# WORKING GROUP A / GROUP DE TRAVAIL A

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**MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE**

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**MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE**

# Introduction to Working Group A / Introduction au Group de Travail A

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The group was organized with the aim of giving voice to all the participants through an initial and final video-recording of the possible and accomplished opportunities to reflect about mathematization and modelling of everyday context as a didactical principle. For the nine papers, we choose to limit the presentation to ten minutes, leaving 5 minutes for reaction prepared in advance by another participant of the group, and ten minutes more for a generalized discussion.

Discussions helped to analyse what qualifies a real-world context as a point of departure and/or a point of arrival of a didactic arrangement, the relevance of the authenticity of everyday contexts, which material arrangements support students' learning of mathematics by mathematization, and which epistemologies of mathematics are built into particular didactical principles of mathematization. The group felt that this allocation of time and roles was extremely productive, generating fruitful discussions and great involvement among the members of the group, as was illustrated in the last moments of the final video recording.

Reflecting on what qualifies a real-world context for didactic arrangements, differences among the active roles of students (from Kindergarten to High School), in-service teachers, teacher trainers, experts and researchers were discussed. Five of the nine papers addressed mathematization as a didactical principle with different emphases in the modelling process. Those didactical principles were: the interpretation and application of the theoretical notions of non-Euclidean geometry (experience authored by Bini); giving meaning to three keywords -whole, unit and quantity- (presented by Rottoli); diverse mobilizations in each of the different stages of the modelling cycle of the mathematical and physical epistemological knowledge (research work defended by Mouted); the transference of mathematical knowledge between micro-contexts in successive cycles of modelling (research co-authored by Tsitsos and Stathopolou); and *a priori* analysis of a problem posed in life science context (defended by Yvain). Two papers considered both the didactical and social principle of mathematization at the same time, placing at the core how a contextual situation is explored, modelled and validated jointly by researchers, teachers, students and experts (discussed by Giménez), or aiming to raise critical consciousness within social and cultural relevant contexts for students, teachers, teacher trainers and families (co-authored by Anhalt and Turner). A third paper enhanced the social and didactical principle of decisions made, using mathematically-informed thinking to empower students in everyday situations of risk and probability (presented by Serradó).

The didactical arrangements presented used a variety of different artefacts, physical experiences, materials or learning spaces to promote students' horizontal and vertical mathematization. Although, initially, the physical experience of preparing a fruit salad was designed as an opportunity to construct the notion of fraction, the discussion recognised the potentialities of the project to analyse the difference between discrete and continuous conceptions of number. Meanwhile, the fruit salad was an opportunity for the horizontal mathematization. Analysing the non-Euclidian geometry of an orange was convenient for understanding the mathematical model, as a consequence, deriving the formula for the surface area of a spherical triangle, and providing students opportunities for critical thinking about their interpretation as residents of a giant sphere. The reaction about the usefulness of the orange, as an artefact used in the virtual world to understand the real world, questioned the didactical possibilities of beginning with a discussion that is about the meaning of being a resident of the Earth to promote horizontal mathematization. In spite of understanding the conditions as favouring the devolution of horizontal modelling, *a priori* analysis of a "didactic engineering" activity - using the fictional reality about the growing of a tree - was discussed. Two didactical principles for this horizontal modelling were evidenced: the implementation of a didactical device, and the elaboration of a specific situation. In this case, the specific situation came from a biologic epistemological study.

Moreover, the physical epistemological study of the representation of space-time of Minkowski provided information about the differences of *a priori* analysis, using pen and paper and dynamic geometry implemented when transferring the mathematical results to the reality. The theoretical framework proposed by Mouted to analyse this transference considered three parallel planes: the physical and mathematical epistemological, and the cognitive. However, it was discussed whether it would be necessary to consider a transversal technological plane to understand the transitions between the epistemological and cognitive planes. Tsitsos' work suggested the importance of conjecturing, representing and validating mathematical practices through the use of physical artefacts and dynamic geometry. Those devices allowed understanding the mathematical model involved while abstracting the mathematical concept of variance.

To sum up, the important understanding of the relevance of the authenticity of the contexts that provide the basis for mathematization in realistic situations and modelling means widening the contextual view to the epistemological, cognitive, technological and social perspective. From a contextual view, the authenticity was analysed through the differences when real, virtual real, fictional real situations are the point of departure and/or arrival that builds on mathematization when involved in modelling processes. We also considered the authenticity of the representations, artefacts and theoretical references, which allow for interconnecting the extra-mathematical epistemological plane (biology, physic, historic, cultural, etc.) with the mathematical one through didactical arrangements. Those didactical arrangements (designed, implemented and analysed, *a priori* and/or *a posteriori)* examine the non-neutral intersection between the technological and cognitive plane. The cognitive authenticity should derive from the understanding of the purposes of social and cultural practices for actively engaging the different actors (students, teachers, teacher trainers, researchers, experts, etc.) in mathematization and modelling of everyday contexts.

This wider approach to the relevance of authenticity is a challenge to reflect about: how and when we take into account the extra-mathematical epistemological, technological and cognitive planes that intersect with the modelling process.

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# A didactic approach in mathematical modeling: raising critical consciousness within social and culturally relevant contexts

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**Abstract**. This paper reports on a year-long professional development project, Mathematical Modeling in the Middle Grades ($M^3$), that created unique opportunities for thirty mathematics teachers and teacher leaders in grades 5-8 to explore mathematical modeling through culturally relevant community contexts. The teachers were from rural schools near the U.S.-Mexico international border with diverse student populations. The work of $M^3$ focused on interconnecting mathematisation as a social process and as a didactic principle in a professional development setting. The goals of $M^3$ was to build teachers' background knowledge in mathematical modeling and to prepare them for implementing modeling tasks in their classrooms that focus on students' mathematics knowledge and leverage cultural and community contexts. Teachers learned about and utilized the modeling process while exploring local contexts in the communities in which they teach. They implemented modeling tasks which advantageously used their students' background knowledge as part of the solution process resulting in successful student engagement in the mathematical modeling process. The student models yielded creativity, interesting mathematics, and revealed their understanding of real-world community issues, thus raising awareness and critical consciousness.

**Résumé.** Ce papier annonce sur un projet de développement professionnel d'un an, un Modelage Mathématique en Qualités du Milieu (M3), qui a créé des occasions uniques pour trente enseignants de mathématiques et chefs d'enseignant dans les qualités 5-8 pour explorer le modelage mathématique par les contextes de communauté culturellement pertinents. Les enseignants étaient des écoles rurales près des Etats-Unis-Mexique la frontière internationale avec les populations étudiantes diverses. Le travail de M3 s'est concentré à raccorder mathematisation comme un processus social et comme un principe didactique dans un cadre de développement professionnel. Les buts de M3 étaient de construire la connaissance de base d'enseignants dans le modelage mathématique et les préparer à exécuter des tâches de modelage dans leurs classes qui se concentrent sur la connaissance de mathématiques d'étudiants et exercent une influence culturel et les contextes de communauté. Les enseignants ont appris de et ont utilisé le processus de modelage en explorant des contextes locaux dans les communautés dans lesquelles ils enseignent. Ils ont exécuté des tâches de modelage qui ont utilisé avantageusement la connaissance de base de leurs étudiants dans le cadre du processus de solution ayant pour résultat l'engagement étudiant réussi dans le processus de modelage mathématique. Les modèles étudiants ont produit la créativité, les mathématiques intéressantes et ont révélé leur compréhension d'éditions de communauté de monde réel, en levant ainsi la conscience et la conscience critique.

## 1. Introduction

Mathematical modeling demands that students apply the mathematics they know and their background knowledge to solve problems in everyday life situations. This is due to the contextual nature of mathematical modeling activities, which can accommodate students' experiential interpretations that lead to their own assumptions and decisions for their models. Moreover, modeling allows students to experience the type of problem solving that mathematicians engage with in the real world and supports investigations of critical social issues. This offers opportunities for leveraging students' cultural competencies through mathematical modeling tasks with the potential for meaningful learning. The process of bringing everyday knowledge and scientific knowledge together through modeling can be especially effective when the situation context is relevant to the students' communities and lived experiences (Gay 2000; Ladson-Billing 1995). These connections make it possible for students to participate in meaningful mathematics learning that is connected to their background (Anhalt, Cortez, & Smith, 2017).

