepts, that were part of the research on continued education of math teachers for the use of digital technologies, developed in the Education Observatory Program, from Anhanguera University of São Paulo, funded by CAPES. The methodology used in the studies was developed from the perspective of qualitative analysis involving formative actions aimed at mathematics teachers who work in Basic Education.

Programming Activities

To understand some of the educational implications of the programming activity, Valente (2002), created an explanatory model of the spiral of learning. This spiral is formed by a movement of thoughts and actions involving : description-execution-reflection-debugging- (new) description, which occurs when the subject interacts with the computer to solve a given problem.

To program, the subject describes by the commands of the computational language, the problem resolution, and the computer executes the commands on the screen immediately. The subject may confront and receive feedback, about what he thought and described through the result expressed on the computer screen.When the result is the expected one, the learning spiral can continue without necessarily provoking reflection and debugging about the problem resolution.But if the outcome is different from the expected, the subject (either spontaneously or with the mediation of the teacher) is to reflect on the problem description and debug what he thought regarding the application of concepts (mathematical or computational) and strategies.

In this cyclical movement in a spiral, when the outcome is different from the expected, it means that something is wrong, but in the activity of programming, the error has a different connotation, it is seen as part of the learning process.The analysis and understanding of error is what lead to debugging leading the subject to think about the content represented and its manner of representation.

It is noteworthy that in the process of programming, in every moment of encoding and decoding occurs the translation : the mathematical language being encoded into the language of software. This movement requires from the subject the mathematical understanding and its representations. Hence the potential of the programming activity in the process of construction of knowledge.

This paper presents two episodes occurred in the course of continued education of mathematics teachers using the programming activity. In one of the episodes, the Geogebra software was used, involving the concept of Axial Symmetry (Pupo, 2013) ; in another, the Scratch programming language was used for creating a microworld exploring the idea of generalization (Rocha, 2013). We highlight in these episodes, the aspects that showed the potential of the programming activity and the joints with the mathematical and pedagogical content.

This mental process is more complex because it requires thinking in the mathematical concepts involved and at the same time, in the form to represent them, translating them into the logic of computationallanguage. When this occurs, it means that there was integration between the knowledge of specific content, in this case, mathematical and technological (computational).

Episode 1 : Programming with Geogebra

In this example,the researcher-trainer hid the "Symmetry" button of Geogebra to problematize a situation in which teachers should build (program) a button, which had the function of finding the symmetry axis of a line segment.

For this, the teachers wrote the script of the resolution relating the properties of the concept of axial symmetry, and the logical structure of the programming language, without viewing the functions (Points, Lines and Circles) available in Geogebra. For example :

This mental process is more complex because it requires thinking in the mathematical concepts involved and at the same time, in the form to represent them, translating them into the logic of

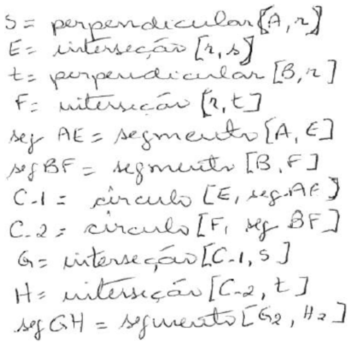

FIGURE 5.35 – Example of script written by a teacher out of the computer

computationallanguage. When this occurs, it means that there was integration between the knowledge of specific content, in this case, mathematical and technological (computational).

Episode 2 : Programming with Scratch

In this example, the teacher-researcher developed actions for the teacher to learn by doing through the programming with Scratch Language. The pedagogical focus was to foster the construction of micro worlds on mathematicscontent, more specifically exploring the generalization by default with numerical sequences.

In this process teachers raised a number of conjectures about problem situations that could highlight the regularities of numerical sequences. Then they went on describing their algorithms using language commands for creating the micro world, articulating mathematical and computational concepts.

However, in this process of creating the micro world, the teachers were urged to review and think about their own pedagogical practice. This happened during programming, when the teacher used the conditional "if" to foresee how the program could give feedback to the student.This feedback had to be programmed in case the student when interacting with the micro world did not set the resolution of the problem addressed. The teacher should describe, via language commands, a kind of suitable intervention to facilitate the student learning.That moment was crucial for triggering the reflective process about the teacher practice, involving analysis of what might have caused the error of the student, i.e., what gaps the student had regarding the mathematical content involved, and how the teacher could intervene so that those were overcome.

This episode showed that there was an integration of specific content knowledge, in this case, mathematical, technological (computational) and pedagogical of the teacher.

Final considerations

The two episodes have shown the potential of the programming activity in terms of having provided the teacher in the context of continued education to experience the appropriation of digital technologies going beyond the operationalmastery of their resources.

At the focus on that such research were developed favoring formative actions based on the constructionist approach, we found the possibility of the teacher of Basic Education to rebuild his professional knowledge from the perspective of TPACK (Technology, Pedagogy and Content knowledge) from Mishra and Koelber (2006).

Understanding TPACK and its incorporation in the proposed continued education courses for Mathematics teachers, according to the authors (Prado and Lobo da Costa, 2013) have contributed to the integration of digital technologies into the curriculum, specifically mathematics, as well as encourage the

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5.11 Early childhood spatial development through a programmable

Cristina Sabena

Résumé : Cette contribution porte sur les relations entre espace vécu dans l'expérience quotidienne vs l'espace comme une notion mathématique. S'appuyant sur la recherche psychologique sur la conceptualisation spatiale chez les enfants, quatre expériences ont été menées à l'école maternelle. Les expériences ont été réalisées avec un robot programmable avec une abeille-forme. De l'analyse macro et micro des activités, les résultats montrent i) la façon dont les activités robots ont un potentiel didactique à l'égard de la coordination entre les systèmes de référence égocentriques et allocentriques, ii) le rôle crucial de la parole orale et du gestes dans l'exécution des tâches de résolution de problèmes impliquant les relations spatiales avec le dispositif programmable, et iii) l'émergence de deux conceptualisations spatiales différentes au cours des activités : une est statique et global, et une autre dynamique et basé sur les chemins.