## 2. Theoretical Perspectives

Mathematical modeling has gained prominence in the K-12 mathematics curriculum in the United States, in part due to emphasis in the Common Core State Standards for Mathematics (CCSSI, 2010) and the Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) report (Garfunkel & Montgomery eds., 2016). Due to its open-ended nature, non-linearity, and multiple solution paths and strategies, mathematical modeling tasks allow for a broader range of students to be successful than in the context of a more standardized mathematics curriculum (Anhalt, 2014; Lesh & Lehrer, 2003). In particular, modeling problems require that students critically analyze situations.

As teachers often have had limited exposure to mathematical modeling in their preparation, there is a pressing need in teacher education for increased learning opportunities nested in real-world contexts that allow critical conversations surrounding social issues. Our work draws on Ladson-Billings (1995) Culturally Relevant Teaching (CRT) framework, which focuses on student academic achievement, cultural identities, and critical consciousness. The $M^3$ project explicitly connects mathematical modeling to equity-based culturally responsive mathematics teaching. This comprehensive teaching approach addresses mathematics, mathematical thinking, community-based funds of knowledge, and cultural identity to support student mathematics learning and engagement (Aguirre & Zavala, 2013; Bartell, 2013). Teacher professional development with an explicit commitment to culturally responsive mathematics teaching focuses on learning experiences that enable teachers to deepen their understanding of mathematics, make connections to student lived experiences, and help students experience mathematics as an analytical tool to make sense of, critique, and positively transform the world (Aguirre & Zavala, 2013; Gutstein, 2006; Turner et al., 2012; Simic-Muller, 2015).

## 3. Background on the Mathematical Modeling in the Middle Grades ($M^3$) Project*

This paper reports on a year-long professional development project, Mathematical Modeling in the Middle Grades ($M^3$), that created unique opportunities for thirty mathematics teachers and teacher leaders in upper elementary and middle grades to explore mathematical modeling through culturally relevant community contexts. The teachers were from rural schools near the United States-Mexico international border with diverse student populations. The goal of $M^3$ was to prepare teachers for implementing modeling tasks in their classrooms that focus on students' mathematics knowledge and leverage cultural and community contexts, resulting in successful student-created models, and consequently, raising critical consciousness in real-world community issues. The work of $M^3$ focuses on interconnecting mathematisation as a social process and as a didactic principle in a professional development setting. The teachers' commitment to learning about mathematical modeling and meaningful ways to teach modeling to diverse student populations was key to the success of the $M^3$ project. The project consisted of 65 hours of professional development spread over two summer institutes and monthly academic year meetings.

The initial summer institute consisted of establishing a professional learning community in which all ideas were valued, therefore, listening skills and active participation became central. Participation in collaborative groups consisted of a Launch-Explore-Summarize approach (Schroyer 1984) as teachers worked through the cyclic nature of the mathematical modeling process: (a) *launch* a context-rich problem to capture interest; (b) *explore* with background information to create and operate on a mathematical model; and (c) *summarize* to share outcomes that could inform recommendations for future action. We began with simple modeling problems in non-specific contexts that grew to more complex problems as teachers transformed their curriculum to include modeling tasks with local community contexts as settings for problems.

During the academic year, the teachers participated in monthly study groups, which focused on analysis of student work as they implemented mathematical modeling tasks in their classes. Sessions also included opportunities to discuss teaching mathematical modeling, and its challenges and benefits for student learning. The teachers expressed interest in developing tasks with more relevant contexts involving serious issues impacting the community.

During the second summer institute, the exploration of more complex modeling problems increased as we investigated local community issues such as border crossing time, produce imported from Mexico to the U.S., and landfill costs for processing produce waste. These topics impact the lives of the families in these communities on a daily basis. An example of a modeling task based on a local community issue of produce waste and landfill costs is in 'figure 1'.


**Figure 1**. Local foodbank waste and landfill costs[1]

- Using the data provided, find a way to approximate the waste by Borderlands from 2004-2012.
- What was the cost to the county each year from 2001-2013?
- What are ways that Borderlands Food Bank could reduce the amount of produce being dumped in the landfill?

Background information revealed that produce near expiration date that comes across the international border is donated to local food banks for distribution to low income communities, yet the food bank cannot manage all of the donated produce, so it often must dispose of some produce at the local landfill. Unfortunately, the cost of processing the wasted produce at the land fill must be paid for by the food bank or the local county taxes. The teachers were concerned because food waste and county taxes impact the whole community.

## 4. Findings

Throughout the project, the teachers expressed enthusiasm for teaching mathematical modeling. Teachers learned and utilized the modeling process and valued carefully created modeling activities. Teachers changed their views of curriculum development by incorporating engaging modeling activities that advantageously use their students' background knowledge as part of the solution process. Our project results show that teachers were able to successfully engage students in the mathematical modeling process as they participated in culturally relevant contextual tasks in mathematical modeling. Their models yielded creativity, interesting mathematics, and revealed their understanding of the situations.

Teachers reported that problems that were explored promoted civic awareness and reflection of a broad range of issues facing local communities. The teachers indicated that they learned that real-life issues can be adapted to create engaging mathematical modeling problems and that different views of the problem can lead to different acceptable solutions. This is an important realization and a feature of modeling tasks that does

[1] http://www.nogalesinternational.com/news/landfill-cap-puts-pressure-on-food-bank/

not occur in traditional word problems (Anhalt 2014). In mathematical modeling, the assumptions are central to the process as they determine in large part the construction of the model. However, even under the same assumptions, the modeling approach may vary, as they learned throughout the project. Discussion on the topic of assumptions was at the heart of balancing the didactic fictionality and the reality of the social phenomenon. The teachers produced quality work in mathematical modeling through posters incorporating their group thinking and solutions. Based on their reflections, the teachers felt that learning about mathematical modeling changed their way of thinking about curriculum in that modeling offered students opportunities to take ownership of the problems, especially when the contexts are socially and culturally relevant.

A subset of teachers took the initiative to make a presentation to their school administration advocating for curriculum that incorporated mathematical modeling with cultural relevance for students. They presented their students' work and learning in mathematical modeling as evidence to support their claims. This event proved to be a compelling testament to the transformation brought about by their participation in the $M^3$ project.

## 5. Conclusions

Mathematics teacher education should seek to challenge the status quo by creating professional learning activities that combine mathematical modeling and critical social awareness in ways that help teachers and their students raise civic advocacy, think more critically about what is happening in our communities and take transformative action. Consequently, teachers can increase student learning of rigorous mathematics, nurture students' cultural identities, and instill critical consciousness. Explicitly prioritizing rigorous mathematics investigations that address all three goals has been a challenge, particularly because it means to confront potential implicit bias about specific students and their communities or uncover social, economic, and environmental inequities in local communities and in broader contexts. An emphasis on mathematical modeling has the potential to simultaneously grow teachers' and students' mathematical knowledge and critical consciousness if the experiences include analysis of local community problems that promote activism in students.

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# A non-euclidean clockwork orange: from reality to mathematics and back

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**Abstract**. This paper presents a teaching experience in non-Euclidean geometry involving the use of artefacts and physical experiences. The teaching practice was set up for a group of 25 4th-year high-school students. The aim of this activity was to encourage the "process of translation from 'reality' to mathematics and back". Further, the students were stimulated to evaluate the authenticity of the proposed context, which was designed for learning about spherical geometry and its value in helping students understand and organise the abstract key facts of the topic. The overall aim was to shift back to the starting point and apply the acquired theoretical knowledge in order to interpret a real-world problem.

**Résumé**. Cet article présente une expérience pédagogique dans la géométrie non-euclidienne impliquant l'utilisation d'artefacts et d'expériences réelles. La pratique pédagogique a été mise en place pour un groupe de 25 élèves du quatrième année de l'école secondaire. Le but de cette activité était d'encourager le "processus de traduction de la 'réalité' aux mathématiques et retour". En plus, les étudiants ont été stimulés à évaluer l'authenticité du contexte proposé, conçu pour apprendre la géométrie sphérique et sa valeur pour aider les élèves à comprendre et à organiser les faits clés abstrait du sujet. L'objectif général était de revenir au point de départ et d'appliquer les connaissances théoriques acquises afin d'interpréter un problème réel.

## 1. Introduction and theoretical framework

According to Jablonka and Gellert's reflection on mathematisation and demathematisation (Jablonka & Gellert, 2007), "if classroom talk concentrates on the public language of description in order to help students construct meaning, then the mathematical knowledge of students tends to remain in the public domain of its origin. If, on the other hand, the classroom talk is mainly esoteric, then the individual construction of meaning appears to be more difficult."

In high-school, non-Euclidean geometry is often treated in a particular way: from an epistemological point of view, it is a highly significant subject. However, there is a risk of losing the cognitive ties to reality if the subject matter, as taught in schools, is kept strictly in the abstract realm. Further, the key role of non-Euclidean geometry, considering the interpretation of geometry in real-world settings, is endangered by the demathematisation of such by, for example, Google Maps and GPS navigation.

The overall aim of this teaching experience was to encourage the "process of translation from 'reality' to mathematics and back" called for by Jablonka and Gellert (Jablonka & Gellert, 2007). In addition, the purpose was to evaluate the effectiveness of the authenticity of the proposed context for the learning of spherical geometry. The evaluation included the teaching experience's value in terms of its support for students' understanding and organisation of the key abstract facts about non-Euclidean geometry. Further, the students were questioned about their newly developed ability to shift back and apply the acquired knowledge to interpret the real-world problem that was given to them.

## 2. Method and activity

This learning experience has been inspired by Lénárt's work on comparative geometry and by the article "Grapefruit Math" by Evelyn Lamb, published in the Scientific American online issue in May 2015. The added value of this specific project is the use of artefacts and physical experiences that effectively support students' learning by means of connecting abstract concepts of thought and discoveries to real world tangible results.

The activity was carried out with a class of 25 18 y.o. students attending the 4th year of Liceo Scientifico. The results were gathered by the teacher through the observation of classroom discussion and the assessment of students' homework.

After an introductional theoretical approach to non-Euclidean geometry, which primarily entailed the discussion on the role of the fifth postulate and the consequences of its negations with reference to the validity of the "classical" theorems about parallels and transversals and the sum of a triangle internal angles, students were asked to bring an orange, three elastic bands and a protractor for the upcoming lesson.

In said lesson they were confronted with the following tasks:

- Draw three points on the orange that do not lay on the same great circle.
- Put two elastic bands, each passing through two of the three points as in 'figure 1', and derive the expression for the area of each lune.

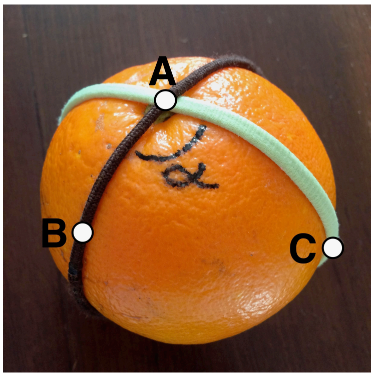


**Figure 1.** from Elena's work: the construction of the lune

- Add a third elastic band to form a spherical triangle as in 'figure 2'.

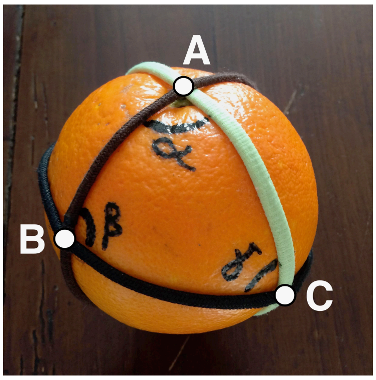


**Figure 2.** from Elena's work: the construction of the spherical triangle

- Use a protractor to measure the internal angles of the spherical triangle, add them up and contextualise the result.
- Derive the formula for the surface area of the spherical triangle.

The final questions encouraged critical reflection on the subject matter that required from the students to switch from the real-world model (the triangle on the orange) to the mathematical model (the formula for the surface of the triangle). After the students had completed these tasks, they were confronted with those follow-up questions, which aimed to link the mathematical interpretation to their everyday life:

- What happens if the triangle on the sphere gets smaller?

- What does that have to do with your everyday life as a resident of a giant sphere?

The students discussed their results and findings in class. After that, each student summarised her/his conclusions in a written essay that was finally submitted through the school's Moodle platform and assessed by the teacher.

## 3. Results and conclusion

The key aspect of this learning experience lies within the fact that "the essence of the process is that students are searching for the truth themselves, through experiments with palpable and virtual models, and through discussion or debate with one another or their teacher." (Lénárt, 2009).

The outcomes of this activity proved the efficiency of mathematical modelling as a didactic principle. On the one hand, the students were able to assess the mathematical behaviour of great circles as "straight lines" on the sphere by themselves. Thereby, the theoretical assumption presented in class beforehand gained a contextualised meaning due to personal experience. The overall, shared comment was:

*I couldn't believe that the rubber bands stayed put only on great circles!*

On the other hand, it clearly showed that by integrating real objects in order to draw in on abstract mathematical concepts, the students were more willing to apply the mathematical concept to real world observations thereafter. Further, this made them get an idea of why – in a small scale - we cannot identify our reality as non-Euclidean, as Chiara neatly pointed out in her paper:

*This happens because the surface area of the shrunk spherical triangle is so small compared to the one of the sphere that the curviness of its sides becomes negligible and it is perceived as a Euclidean triangle.*

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# Le cadre théorique l'ETM étendu : analyse d'une séquence utilisant la relativité restreinte

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**Abstract.** We want to show how the theoretical frame of the extended MWS allows to analyse the tasks operated during the process of modelling. The frame of the extended MWS allows to show, through the example of a special relativity teaching sequence in a grade 12 class in France, which are the interactions between the cognitive plane and the epistemological planes of the physics or the mathematics. The geneses, the association of geneses and the interaction between the epistemological plans of the mathematics and the physics can be clarified for certain stages of the cycle of modelling.

**Resumé.** Il s'agit de montrer comment le cadre théorique de l'ETM étendu permet d'analyser les tâches mises en œuvre lors du processus de modélisation. Le cadre de l'ETM étendu permet de montrer, au travers de l'exemple d'une séquence d'enseignement de relativité restreinte en terminale S en France (grade 12), quelles sont les interactions entre le plan cognitif et les plans épistémologiques de la physique ou des mathématiques. Les genèses, l'association de genèses et l'interaction entre les plans épistémologiques des mathématiques et de la physique peuvent être explicitées pour certaines étapes du cycle de modélisation.

## 1. Présentation du cadre théorique de l'ETM étendu

L'espace de travail mathématique (ETM) a été développé afin de mieux comprendre les enjeux didactiques autour du travail mathématique dans un cadre scolaire (Kuzniak et al., 2016). L'ETM comporte deux niveaux : un de nature cognitive en relation avec l'apprenant et un autre de nature épistémologique en rapport avec les contenus mathématiques étudiés. Le plan épistémologique contient un ensemble de représentamen (signes utilisés), un ensemble d'artéfacts (instruments de dessins ou logiciels) et un ensemble théorique de référence (définitions et propriétés). Le plan cognitif contient un processus de visualisation (représentation de l'espace dans le cas de la géométrie), un processus de construction (fonction des outils utilisés) et un processus discursif (argumentations et preuves). Le travail mathématique résulte d'une articulation entre les plans cognitifs et épistémologiques grâce à une genèse instrumentale (opérationnalisation des artefacts), une genèse sémiotique (basée sur le registre des représentations sémiotiques) et une genèse discursive (présentation du raisonnement mathématique). Les différentes phases du travail mathématique associées à une tâche peuvent être mises en évidence par la représentation de trois plans verticaux sur le diagramme de l'ETM. Les interactions de type sémiotique-instrumentale (sem-ins) conduisent à une démarche de découverte et d'exploration d'un problème scolaire donné. Celles de type instrumentale-discursive (ins-dis) privilégient le raisonnement mathématique en relation avec les preuves expérimentales. Enfin, celles de type sémiotique-discursive (sem-dis) sont caractéristiques de la communication de résultats de type mathématique. Le diagramme des ETM a été adapté (Moutet, 2016) en rajoutant un plan épistémologique correspondant au cadre de rationalité de la physique (figure 1). Il a été choisi de ne garder qu'un seul plan cognitif car les spécificités du plan cognitif des deux disciplines en jeu (physique et mathématiques) n'ont pas été particulièrement étudiées dans le cadre de l'étude exposée dans cet article. Le cadre de l'ETM étendu permet d'analyser finement les interactions entre les différents cadres de rationalité et le plan cognitif de l'élève puis de qualifier la nature du travail réalisé par l'élève ou celui qui lui est demandé.