Abstract : This contribution addresses the relationships between space as lived in everyday experience vs space as a mathematical notion. On the background of psychological research on spatial conceptualisation in children, four teaching-experiments have been conducted in kindergarten school, in order to investigate children spatial conceptualization. The teaching-experiments have been carried out with the use of a programmable robot with a bee-shape. From the macro and micro analysis of the activities, results show i) how the robot-based activities have a didactic potential with respect to the coordination between egocentric and allocentric reference systems, ii) the crucial role of verbal speech and gestures in carrying out problemsolving tasks involving spatial relationships with the programmable device, and iii) the emergence of two different spatial conceptualizations during the artefact-based activities : a static and global one, and a dynamic and paths-based one.

Introduction

In recent times, research on early years mathematics is emerging (see for instance the new Thematic Working Group in CERME, http ://www.cerme8.metu.edu.tr/wgpapers/wg13_papers.html), with a great attention devoted especially to the development of whole numbers competences, as witnessed also by the forthcoming Icmi Study to be held in Macao 2015.

This contribution focuses on the development of spatial conceptualization in young children, addressing the delicate relationship between space as lived in everyday experience vs space as a mathematical notion. On the background of psychological results on spatial conceptualisation in children, and taking a multimodal perspective on mathematics teaching and learning, an experimental study has been conducted in kindergarten school. Using the teaching-experiment methodology and qualitative data analysis of video-recordings and written materials, the study explored the didactic potentialities offered by a programmable robot with a bee-shape, with respect to children development of spatial competences.

Theoretical framework

The complexity of children conceptualization processes related to space has been pointed out by research in phycology and education for several years. Great differences in different theorizing in the field prevent researchers from reducing these processes to simple and linear models of learning, based on rigid pre-determined steps. Concerning spatial relationships, we can consider three different fields of experiences, which correspond to three different kinds of space, requiring each specific perceptive and exploration modalities (Bartolini Bussi, 2008) :

- The body space, that is the internal reference frame relative to the awareness of body movements, its parts, and to the construction of the body schema;

– Specific external spaces, including different kinds of living spaces (the house, the town, the school,...) and different representative spaces (the sheet of paper, squared papers, the computer screen, ...) ;
– Abstract spaces, that are the geometrical models developed within mathematics science in its history.

The first two kinds of spaces refer to actual spaces in real world, the latter one belongs to the world of mathematics. Such a categorization must not be thought as a sort of hierarchical scale, or as a developmental sequence. On the contrary, according to Lurçat, "it appears difficult to imagine a development in which the body schema is constructed before, to allow then the knowledge of external world" (Lurçat, 1980, p. 30, translation by the author). As a matter of fact, several studies (Lurçat, 1980 ; Donaldson, 2010) agree in recognizing a fundamental role in the experiences that the child does both in his/her family and in specific educational settings (such as the school, included kindergarten), and go beyond linear models, which position abstract space at the end of a developmental process (in the stage of formal operations, in the Piagetian case). Recent strands in cognitive sciences place perception and everyday experiences with the body as grounding pillars for more abstract knowledge conceptualization, included the mathematical knowledge. In particular, the embodied cognition perspective (Lakoff & Nùñez, 2000) proposes a model for the "embodied mind", as a radical criticism of the dualism between the mind and the body of classical cognitivist approaches.

If mathematics is no longer a purely "matter of head", it becomes of paramount importance to carry out mathematical activities in suitable contexts in which children can interact with different kinds of space and spatial thinking. Concerning the external space, we can distinguish between macrospaces and micro-spaces (Bartolini Bussi, 2008) :

– macro-spaces are those in which the subject is embedded (the subject being part of the macro-space) ; their exploration is carried out through movement, and their perception is only local and partial, requiring usually to coordinate different points of views ;
– micro-spaces are external to the subject ; their exploration is carried out through manipulation, and their perception is global.

A park is an example of macro-space, whereas a sheet of paper and a book page are examples of micro-space. As an intermediate category, called meso-space, can be considered the big posters often used in classroom for group-work : children can enter into them, but also look at them at distance. The essential aspects in this distinction are the different modalities of perception and exploration : the school garden, for instance, can be an example of macro-space-when the child is playing within it-or of micro-space, when the child is observing it from a window above.

The body space and the external space differ from abstract space, in that they can be perceived and explored, but also by being featured by fundamental directions (vertical and horizontal) and by typical objects (e.g. a door in a room, a fridge in a kitchen). On the contrary, geometrical abstract spaces do not have any privileged direction (they are isotropos), nor special points (they are homogeneous). When reference systems like the Cartesian ones are introduced, not only metrics, but also privileged points (the origins, the axis) and special directions are established in the geometrical space. This kind of reference system can be considered objective or absolute, in the sense that it does not depend of the position of the subject using it. Objective references are the product of the historical-cultural development of society and need to be introduced by the teacher starting from the subjective references, which depend on our positions and according to Lurçat (1980) also on our ways to project our body schema into objects. Subjective reference systems can be egocentric, if the description is provided according to the subject position (e.g. "to my left") or allocentric, when the reference is made with respect to another object or person (e.g. "to the left of the house"). While Piaget and Inhelder in the fifties (Piaget & Inhelder, 1956) claimed that children until 8-9 years of age are incapable of decentralize with imagination and so of correctly using allocentric references, following studies have refuted this conclusion, and proved that also children aged 3 are able to decentralize, if faced with problems comprehensible to them (for a discussion, see Donaldson, 2010).

On the base of this discussion of results from psychology, we can ground the hypothesis that the reality faced by young children (and indeed, by all of us) is full of cognitively-different spatial contexts, which need to be addressed in specific ways in order to develop the related specific competences. As a matter of fact, as Lurçat says,

not all spatial behaviours necessarily imply a knowledge on space. In order to have knowledge, a suitable activity is necessary : for instance, going in a place, locating objects, positioning in the space of places and objects [...]. As in other pshichical fields, it does not exist an age for the development, which can be considered independent from the concrete conditions of existence (Lurçat, 1980, p. 16, translation by the author).

An educational implication of this perspective is that in order to develop the necessary different spatial competences, children are to be involved since their early childhood in dedicated activities with dedicated task design. For instance, in order to foster the passage from ego-based to allocentric references and lay the foundations of objective reference systems, is it possible to carry out already in the kindergarten activities on the change of points of view, as the realization of maps of familiar places.