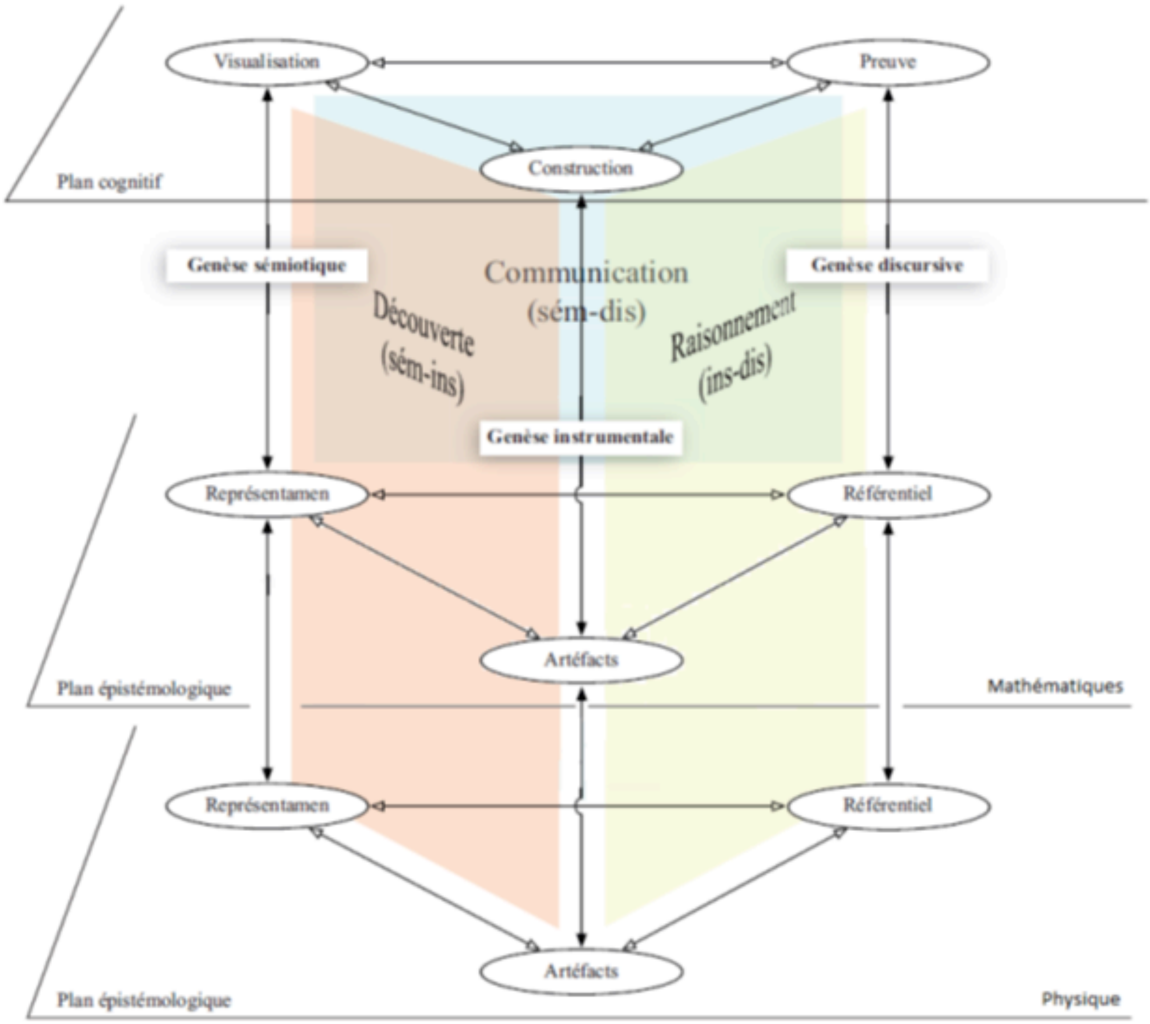


**Figure 1.** Modèle de l'ETM étendu

## 2. De la « situation modèle » aux « résultats réels »

Nous nous sommes basés sur le cycle de modélisation (Blum & Leiss, 2005) pour analyser une séquence d'enseignement (Moutet, 2016) portant sur le changement d'ordre chronologique d'événements en fonction du référentiel dans le cadre de la relativité restreinte (de Hosson, 2010). Elle est destinée à des élèves de terminale S (grade 12). Deux référentiels liés à deux observateurs, Armineh et Daniel, sont utilisés. Armineh conduit une voiture se déplaçant à une vitesse proche de la vitesse de la lumière par rapport à Daniel. Ce dernier se trouve sur le bord de la route à côté de trois flashs lumineux $S_1$, $S_2$ et $S_3$ associés à trois événements particuliers, et initialement connus dans le référentiel de Daniel (figure 2). Le diagramme d'espace-temps de Minkowski a été construit par les élèves et utilisé en classe à l'aide d'une activité papier-crayon relativement guidée par l'enseignant. Le logiciel de géométrie dynamique GeoGebra a permis par la suite, au travers d'une activité dans laquelle les élèves étaient en autonomie, de réinvestir le diagramme d'espace-temps de Minkowski. Il s'agit d'utiliser avec GeoGebra une autre genèse instrumentale afin de pouvoir effectuer une analyse *a priori* différente de l'activité papier-crayon à l'aide du modèle de l'ETM étendu. Le diagramme de Minkowski permet de représenter le repère (xOc.t) relatif au référentiel de Daniel et le repère (x'Oc.t') relatif au référentiel d'Armineh. Cette dernière se déplace à la vitesse v de 0,6 fois la vitesse de la lumière dans le vide (on considère que la vitesse de la lumière dans le vide est à peu près égale à celle dans l'air) par rapport à Daniel suivant un axe (Ox). Les droites (Ox) ou (Ox') correspondent à la route dans les référentiels de Daniel ou d'Armineh (figure 3).

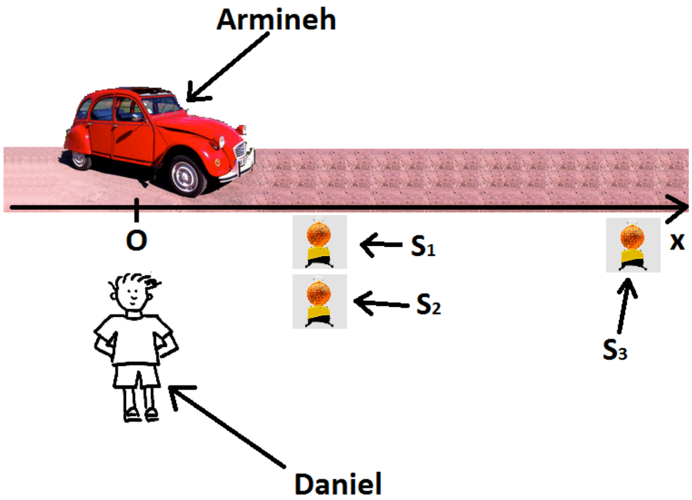


**Figure 2.** Le "modèle réel" de la situation

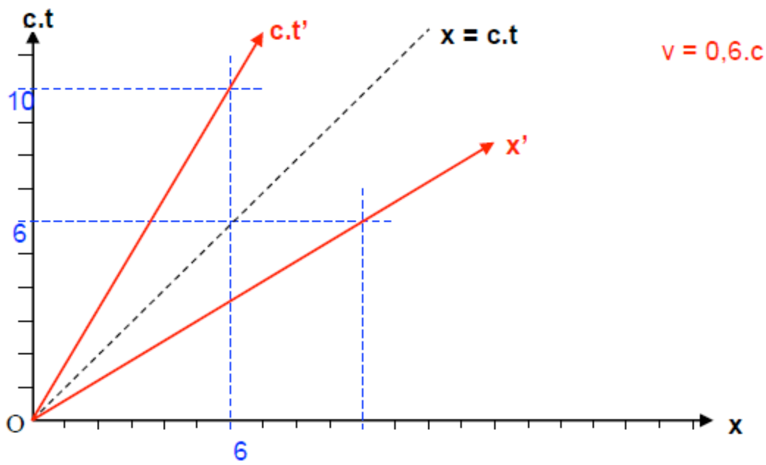


**Figure 3.** Diagramme de Minkowski pour v = 0,6.c

Le curseur de GeoGebra permet de modifier les conditions expérimentales en changeant la vitesse v, ce qui permet également une genèse sémiotique différente par rapport à l'activité préliminaire papier-crayon. Le plan épistémologique des mathématiques et le plan cognitif sont mobilisés lors de la construction du curseur. Les axes Ox' et Oc.t' sont modifiés en fonction de la vitesse v, ces deux axes se rapprochent de la droite x' = c.t' lorsque la vitesse v se rapproche de c. Le plan épistémologique de la physique est également mobilisé lorsque les élèves concluent sur l'ordre chronologique des événements suivant les deux référentiels (figure 4 et figure 5). L'utilisation du logiciel GeoGebra amène donc aussi une genèse discursive différente par rapport à l'activité papier – crayon. Le modèle de l'ETM étendu permet de réaliser l'analyse *a priori* de chacune des tâches à effectuer par les élèves et il nous a permis également de tester avec succès l'analyse *a posteriori* du travail effectué par quatre élèves.

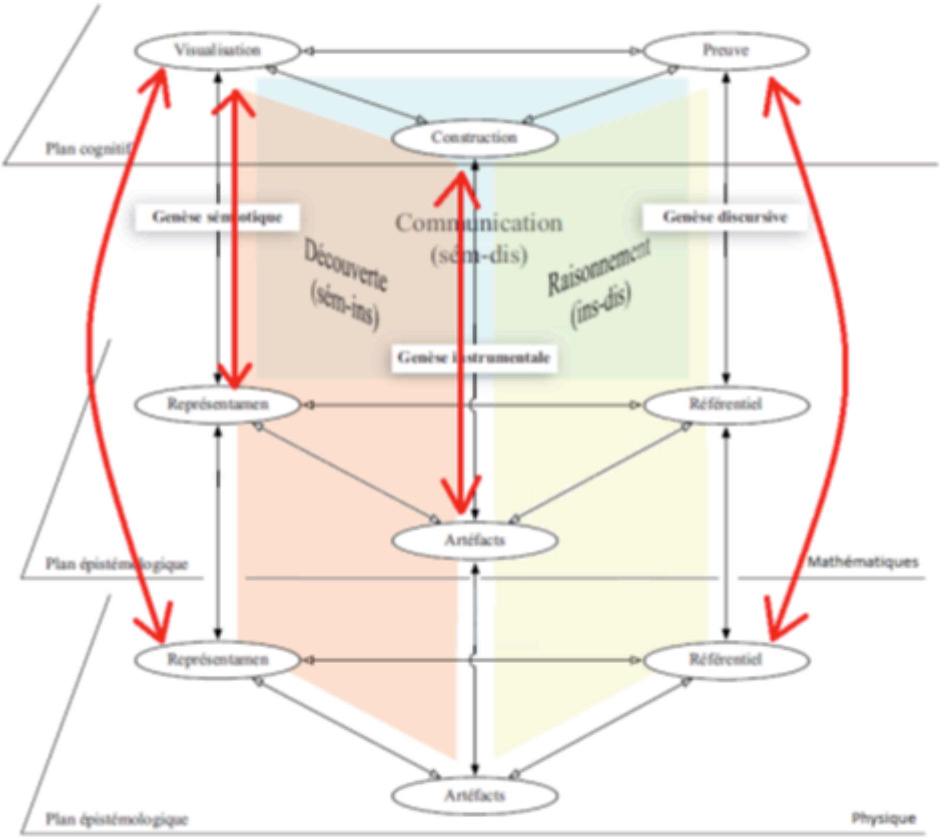


**Figure 4.** Analyse de l'utilisation du curseur avec GeoGebra

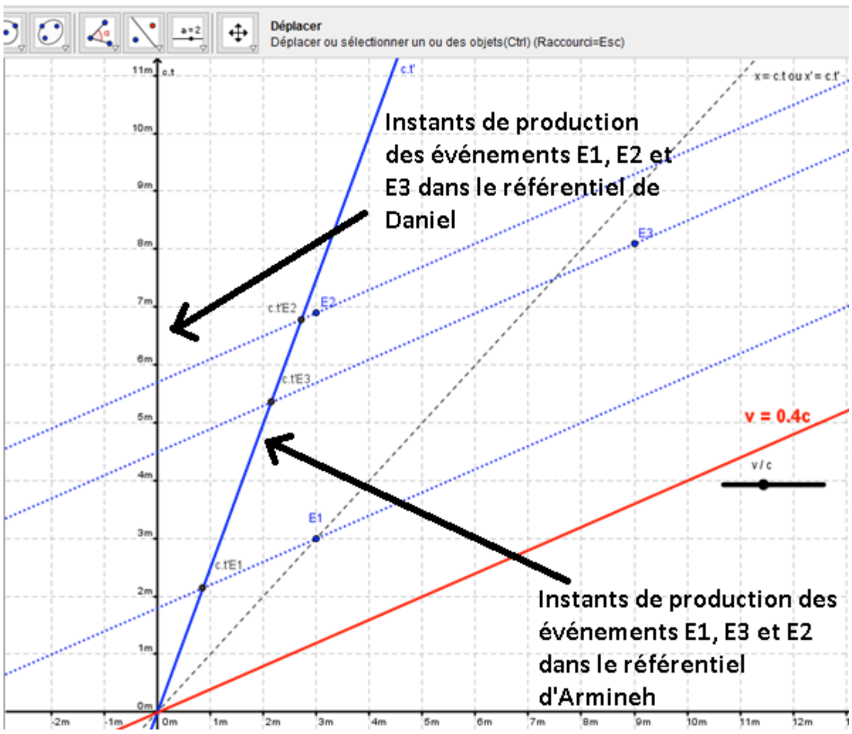


**Figure 5.** Diagramme de Minkowski avec GeoGebra

## 3. Conclusion

Le cadre de l'ETM étendu nous a permis d'analyser les tâches associées à certaines étapes du cycle de modélisation. Il permet de prendre en compte la mobilisation des plans épistémologiques des mathématiques et / ou de la physique pour chacune des tâches tout en tenant compte des genèses mobilisées. Le cadre de l'ETM étendu nous a également permis de montrer que le logiciel GeoGebra développe des genèses spécifiques par rapport à une activité papier - crayon. Une nouvelle génèse sémiotique permet une visualisation du changement des coordonnées temporelles des événements en fonction de la vitesse v d'Armineh par rapport à Daniel. Une nouvelle genèse instrumentale correspond à la manipulation du logiciel de géométrie dynamique avec la fonctionnalité curseur permettant de changer simplement les conditions expérimentales. Enfin une nouvelle genèse discursive permet de conclure sur l'ordre chronologique des événements en fonction du référentiel d'étude et de la vitesse v. Nous envisageons, par la suite, d'analyser grâce au modèle de l'ETM étendu ou à une de ses évolutions, les tâches mises en œuvre à chacune des étapes du cycle global de modélisation lors de séquences utilisant la relativité restreinte ou la mécanique classique. Des résultats préliminaires tendent à montrer que les genèses ainsi que les plans épistémologiques des mathématiques et de la physique ne sont pas mobilisés de la même façon en fonction de l'étape du cycle de modélisation. Des études utilisant la chimie sont également envisagés en tenant compte cette fois-ci du plan épistémologique de la chimie à la place de celui de la physique.

**Remerciements**

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# Mathematisation in the universe of fractions: didactic principle

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**Abstract.** In this presentation that concerns the familiarization of children of primary school with the concept of fraction, we focus on the activity of preparation of Macedonia by the children of two fifth grades. The discussion on the construction of the Whole, results in flexibility for children in giving meaning to the keywords "Whole", "Unit", and "Quantity" in accordance with the specific situations. The flexibility is fostered by the fact that our didactic proposal is a mathematisation process, as manifested by the presence of three insights: (a) the starting point is an elementary act that is simultaneously principle and trace; (b) Euclidean division puts ideas into relation and coordinates them; (c) the new didactic universe of fractions is coherent and consistent, and it safeguards the confidence of the children.