Along with meaningful experiences in different spaces, language constitutes a second fundamental source of knowledge. The key role of verbalization not only as a communicative means but also for thinking processes has widely been discussed in Vygotskian studies (e.g. Vygotsky, 1934). Focusing on spatial development, Lurçat (1980) points out that *It seems hard to separate, in the appropriation of the environment realized by the young child, these two sources of knowledge, the one practice, the other verbal, since both converge early in the first months of life* (ibid., pp. 15-16, translation by the author).

More recently, the role of embodied resources such as gestures, gazes, and body postures in thinking processes (and of course in communicative ones) has been pointed out in psychological literature with cognitive and linguistic focus (McNeill, 1992, 2005). For what concerns spatial tasks, iconic and pointing gestures come to the fore : iconic gestures are those ones which resembling the semantic content they refer to, pointing gestures are usually performed with the index forefinger and have the function of indicating something in the actual context. The study of gestures and embodied resources in synergy with verbal language has gained a certain attention also in mathematics education, in an increasing variety of contexts, such as : students solving problems (Radford, 2010), students and teachers interacting (Arzarello et al., 2009 ; Bazzini & Sabena, 2012), the teacher's lectures (Pozzer-Ardenghi & Roth, 2008), considering not only the semantic but also the logical aspects of mathematical thinking (Arzarello & Sabena, in press). In particular, I refer to the multimodal approach, which underlines the coexistence and interplay of the diverse intertwined modalities as semiotic resources in the classroom context (Arzarello & al., 2009 ; Sabena et al., 2012).

Besides language and embodied signs also symbols, drawings and graphical representations of various kinds acquire great importance in mathematics activity and mathematics education. Written signs situate in a very specific way in the child's space of reality : generally, school lessons heavily exploit bi-dimensional micro-spaces, such as the blackboard, the book sheet, or more recently the computer/tablet screen. The passage from experience and perception in the tri-dimensional (macro-) space to these representation spaces is a very complex process, so far little studied in literature. Also at school, this passage is often taken for granted and in many cases written representations are used but not problematized. In such a passage, the use of artefacts can be exploited as didactic resources in the development of children spatial competences.

Methodology

On the base of the outlined theoretical frame, an experimental study has been planned and carried out in a kindergarten school in Northern Italy. The study is based on the teaching-experiment methodology, and the activities have used a programmable robot with a bee-shape (Fig. 5.36, right), a technological artefact new to the children. The robot is a kind of tridimensional version of the Logo turtle : it can move on a plane with 15cm-long steps, which can be programmed through keys placed in its upper part (Fig. 5.36, left). Besides onward and backward steps, also right and left turns and a pause of one second can be programmed. A specific button ("clear") allows the user to clear the memory from past commands.

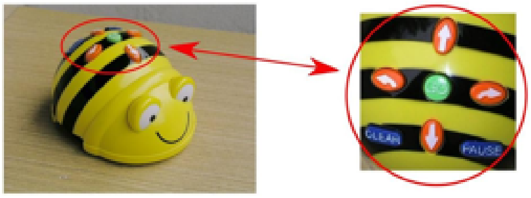

FIGURE 5.36 – The programmable robot used in the teaching-experiment.

The teaching-experiments were inserted in the usual schedule in the classrooms, with the collaboration of four teachers, four Master students in Primary school education, and the author. Being inserted in the school activities, the experiments had didactic as well as research goals. From a didactic point of view, the activities had the general goals of promoting competences related to problem-solving, logical and spatial thinking. These competences were linked to the use of a new artefact, in the context of exploring it through a playful environment. Concerning spatial thinking, the passage from egocentric to allocentric reference systems was particularly at stake. A new feature for the children was the possibility to program in advance the movements of the robot, and to check their choices, by means of observing the actual movement. Anticipation and control processes could therefore be stimulated and developed.

Children were organized in groups of about 10-12, with one or two bee-robot at disposal. After an initial try with groups of 3-4-5 years old children, the experiment was continued only with 5-years old children, corresponding to the last year of kindergarten. The robot, in fact, resulted too complex for younger children to be used according to its commands.

For each group, the activities developed along 5 to 6 one-hour meetings [1], for a period of about one month. Most of activities involved the whole group, with the coordination of the teacher, and only in some cases individual work was required (e.g. to produce a drawing). The first meeting was always dedicated to the introduction and exploration of the new artefact. Due to the exploration character of the experimental study, in the following meetings different kinds of path were built or chosen, to be travelled by the bee-robot. Only in one classroom, a specific written symbolism was also introduced in order to record the programmed steps. In spite of different choices regarding the paths for the robot, some key-features were agreed with the teachers and kept in all groups : the active participation of the children in social context, a playful atmosphere, the alternation of activities with the artefact with reflection phases, and the attention to verbalization and to multimodal resources.

This didactic dimension intertwines with the research one. The study had mainly an explorative character of the potentialities of the artefact-based activities with respect to

1. In Italy, usually we use the term "lesson" starting from Primary school, were formal education begins (also with textbooks, notebooks, and so on). In kindergarten, activities unfold in a less formal way. To keep this specificity, I use the term "meeting".

- the change between different reference systems, considering in particular egocentric vs allocentric and subjective vs objective references ;
- the passage from experiences in macro-space to the use of graphical representations in micro-spaces ;
- the activation of anticipation and control processes ;
- the role of natural language and multimodal communication, and the exploitation of these resources by the teacher as didactic resources ;
- the introduction of symbolic codes.

The activities have been video-recorded and the obtained videos have been analysed in detail. Furthermore, children written drawings related to the activities have been collected and analysed.

Analysis

The initial exploration of the robot has been carried out letting the children play with the robot, and deciding how to move it. The resulting movement was therefore not pre-determined by the teacher with specific paths. In some groups, the activity was organized around a table, while in others children were sitting in a circle on the floor (Fig. 5.37, left) : the resulting delimitation of space produced a sort of meso-space, since the children could globally perceive it with their sight, but also enter into it and explore it with their body. One of the games played in this context was "sending the bee to my friend (name)". In this game, each child had to name a friend, and to program the bee so to be able to send it where stated. Everyone always started positioning the bee in front of him/her, parallel to him/her, hence the reference system introduced by the robot (allocentric system) was initially coincident with the child one (egocentric system). In all cases, children chose friends sitting opposite to them, and programmed only straight routes. Quite often, the estimated number of steps was insufficient to cover the distance to reach the goal : the child had to move and program the robot again. Also in this case, we observed that children positioned their body and head in a way to have the same point of view of the bee, i.e. the same reference system (Fig. 5.37, right).