**Résumé.** Dans cette présentation qui concerne la familiarisation des élèves de l'école primaire avec le concept de fraction, nous nous concentrons sur l'activité de préparation de la Macédonia par les élèves de deux cinquièmes. La discussion sur la construction de l'«Entier» produit flexibilité pour les élèves qui donnent un sens aux mots-clés «Entier», «Unité» et «Quantité» en fonction des situations spécifiques. La flexibilité est favorisée par le fait que notre proposition didactique est un processus de mathematisation, comment la présence de trois renseignements montre: (a) le point de départ est un acte élémentaire qui est à la fois principe et trace ; (b) la division euclidienne met les idées en relation et les coordonne ; (c) le nouvel univers didactique des fractions est cohérent et « consistent » et il sauvegarde la confiance des élèves.

## 1. Introduction

We are a working group on didactics of mathematics that has structured its specific method we like to mark with EEE: Exploring[2], Enquiring[3], Evaluating[4]. This method is emerging in his own form during our activities on familiarization[5] of children of primary school (starting from grade 3) with the concept of fraction. Our didactic proposal is aimed to introduce the concept of fraction and its fundamental properties by starting from an act of mathematisation and by following a coherent didactic path, which avoids the formation of inhibitions[6].

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[2] Our exploring on the "familiarization" of children with fractions has lasted two school cycles (= 10 years) and has allowed to find a synthesis between the indications obtained both from scientific literature and from teaching practice.

[3] Enquiring is characterized by the constant interaction between two distinct groups of reflection. In the first group, individual educational acts are discussed, before and after their presentation to the classes. This group is gradually enriching by pages of children's exercise books, which become explicit objects of noticing. The second group takes care of the "reflective, philosophical practice". It is the activity of this group that allows us to give completeness to our enquiry.

[4] Evaluating. We have planned different times of evaluation of the effectiveness of our teaching proposal; times managed both by us and by others.

[5] Familiarization differs from the teaching/learning process because it favors the formation of "correct intuitive representations" rather than the learning of formal rules.

[6] Inhibitions make the teaching/learning of fractions persistently unsatisfactory, as it is frequently claimed in

We have already presented our project at various conferences[7], in which motives, objectives, stages of our proposal have already been the object of discussion. During the current school year, the classroom activities related to our enquiring are going to be completed. Our enquiring has begun in 2014 with the two third classes of primary school in Locate di Ponte San Pietro. During the current school year, we are enquiring with two fifth classes in Locate, with one fourth class in Milan, and with one third class in Sondrio.

This is therefore the time for final considerations on the implementation of our project into the classrooms. In this presentation, we choose to emphasize some considerations concerning the features that distinguish our teaching proposal as process of mathematisation. We start from a problematic situation used in the classrooms for assessing the key competencies in giving meaning to the keywords "Whole", "Unit" and "Quantity" in accordance with the specific situations.

### *1.1 Determination of the "Whole", of the "Unit", and of the "Quantity".*

In the scientific literature, the names "Whole", "Unit" and "Quantity" are not always used with univocal meanings. In our approach, we refer their meanings to their "forms" (Whole = n/n, Unit = 1/n, Quantity = m/n) by a process of mathematisation that have involved the activities throughout the entire third class. The first step of this process consists in identifying the act of comparison by a pair of numbers; in this step the Unit is qualified as Common Unit. In the second step the fraction is specified as comparison between the Quantity and a special Quantity, called Whole. It is in this step that the Whole acquires its form n/n; then the Unit and the Quantity respectively take their form 1/n and m/n. These forms are explicitly used in all the activities.

After reporting the teaching situation of preparing the Macedonia, the process of mathematisation will be the heart of our reflection.

## 2. Problematic situation: La Macedonia (the fruit salad)

This problematic situation has been proposed at the end of school year, in each of the two classes participating in the project. Each class is formed by sixteen children.

For the year-end party, each class prepares the Macedonia. Children bring fruit from home: 6 bananas, 5 apples, 6 pears, 2 bags each with 4 oranges, 1 basket of Kiwi, 1 kg of strawberries, 1 pineapple, 1 bag with 4 lemons, 100 g of sugar.

The class is divided in four groups. Each group must prepare its Macedonia according to the following recipe.

Cut into small pieces the following fruits:

3/2 bananas, 5/4 apples, 1+1/2 pears;1/2 oranges, 1/4 kiwi, 25% strawberries; 1/4 pineapple.

Add 1/4 sugar. Squeeze 1/4 lemons. Mix.

The activity begins with *a discussion to determine the "Whole" for each ingredient.* Consequently also "Unit" and "Quantity" result determined. The teacher leads the discussions within each group.

### *a. With regard to the bananas, the apples and the pears.*

Naturally the children choose the individual fruit as Whole. Children record the Wholes ($W_B$=1banana; $W_A$=1 apple; $W_P$=1 pear).

The Quantities coincide with the fruits brought from home. Regarding these fruits, the Quantities are larger than the Whole. Children record the Quantities (B=6 bananas; A=5 apples; P=6 pears).

The discussion concerning the Unit is most significant in this first part. Thanks to the activities related to the form of the Unit and carried out since the third class, the children identify that $U_B$=1/2banana, $U_A$=1/4apple, $U_P$ =1/2pear; then the banana and the pear must be divided into 2 equal parts, while the apple must be divided into 4 equal parts. The fact of obtaining the Unit by partition, echoes the classroom activities with pies, water and so on.

In this part of the activity, the Whole differs from the Quantity, and the Unit is obtained by means of partition. The activities concerning the Unit have been driven by the familiarity with Euclidean division, that

scientific literature.

[7] CIEAEM 66, Lyon (2014), CIEAEM 67, Aosta (2015), XX Congresso UMI, Siena (2015), HPM, Montpellier (2016), ICME-13, Hamburg (2016), SIRD, Milano (2016).

allows children to immediately write 3/2 = 1+1/2 and 5/4=1+1/4, and to identify the Units 1/2 and the Unit 1/4.

*b. With regard to the oranges, the kiwis, the strawberries and the lemons.*

The Whole is constituted by 1 bag or 1 basket ($W_O$=1bag=4/4; $W_K$=1basket=8/8; $W_S$=1Kg=48/48; $W_L$=1bag=4/4).

The Unit coincides with the single fruit ($U_O$=1orange=1/4; $U_K$=1Kiwi=1/8; $U_S$=1strawberry=1/48; $U_L$=1lemon=1/4).

The Quantity is the number of bags or packages (O=2bags; K=1basket; S=1kg; L=1bag).

Here the actual situation promotes the identification of the Whole with the basket, bag, or kg. It is useful to underline that Quantity does not correspond always with the Whole. The fact of identifying the Unit as a single fruit, echoes the classroom activities with Lego, picture cards, eggs packs and so on.

*c. With regard to the pineapple.*

The Whole and the Quantity coincide ($W_{Pi}$=Pi=1). The teacher raises the question of a suitable subdivision of the pineapple "To divide the pineapple into four parts is not effective. How can we distribute it in a more fair way?" A convenient way to divide the pineapple is to cut it into 8 slices: 1/4=2/8; an opportunity to discuss equivalent fractions. The teacher cuts the pineapple in 8 slices. The Unit is a slice ($U_{Pi}=1/8 \rightarrow W_{Pi}=8/8$).

Here the partition has a central role. Notably, reference is made to the appropriate choice of the Unit, that had been highlighted in the activities with water and with pies.

*d. With regard to the sugar.*

The Whole and the Quantity coincide ($W_{Su}$=Su=1 glass). The Unit is obtained by dividing the Whole into 4 parts ($U_{Su}=1/4 \rightarrow W_{Su}=4/4$).

Children have solved themselves this subdivision problem by taking 4 empty glasses and pouring into each glass the same amount of sugar; a translation of the activities with water.

*e. Comments on the activity.*

We summarize the properties of names "Whole", "Unit" and "Quantity" with the different ingredients.

Whole and Quantity coincide with the kiwis, strawberries, lemons, pineapple, sugar; Whole and Quantity differ with bananas, apples, pears, oranges.