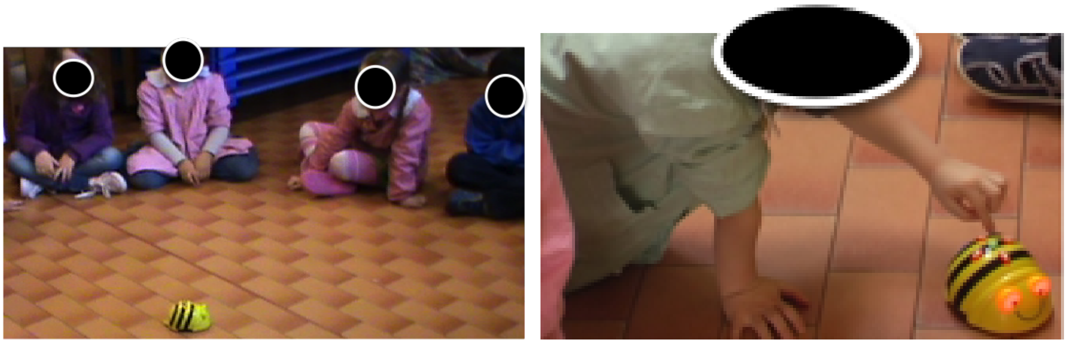

FIGURE 5.37 – Initial exploration of the artefact in the meso-space.

This is the most natural choice, which keeps the cognitive load low. We decided to keep this choice in those activities that focused on specific aspects of the artefact, for instance the lengths of its steps, compared with human steps (Fig. 5.38).

Other games required the imitation with one own body of some movements made by the robot, without any verbal description. The imitation is simple if the child is oriented in the same way of the bee-robot (for instance, if the child is following the robot), because the ego-based reference system is the same as the robot. When the robot is oriented differently with respect to the child, we noticed

FIGURE 5.38 – Egocentric perspective kept during the comparison of steps lengths.

many difficulties. The complexity of the task (proposed without the mediation of the speech) lies in the fact that the children have to adopt not only another reference system, but also a mobile one : the coordination needs to be continuously re-established and controlled, in particular after turns. Only the recourse to verbal utterances to describe the robot' motion (such as 'three steps onwards, turn right, two steps onwards') could help the children to correctly relate the robot movement to his/her own movement. However, verbal indications were of little help for children with difficulties in knowing right from left (a problem for which the bee-robot could not offer any support).

With each group of children, at least one meeting was dedicated to an activity on a poster showing a path to be travelled by the robot. The paths were all structured with lengths multiple of 15 cm (the exact dimension of the robot, and of its steps) and with right angle turns, so to be viable by the robot in an exact number of steps and rotations. These choices were meant to aks the children to program rotations, but at the same time to avoid any problem provoked by non-perpendicular directions. An example is in Figure 5.39, left, showing the "Bee game"[2], a sort of Snake and ladders game. The game setting facilitated the introduction of the rule of 'moving the bee only through its buttons' (and not pushing or rotating it with the hands, as the children were tempted to do...). In our intentions, the race setting would have also fostered the need of programming as many segments of the path as possible, in order to reach a farther place. For instance, if the first roll of the dice gives '3', the children have to program the sequence 'two onwards, turn left, one onward'. However, in our experiments the children did not fulfil this expectation. Indeed, in all groups children preferred to program one segment at a time : in the mentioned example, programming two steps onwards, observing the robot movement, then programming one turn leftwards, observing the turn, and then programming the final two steps. Figure 5.39, right, shows a child while programming this last segment : again, the ego-based perspective is taken by the child in order to carry out the task. Probably we missed the occasion of challenging the children, by introducing an additional rule, such as 'programming the robot sitting always in the black arrow place'. This request would have forced the children to coordinate their egocentric perspective with the moving perspective of the robot (allocentric for the children). Due to the complexity of the task, together with the teachers we preferred not to interrupt the game, and to let the children playing in the way that was more confortable to them.

In another group, the activity with the path (Fig. 5.40) was introduced by a collective discussion guided by the teacher. The bee-robot was not on the scene : the discussion constituted a moment of guided reflection for the children, during which the development of the spatial competences is realized by observing and describing the present scene, but also recalling past experiences with the artefact, and anticipating potential actions through imagination.

2. In Italian the popular game "Snake and ladders" is called "The goose game".

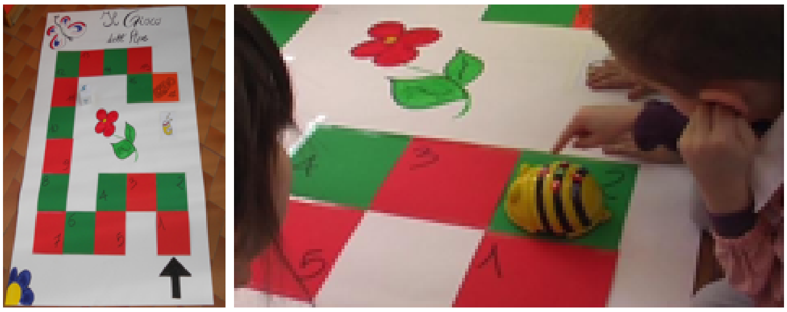

FIGURE 5.39 – 'The bee-game' : Ego-centric perspective to program the movement.

FIGURE 5.40 – The setting of the activity 'Let's help the bee to reach the flower'.

In the following excerpt the beginning of the discussion is reported :

1. Teacher : Today we explore this (looking at the poster). What comes to your mind by looking at this'
2. Stefano : It is a road
3. Viviana : A flower and a house
4. Teacher : And whose is the house ?
5. All the children : The bees !
6. Guido : Because bees live in the flowers
7. Teacher : And how is it this road ? She is silently making a pointing gesture moving her forefinger horizontally, Figure 5.41. Is it straight'
8. All children : Noooo !
9. Stefano : It has some curves (with his hand he is traveling the road, Fig. 5.42)
10. Cristina : It makes like this, then like this (pointing to the road ; also other children are pointing to the road, Fig. 5.43)

Other children do not make any verbal comment, but touch the entire path with their hands (Fig.5.44).

The teacher's questions have the goal to help the children becoming aware of the characteristic of the road along which they will make the bee travel. Though not explicit, they play an important role with respect to the anticipatory thinking needed to program the robot. Furthermore, the children are sitting all around the poster (Fig. 5.40) : the passage from an egocentric perspective to an allocentric perspective is therefore required for the majority of children in order to imagine the bee moving in the path.

FIGURE 5.41 – The teacher's pointing gesture.

FIGURE 5.42 – Stefano's gestures on the road.

FIGURE 5.43 – Pointing gestures to the road.

FIGURE 5.44 – Touching the road.

In their answers, the children initially indicate static elements : the road, the flower, and the house (lines 1-6). Then, asked to describe the road (line 7), they provide a dynamic description, apparent not only in their words, but particularly in their gestures (Fig. 5.42, 5.43, 5.44). The passage from static to dynamic description has been prompted also by the teacher's intervention, exactly by her gesture (Figure 5.41), which consists in a pointing gesture moving slowly horizontally and indicating the path on the poster.

The children verbal descriptions are very poor (besides the indication of curves, only deictic terms), and can be understood only considering the co-timed gestures ; the teacher uses then a trick to push them to elaborate more precise descriptions :

FIGURE 5.45 – Fabio's pointing gesture indicating the arrival.

1. 11. Teacher : And then' Let's do like this : I close my eyes and you tell me how is the road, because I do not know it... Is there a starting point' And an arrival' Explain to me.
2. Fabio : The start is in the house and maybe over there (pointing gestures, Fig. 5.45) where there is the flower it's where the bees go, to make honey, it is the arrival.
3. Teacher : But in this way I would not be able to arrive : you must explain well.
4. Fabio : You must go straight (pointing gesture, Fig. 5.46, left), then turn (moving his entire body, Fig. 5.46, second and third, and making a turning gesture with right hand, Fig. 5.46, right), go still a bit straight, then turn again, go straight and you are arrived at the flower.

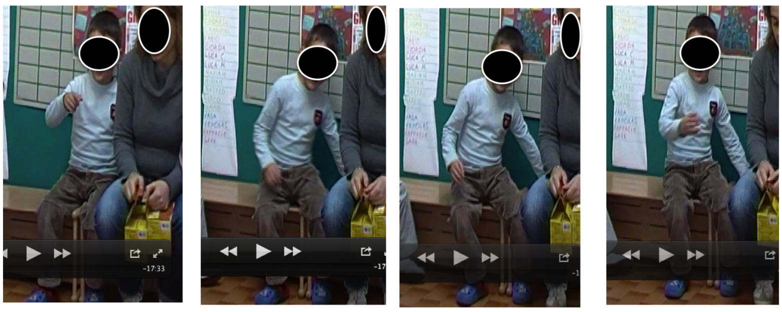

FIGURE 5.46 – Fabio's gestures accompanying the two new spatial descriptors "straight" and "turn". In pictures b-c-d a body rotation is visibly accompanying the hand gesture.

5. Teacher : But I don't know where to turn, how can I understand which part to turn...The children continue to explain mainly with gestures, as in previous lines.
6. Teacher : No, no, if you had to explain it only with words
7. Chiara : Left and right
8. Teacher : Left and right, or towards...Chiara, try ! (The teacher closes her eyes)
9. Chiara : First the bee can start from the flower (pointing gesture, Fig. 5.47) and then makes some curves (other pointing gestures ; the teacher opens the eyes and looks at Chiara) to go home.

10. The teacher gestures in the air a curving path (Fig. 5.48), producing with a voice the sound of a bee. The children are surprised and amused. The teacher adds at the end : Eh no no : explain better, come on, I think that you can make it. Not "I make some curves", but how many curves... I go straight and for how long, rightwards, or towards the benches, towards the door...
11. Chiara : Left and right !
12. Teacher : Left and right... go on Luca, try !
13. Luca : It starts from the house, goes straight, then turns leftwards (Fabio is helping him to decide between right- and left-wards), after it goes a bit straight and then right and it arrives.

FIGURE 5.47 – Chiara's pointing gesture.

FIGURE 5.48 – The teacher's gesture.

In this excerpt we can see the great difficulties met by the children in making a description of the road. The teacher suggests identifying some reference points, such as the starting point and the arrival (line 11). Even if she is visibly keeping her eyes closed, Fabio (lines 12 and 14) and the other children continue to use gesture as a main communicative resource accompanying speech. As shown also by previous gestures (Figg. 5.42, 5.43, 5.44), the house is assumed to be the starting point of the bee path. In line 12 this aspect is made explicit the child, who accompanies his words with a pointing gesture (Fig. 5.45) indicating the flower as the arrival.

In line 14 Fabio, after the teacher's prompt, introduces new descriptive spatial words : "go straight" and "turn". The introduction of the two words is accompanied with two specific gestures : the former is a static pointing gesture made with the extended right forefinger (Fig. 5.46, left) ; it indicates the initial straight segment of the path. The latter is a dynamic gesture made with the full right hand (Fig. 5.46, right) combined with a body rotation. The body rotation is prepared by a little detachment rightward from the Master student (Fig. 5.46, middle), providing the boy with the necessary space to carry out a little leftward rotation. The body movement and the hand gesture are the only semiotic resources expressing the information about the sense of the rotation (leftwards). Then Fabio concludes his description using only words, without specifying any quantification for the straight parts, nor the directions of the turns.

The teacher is constantly pushing towards richer verbal descriptions and in line 16 this goal is made explicit to the children ("explain only with words"). However, it is still not clear for all them : see Chiara still strongly relying on pointing gestures in line 19, Fig. 5.47. At this point the teacher performs a sort of mocking imitation of the descriptions provided by the children (line 20), using both gestures and voice sound : in this way, she is helping the children to become aware of the semiotic resources that they are using, in spite of her request to give verbal descriptions as precise as possible. Right after this pars destruens, the teacher gives a constructive support (pars construens) providing additional linguistic terms to be used. At a non-locutionary level, she is establishing which semiotic resources are to be used in the task, and also helping the children in the chosen semiotic system (the verbal one) [3].