The Unit is a single fruit with oranges, kiwis, strawberries, lemons; the Unit is obtained by partition with bananas, apples, pears, pineapple, sugar.

For the strawberries it is requested to take the 25%. This request provides an opportunity to bring the discussion on the percentage. "Write the requests for kiwis, strawberries, lemons, and sugar using the percentage".

In this activity, several "subconstructs of the construct" of the concept of fraction[8] come simultaneously into play, creating a rich articulation of meanings of the names. Nevertheless, the children have moved with "lightness" and flexibility, giving quiet, serene answers, and getting adequate results.

## 3. The process of mathematisation

The flexibility children have shown in similar situations, is fostered by the fact that our didactic proposal is a mathematisation process: *the confidence given by this latter constitutes the background for the search of new meanings*.

What are the elements that, in our opinion, reveal the presence of a mathematisation process?

To answer this question we let be guided by the Wheeler's insights concerning the clues to presence of mathematisation; insights that we summarize in the following way: (a) structuring (b) putting ideas into relation and coordinating them, (c) searching for universality.

*a. Structuring. Principle as trace. The trace structures the enquiring.*

[8] The five Kieren's "sub-constructs of the construct of rational number" are: part-whole, quotients, measure, ratios, operators. Other possible subconstructs are: proportionality, point on the number line, decimal number, and so on.

***The comparison of two quantities is a pair of natural numbers (ratio).***

Children start their familiarization with fractions from this act of comparison. Beside the other forms of registration of the initial act, they *are instructed to write* A;B = 9;24: "the comparison between the quantities A and B is the pair of numbers 9;24". Just this writing transforms the initial act into an *elementary act*: it originates and founds the subsequent didactic process.

This didactic choice is the result of our "*reflective, philosophical practice*"; a practice that has led us to a historical and mathematical exploration about the "originary" meaning of the concept of measure.[9] However, in classroom practice this elementary act is lived without any trouble.

"Elementary act" must be understood in Euclidean sense: not only it is the initial act, but above all it implicitly directs and structures the didactic process of fractions; it is simultaneously *principle and trace* of the process of mathematisation. The name "trace", which we understand according to Lévinas,[10] emphasizes how this act becomes a structuring factor. Indeed, in our activities on fractions, it has structured the path towards the Euclidean division. Initially it has guided us, without us realizing it: our teaching proposal contained no explicit idea of Euclidean division, but we were unwittingly pursuing its features. It was a surprise for us when it was pointed out that the sign structure we were disclosing, is the Euclidean division. Euclidean division has emerged through a process of interaction between teachers, children and trace, with the latter that continuously has refocused us.

*The elementary act as principle and trace: it is the first clue to presence of mathematisation.*

b. *Putting ideas into relation and coordinating them. Euclidean division as icon.*

**The mathematisation process takes its comprehensive form in Euclidean division.**

"Euclidean division is experienced by the children not as a formula to be memorized, but as the icon[11] of their active process of learning": it is the synthesis concerning the memory of own history of learning and it becomes the core of the universe of fractions.

Euclidean division is the core because (a) all "subconstructs of the construct of rational number", have their roots and find unity in it, (b) and because it engenders a spiral didactic process. This process explores the various contexts keeping Euclidean division in mind and continually comes back to it. [Longoni et al., 2016]

The icon-Euclidean division plays a coordination function and is element of unity in the process of entrusting meanings to the names "Quantity", "Whole" and "Unit". The problematic situation of Macedonia is characterized by the fact that each of these names has different meanings depending on the fruits: the Whole coincides with the single fruit for some ingredients, while coincides with the pack or the basket for others. The Quantity sometimes coincides with the Whole, sometimes not. The Unit is several times obtained by means of partitions, other times it corresponds to the single fruit. The process of signification reflects the multiplicity of faces / sub-constructs of the construct of fraction, but finds unity in the icon - Euclidean division.

*Euclidean division puts "the sub-constructs" into relation and coordinates them: it is the second clue to presence of mathematisation.*

c. *Universality. The new didactic universe of fractions.*

**To build a new didactic universe of fractions, with its own rules and properties.**

This aim has guided us since the exploration stage. It differs from the more common choice to take that whole number knowledge could be utilized by children in their construction of initial fraction concepts.[12] This aim has originated from the consideration of ineffectiveness of teaching and learning fractions, as repeatedly reported in the scientific literature.

The new universe is *coherent*. Coherence is referred, rather informally, to the existence of the structuring principle, and to the role of the core, the Euclidean division, with its potentiality for relating and

[9] More details have been presented at HPM 16 in Montpellier.

[10] The binomial "presence / absence in relation to the other and to the originary" characterizes the concept of trace in Lévinas.

[11] The evocative value we ascribe to the word "icon", refers to art history, to which we have recourse to highlight and enhance deeper meanings that this word has in the context of our cultural training.

[12] "But thinking in terms of integers and integer relations often interferes with the acquisition of rational number concepts".

coordinating.

It is a new universe but also *consistent* with what the children have already learned: it interacts with their knowledge. In our didactic experience, for example, the correlation between the concept of fraction (based on the subconstruct "ratio") and the subcontract "division", has occurred thanks to the used manipulative: with these manipulative (egg boxes, picture cards, etc.) the idea of measure as comparison and the idea of division, interact and guide the children to recognize themselves the fraction as measure and division at the same time.

The universe of fractions safeguards the *confidence* of children: it constitutes the background for the search of new meanings. The activity on Macedonia is a rich process of search of meanings, based on the confidence on the forms/icons that hold the universe of fractions.

*The third clue to presence of mathematisation is the universality, understood as coherence, consistence and confidence on the new universe.*

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# Enquiring and mathematising an authentic situation

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**Abstract.** This paper focuses on analysing the potentialities of using multidisciplinary contexts to enhance the integration of inquiry and mathematical modelling practices. We focus on a particular teaching sequence where modelling becomes an essential tool to analyse several questions emerging from the interplay between Mathematics and History. More concretely, we focus on the design and implementation of a didactic sequence based on an archaeological context with 12-13 years old students. We can see the important role, on the one hand, of the real context that allow students validating their hypothesis and finding answers in their process of inquiry and, on the other hand, of the devices used to facilitate the interaction between the two disciplines.

**Résumé**. Cet article se concentre sur l'analyse des potentialités de l'utilisation de contextes multidisciplinaires pour intégrer des pratiques de recherche et de modélisation mathématique. Nous nous concentrons sur une séquence d'enseignement particulière où la modélisation devient un outil essentiel pour analyser plusieurs questions émergeant de l'interaction entre les mathématiques et l'histoire. Plus concrètement, nous nous concentrons sur la conception et la mise en œuvre d'une séquence didactique basée sur un contexte archéologique avec des étudiants de 12-13 ans. Nous pouvons voir le rôle important, d'une part, du véritable contournement qui permet aux étudiants de valider leur hypothèse et de trouver des réponses dans leur processus d'ingestion et, d'autre part, des dispositifs utilisés pour faciliter l'interaction entre les deux disciplines.

## 1. Introduction

In this paper, we assume the importance of mathematical modelling to inquire into the study of authentic extra-mathematical questions in schools. The main aims of our paper are both reflective and experimental to (1) reflect on the use of multidisciplinary contexts for the teaching and learning of mathematics, posing the starting questions but also providing sense to the whole study, and (2) analyse their affordances to enhance the integration of inquiry based learning (IBL) and the development of modelling students' competences. In particular, this paper focuses on the design, implementation and analysis of teaching sequences based upon archaeological contexts, and their role to improve not only mathematisation itself but also improving inquiry open attitude.

In previous research, we have worked on the design of teaching sequences where mathematics and history appear dialectically fostering inquiry and modelling attitudes of students (Sala, Giménez & Font, 2013; Sala et al., 2015). In all these experiences, we found that without integrating a constant interaction with extra-mathematical contexts, most of the interesting questions raised and the answers found by students could not be emerged. Moreover, we insisted on the importance of these interactions to make mathematical knowledge, on the one side, and historical knowledge, on the other side, progress, complement and validate each other.