If we focus on what the teacher suggests to consider, we see that she is mentioning to quantify ("I go straight and for how long"), to refer to subjective references ("rightwards"), but also objective ones ("towards the benches, the door"). The children will pick up only the subjective references, as Chiara and Luca in lines 21 and 23. In a Vygotskian perspective, we can say that the subjective reference system is in the Zone of Proximal Development for the children, whereas the objective ones is still outside it. The quantification aspects will be taken into account in the subsequent activity on the poster with the bee-robot : its steps will constitute the unit to measure the lengths of the straight parts of the path. Figure 5.49 reports two paths invented and drawn by children, to be followed by the bee-robot. Most children keep the square shapes as in the poster actually used (as in Fig. 5.49, left), whereas few others decided to draw only one single line (as in Fig 5.49, center). The invented paths have often some turning points and some numerical indication referring to the number of steps to be programmed is each trait. There are of course also cases with simpler drawing, as in Figure 5.49, right (showing also difficulties with the numerical sequence).

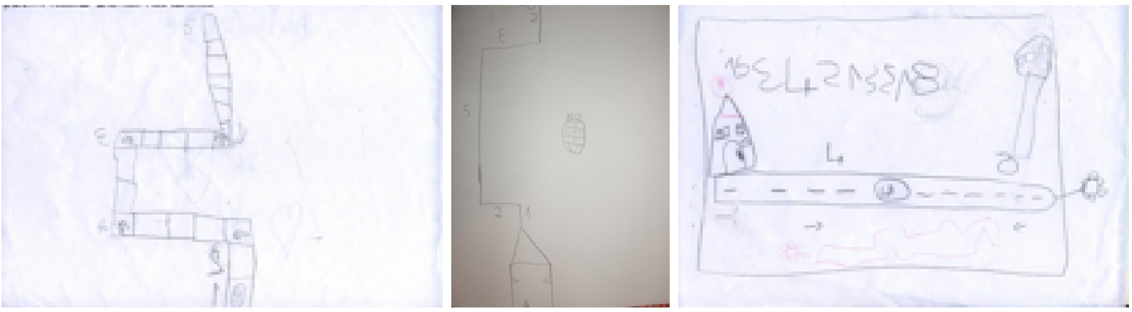

FIGURE 5.49 – Some paths drawn by the children for the bee-robot.

Discussion

The analysis has shown some potentialities of the activities with the bee-robot artefact concerning early childhood spatial development, and mathematical thinking more in general. A first remark regards the intertwining and coordination between egocentric and allocentric perspectives assumed

3. In pragmatics frames, a non-locutionary level is distinguished from the locutionary one : the latter being "what is said", the former "what is conveyed through what is said" (Austin, 1975). The synergic use of the gestural and linguistic resources made by the teacher could be described as a sort of "semiotic game" as described in (Arzarello et al., 2009) and carried out at non-locutionary level. This aspect is only mentioned here since its discussion would divert from the focus of the paper.

by the children in the activities. The egocentric perspective is very often taken by the children to face the proposed tasks. Of course, in order to make sense of what their mates or the teacher were doing with the artefact, the children were often in the need of coordinating their ego-based perspective with the allocentric one assumed by the robot. However, our findings suggest that specific constraints have to be set up on the task in order to 'force' the children to actively work with allocentric perspective : for instance, have the children to imitate the movement of the robot when is not parallel to them, or to program it from a certain place (e.g. from the starting place of a game). Another important feature about reference systems is that the robot constitutes a mobile reference, requiring the children to continuously re-establish and control the coordination, in particular after each turn.

Both ego- and allo-centric perspectives are subjective reference systems, used in the space of reality. As discussed above, geometrical space requires the use of objective references. In the proposed activities, we did not focus on the passage from subjective to objective references. Some hints have been made by the teachers (as the one documented in the excerpt above), but with no success. Our impression is that specific activities need to be designed in order to reach this goal : our future research will try to clarify under which conditions and to what extent it can be reached, and how the various multimodal resources play a role therein.

A second remark concerns two different spatial conceptualizations that emerged during the artefactbased activities : a static and global one, and a dynamic and paths-based one. Considering the excerpt above, the children at first refer to static elements, when they mention the house, the flower, and the road in describing the poster (lines 2-6). Soon after, and fostered by the teacher's intervention through gestures (Figure 5.41) and words (lines 11, 13), a dynamic perspective is brought to the fore : the children use dynamic pointing gestures (also materially touching the poster) and then words referring to the motion along the path (e.g. Fabio in line 14). The two perspectives do not constitute a dichotomy. For instance Stefano in line 9 is blending both of them : his words are referring to a global feature, and the gestures (Fig. 5.42) expressing dynamic ones. In the overall experimentations, gestures have often offered a window into the children's conceptualization of space, and new spatial terms have often been used the first time accompanied by corresponding gestures (as Fabio in line 14).

Evidence of how the experience with the robot paths has influenced the children's conceptualization of space can be seen also in several drawings. I report in Figure 15b and c the drawings made by a group of children, who had used a grid made by straight lines (Fig. 5.32a). In the children's drawings, the grid looses its global features and becomes a sequence of steps.

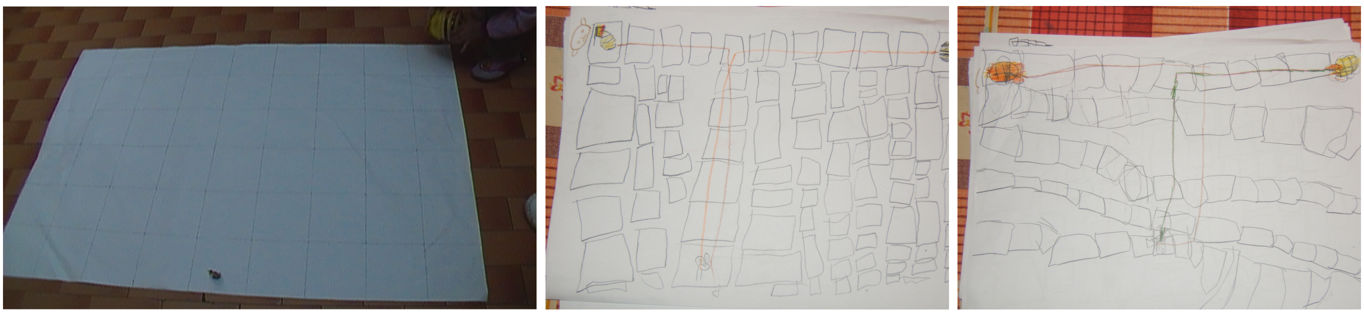

FIGURE 5.50 – The grid and children drawings related to the activity with the grid..

The paths-based perspective has been certainly fostered by the use of the bee-robot, and future research is needed to investigate how it can emerge in other kinds of spatial activities. However, studies in cognitive science within the embodied mind approach have shown that motion constitutes the source domain of many concepts, and that also static objects are often conceptualized in terms of motion4 (Lakoff & Nùñez, 2000).

A last remark concerns more generally the use of programmable robot as didactical means with kindergarten children. In our experiments it became soon apparent that the children did not have much interest in programming using different commands in sequence such as 'three steps forward, turn right, two steps forwards'. Whenever asked of reaching a certain place with the robot, the children adopted the strategy to break down the path into small segments, and to program each of them at a time, observing the separated outcomes. Probably programming an entire long sequence requires cognitive capacities still under construction by the children, who have limited attention, and limited capacity of considering many variables at the same time. But maybe the main difficulty lies in the fact that the goal of reaching a certain place through a single program sequence had not any understandable 'sense' for the children (Donaldson, 2010). We could observe that even when 4 Talmy (2000) has called 'fictive motion' the cognitive mechanism underlying the description of a static object (e.g. a path, in our example) in motion terms (e.g. 'it starts… it goes…'). this goal was proposed within a competitive setting (like a team competition), the children did not undertake it. As a matter of fact, programming a certain artefact using less time as possible can be a goal for adults, which are often under time pressure. In the case of children, pleasure was given in using the robot as long as possible, as in any game. In our task design, we initially underestimated this essential dimension, and not a few times the goals that we had chosen for the activities were completely neglected by the children. Children taught us a lot by neglecting our goals.

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5.12 The use of technology when teaching about the equal sign

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Résumé : En Suède, l'enseignement individualisé est privilégié pendant les cours de mathématiques. Cependant, des études récentes ont montré que les différences dans la réussite des élèves en mathématiques sont en grande partie dues à la qualité de l'enseignement. Cet article traite de la notion de l'objet d'apprentissage par rapport aux objectifs [1] (objectifs d'apprentissage) comme un moyen de promouvoir l'apprentissage des mathématiques. Une étude dans une salle de classe de quatrième où les élèves ont des difficultés avec le symbole égal est utilisé pour discuter comment la technologie peut affecter l'apprentissage. Le cadre théorique de l'analyse de cette étude est la théorie de la variation. L'hypothèse théorique est que l'apprentissage est toujours l'apprentissage de quelque chose, de sorte que la probabilité d'apprendre suppose une expérience de la variété. L'enseignant dans l'étude a utilisé un « tableau blanc »pour soutenir l'apprentissage du symbole égal chez les élèves. Cependant, la manière restreinte dont la technologie a été utilisée a abouti à ce que les élèves n'ont pas reçus les expériences nécessaires de la variabilité essentielles pour obtenir une conception plus large de la notion et comment elle a été utilisée.

Abstract : In Sweden, the focus in mathematics lessons has been on individualising teaching. However, recent studies have shown that differences in students' mathematics achievement are largely due to the quality of the teaching. This paper discusses the notion of the object of learning in relation to goals1 (learning objectives) as a mean to promote learning of mathematics. A study in a Year 4 Swedish classroom where students had difficulties with the equals sign is used to discuss how technology may affect learning. The theoretical framework for the analysis of this study is variation theory. The theoretical assumption is that learning is always the learning of something, so the ability to learn presupposes an experience of variation. The teacher in the study used an interactive whiteboard to support the students' learning about the equals sign. However, the restricted way the technology was used resulted in the students not being provided with the necessary experiences of variability required to gain a broader conception of the concept and how it was used.

Introduction

In Sweden, a new curriculum, known as Lgr11 (Skolverket, 2011) was implemented in 2011. This was a direct result of Sweden's falling standards in international comparison tests.

Important starting points for changing the curriculum have been partly national and international didactic research, and the results of Skolverket's national evaluation of the teaching of mathematics, NU-03. Also international evaluations of Swedish pupils' mathematical skills, TIMSS and PISA, have formed a basis. Additional important points have been analyzes of the results from tests in mathematics for grade 3, grade 5 and grade 9 and Skolinspektionens review of the teaching of mathematics in 2009 (Skolverket, 2012, p.6, my translation).

One difference with this curriculum compared to the previous one, is the way the goals are described. An intention with the new curriculum was to emphasise the goals for acquiring core content. Lobateo, Hohensee, Rhodehamel and Diamond (2012) surveyed the mathematical content standards in curricula from seven countries and found that they emphasised procedural knowledge and did not emphasise conceptual knowledge. They argued that 'curriculum development has been marked by a tradition of identifying learning goals from the perspective of sophisticated mathematical expertise' (p.86), generally from a research understanding.

One of the goals in the content for pre-algebra in the new curriculum is about the equals sign- 'Mathematical similarities and the importance of the equals sign' (Skolverket, 2011, p. 60). However, it is well known that many students have conceptual difficulties with the equals sign (see for example, Bell, 1995). Developing an understanding of the equals sign as a relation is critical. Students need to understand that if the same number is added to both sides of the equation, the relationship between the sides remains the same (Kieran, 1992). Students who view the equals sign as an operation, read the information from right to left. Thus, they have difficulties with a task like $7 + 3 = \ldots + 4$, because it requires them to read the entire task before they can solve it. Students thus need to see the task as a whole and not as individual parts.

One explanation for why so many students consider the equals sign only as an operation is that it is introduced early, and then left without reflection in later school years. Teachers generally take for granted that students, once introduced to the concept, do not need any further instruction about it (Knuth, Alibali , Hattikadur, McNeil & Stephens, 2008).

Theoretical framework

The theoretical framework is variation theory (Bowdon & Marton, 1998 ; Marton & Booth, 1997 ; Marton & Tsui, 2004). It is a framework for guiding pedagogical design for improving teaching and learning (Lo & Marton, 2012). Learning is considered as being neither connected to the individual nor the social context, but to a conjunction of the two, that is a non-dualistic ontology that brings the learner and the learned phenomenon into the same world of experience.

In variation theory, learning involves discerning specific features of an object of learning and how an object of learning is pedagogically treated. An object of learning is a specific capability that students are expected to develop during a lesson. It is the compound of two aspects : the direct and indirect object of learning. The former is defined in terms of content, such as the formula for the triangle area (b'h/2), and the latter refers to the kind of capability the students are supposed to develop, such as being able to do something with the formula. For example, an object of learning is to develop the students' understanding of the triangle area. This is the direct object of learning. The indirect object of learning is how one wants the students to understand and elaborate with the direct object of learning. One way is to teach the students to remember the formula. Another way is to support the students to understand the formula, for example, by contrasting it with the formula of a rectangle, or by making them aware that every line on a triangle can become a base because the base is always perpendicular to the height. A learning object can be described in terms of a goal, but by taking into account the ways that the students already discern the object of learning it becomes empirically grounded. Lobateo & al. (2012) showed how empirically grounded conceptual learning goals serve as useful guides for teachers to better understand students' mathematical understanding.

The ability to learn presupposes an experience of variation. According to variation theory, learning is seen as a differentiation ; in contrast to for example a view where learning is seen as enrichment (similarities). For a learner to notice similarities one must discern what is alike and to discern something one must experience differences. Hence, differences can only be experienced through identifying similarities. It can be difficult for students to learn without variation (Donovan & Bransford, 2005).

The use of technology

Given the increased use of technology in mathematics classrooms, it is interesting to explore how it can be used to provide the variation needed for students to understand an object of learning, such as the equals sign. Although technology constrains the kinds of representations possible, newer technologies often afford newer and more varied representations and greater flexibility in navigating across these representations. (Mishra & Koehler, 2006,). In Sweden, many classrooms now have interactive white boards with an expectation by the government that use of these, like other technologies, will improve students' understandings. However, this simplistic assumption needs problematising. Technology, like concrete materials (Uttal & al., 1997), may not produce the mathematical concepts that teachers or governments expect. Research that suggests that teachers gain the most effective pedagogical advantage from Interactive Whiteboards if they use a range of interactive features (Miller, Glover, & Averis, 2005), assumes that good teaching is related to how many features of the Interactive Whiteboard are used. Instead one has to look for how the critical features can be made visible for the students.

Methodology

The data that is discussed in this paper is drawn from a longitudinal study (2010-1012) that was part of a professional development project based on Learning Study (LS). In this wider study, teachers, were interviewed about professional development in mathematics. The teachers had participated in at least one LS before. The group meet for two hours every other week during a period of two months. In this period, one of the teacher's mathematic lesson were video filmed three times. Each lesson was planned as well as revised by the group of teachers. Instruction of variation theory was provided to the teachers during the study. Immediately before the study started, the teachers had an Interactive Whiteboard installed in their classrooms. Although they had received some professional development on how to use the Interactive Whiteboard this was quite limited.

For this paper I re-examined one of the LS that took place in spring 2012. The class had children in Year 4 and was taught by an experienced female teacher. The data included three videotaped lessons on the equal sign. In the analysis, I looked for how the teacher made use of the Interactive Whiteboard in regards to the object of learning. E.g. how did the use of an Interactive Whiteboard bring to fore the object of learning?

Discussion

The teacher in the study anticipated that the students knew how to use the equals sign since they had worked with it previously and did not feel that they had any problems when completing calculations. However, a screening test showed that the students had problems with the meaning of the equals sign. Especially when the tasks did not resemble those from their textbook. A task of this kind was $36 + 28 = 40 + \ldots$. In the task, $\ldots - 4 = 4 - 2$, only a few students gave the correct answer, 6. Some students gave the answer 8 which suggests that they ignored the - 2, thus arriving at the equation $8 - 4 = 4$. More than half of the students gave 2 as the answer to the question. This suggests that the students considered the task to be an addition, $2 + 4 = 4 + 2$. Some student did not answer the question. Hardly any of the students answered the task $\ldots - 17 = 37 - 10$. The students recognised one side of the equals sign as the answer to the other, thus showing an operational, rather than a relational, understanding of equals sign.

As a result of this, the lessons were planned with the critical aspects identified in mind. The use of contrast, about how something cannot be, and the order of the numbers in addition and subtraction, was used in all lessons and were supposed to force the students to pay attention to why it has to be equal on each side of the equality sign. For example, the following examples were provided to the students :

$$8 + 10 = 10 + 8$$

$$8 - 10 = 10 - 8$$

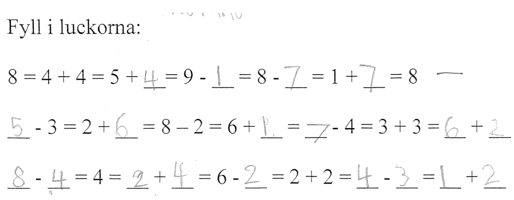

FIGURE 5.51 – Interactive Whiteboard

In this sequences the teacher used the Interactive Whiteboard in all three lessons. Here the Interactive Whiteboard became a hindrance to the students' understanding because of the way it was used. When the students gave examples for a solution on a problem put up on the Interactive Whiteboard, e.g. $8 - 10 = 10 - 8$ the solutions were not put on the board but only talked about until the students agreed that in was not an equality. Hence not using the technology to bring the students attention to the critical aspects needed to be discerned by the students. In order to make the critical aspect of the equal sign into fore the teacher could put up the students different answers on the Interactive Whiteboard in order to contrast the different answers. This was problematised in the meeting between the lessons but the teacher did not want to write something on the Interactive Smartboard during the lesson. She wanted to use the 'slides' prepared in advanced.

In all of the post-tests the students still discerned the equal sign as an operational sign.

Conclusions and implications

The screening results made the teachers change their beliefs about their students' knowledge about the equal sign. Consequently, the teacher started to problematize the goals and consider it as an object of learning. Although the teacher did provide students with different ways of conceptualising the equals sign, she did not seem able to use Interactive Whiteboard in more than a superficial manner. Given that an Interactive Whiteboard has the possibility for providing many different ways of conceptualising the equals sign, it is suggested that professional development with teachers needs to include possibilities for developing their and their students' use of different resources.

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