In the following sections, after introducing the most important theoretical aspects we take into account for the design and analysis of the teaching sequences, we focus on the particular case of one inquiring into the kind of building that is Roman ruins found in Badalona (Baetulo in the Roman time) can correspond to. In this paper, we focus on the following questions: Which is the role of the extra-mathematical context (as starting, intermediate and finish point) along the inquiry project? Which didactic devices were more useful to promote the dialectics between inquiry and modelling, helping students to progress in the mathematical learning through modelling? In addition, how the constant interaction between history and mathematics helped to integrate a rich validation of mathematics to provide them completeness and functionality?

## 2. Theoretical issues

The investigation that supports this paper takes into account different theoretical aspects. First, the notion of basic competence that the Catalan curricula guidelines (similar to the NRC, 1996) include, particularly we focus on the notion of inquiry competence with the objective of assessing the students' progression related to improve their inquiry skills. These inquiry skills are hardly related to modelling approaches, and sometimes it is difficult to find the differences between both processes (Artigue & Blomhoj, 2013). From our viewpoint, mathematical modelling processes must be placed at the core of mathematical activities and of the didactic sequences to ensure the development of important processes and competences assumed in the Catalan curricula, in particular, to ensure the development of inquiry competences (Sala et al., 2017).

In order to guide the mathematical and didactic design of the teaching sequence, we use of the notion of the study and research paths (SRP) as epistemological and didactic model (Chevallard, 2015; Barquero & Bosch, 2015), to face the problem of moving towards a functional teaching of mathematics and, particularly, where mathematics are conceived as a modelling tool for the study of problematic questions. About their main characteristics of the SRP, the ones that were central for their design and which will be later explained, are the following: (a) the starting point of an SRP is a 'lively' generating question with real interest for the community of study. In our case, the generating question was about: *Which kind of building could the ruins discovered in Badalona correspond to*?; (b) during an SRP, the study of the generating question evolves and opens many other 'derived questions' which traces the likely paths to follow in the open-inquiry project, such as: questions about the geometrical shapes of the ruins, or of the public Roman buildings of the epoch, ways to simulate geometrical models corresponding to the building, ways to estimate the dimensions of the buildings, among others; (c) to enhance mathematical modelling in the SRP, the experimental milieu may be progressively enrich by many elements: questions posed, answers provided, different resources (class discussions, physical experiences, technological artefacts, etc.), but also by the integration in the study of external pre-establishes knowledge or answers that can come from mathematical (what is a circumference? an ellipsis? how to estimate their coefficients?) or from history or archaeology (reports from experts, documented ways how Romans built the buildings, etc.).

## 3. The historical context and its epistemological and didactic role

The starting situation which was presented to students was the discovery of some Roman ruins in a Badalona's suburb by the archaeologists' team of the city Museum. The archaeological report that was provided explained that these ruins could belong to an ancient Roman building, but *what kind of building could it have been? Could be a theatre? A circus? An amphitheatre?* Being inspired by a real archaeological investigation, the context used to design the didactic sequence was an *authentic* situation (Vos, 2011) in the sense that the situation introduced to the students is not created for any educational purposes even though some elements are included for educational purposes.

Three phases can be distinguished in its implementation. In the first phase, the archaeological question was presented to the students as the generating question of the inquiry. At this stage, they could search for information in the web, they had access to the map of the zone where the ruins were found and to the original curved Roman wall found. Moreover, they had accessible an adaptation of the real report of the experts that discovered the ruins. Some questions emerged after analysing the different resources: *What do the geometrical shape of the partial Roman wall discovered determine?* When the students enquire into the information about the Roman architecture, they found there were few buildings that had a curved part. For instance, theatres were circular, amphitheatres were elliptic or circuses had a part circular and other part quadrangular. So that, the students got easily into a geometrical mathematising process. Some students used arguments, based upon the geographical principles: "*Using Google-Maps we can see the shape of the ruins found. The houses and the streets have also curious shapes. It looks like a circle! Therefore, we think there could be a theatre under the current buildings*". In this moment, the students could formulate their first hypothesis according to the Roman wall found. Then, during one session, the group experimented with a replica of a Roman wall. There, they could try out where could be the centre of geometrical shapes, such as: a circle, an ellipsis and other conics. They reach some conclusions like: "*It is possible that the curb was an ellipsis, but if it is so, it only matches near the focus, the rest does not have the same curvature*".

In the second phase, most of students agreed that the most likely shape was a circle and that the building corresponded to a theatre. Then, the teacher guiding the inquiry let them know about the Vitruvius cannon and make accessible to students the information about how different buildings where design by the roman architect, Vitruvius. Students could then improve their initial hypothesis and design their model of the theatre following the canon. In order to draw it properly, they could use the Geogebra as artefact to simulate geometrical models. In the third phase, the students could validate and contrast their hypothesis and first answers by matching the geometrical model simulation with the real image or map of the area studied where the ruins had been discovered. At the end of this last phase, students could interview an expert of the archaeological team of the Museum who let them know about the procedure experts follow in an archaeological inquiry. Which are the tools that make possible the modelling approach?

## 4. Devices to promote the dialectics between inquiry and modelling

In a first phase, the students contrast with the real world, using tools to compare and contrast forecasts provided by the mathematical models with real data. The overlapping Google Map of the zone, over Roman imagined map gives opportunities to conjecture that the wall is part of a circle. The adaptation of the experts' report given to students offer information about the different elements: the curved Roman wall that could belong to the front closure of the theatre, the "proscaenum" near the centre where it is possible to find a small street, and two small squares on the opposite points of diameter would be the exit of the "vomitorium". The students make conjectures about geometrical models through such devices, provided by real data, using profits of codisciplinary approach, to find elements coming from the history, to find mathematical (geometrical) results.

During the second phase, the students essay to proof the conjectures by going to the rules done by the Roman architect as Vitruvius. It emerges the need to use representations according the rules of the history. In this stage, the technology had an essential role in order to allow students constructing a geometrical model using the software Geogebra for validating their first hypothesis. This part of task allowed students to interpret their models considering the specific context and verify if their constructions and hypothesis were suitable. The groups must write a report about their findings and inquiry process. Such device, allowed the teacher to follow the progress of inquiry teams, and to assess the global coherence of the work done, and the final results.

In the third phase, the students have done a lot of work, having many proposals about which building corresponds to the ruins after testing them on the map of the zone studied. They value the opportunity to met a specialist of the Museum, who can contribute to their proposals and clarify the questions appeared. During this interview, the students could contrast and validate their results about the model selected.

## 5. The role of validation to provide completeness and functionality to mathematics

One crucial characteristics of the project is the constant interaction between real context and date with the mathematical tools emerged. At the end of each phase, the students ask to check the validity of the mathematical artefacts used. Such validation moment has a functional role that helps students to raise new questions to continue with the study. For instance, to find if the model build with Geogebra serves to fit other similar ruins, to study Romanization processes, to compare the findings with Greek buildings, etc. This process of inquiry enables them to work differently when facing new other problematic situations. We observed that the way to work (work in collaborative teams, using inquiry and modelling terminology, writing reports involving mathematics and history, defending the arguments and justifications, etc.) empower the ability of organizing mathematical knowledge.

Using Inquiry Based Learning and modelling approach, the students get 'deeper' into the mathematics, or arrive at 'higher' levels of abstraction. There is a didactical intention of "going forward and backward to the real world" in order to promote new questionings. The answers give sense not only to know about but giving sense to the community, not only looking for historical and mathematical legitimation. It also empowers the idea of mathematics to be used for applications. Both, didactical and mathematical aspects of mathematisation, could be empowered in our experiences.

## 6. Conclusions

Due to the contribution of the historical information, the problem became achievable at the students' level of mathematical knowledge. The methods to find these solutions only from the mathematical point of view are totally beyond the powers of secondary school students'. The study and research paths (SRP) act as a didactic device to facilitate the inclusion of mathematical modelling in educational systems, and more importantly, to explicitly situate mathematical modelling problems in the centre of teaching and learning processes (Barquero, Bosch, & Gascón, 2008). Along the SRP the students were asked to build up their own 'piece of history' through: the formulation of hypotheses based on the analysis of real data and historical information.

Historical events lead to use closed models to justify the use of mathematical objects and no more. Nevertheless, the interaction among disciplines open minds of students as a didactical principle. It gives sense to mathematical models and enriches the idea of mathematical applications.

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# Empowering students' decision making in an everyday risk situation

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**Abstract.** In the last years, the interest in statistics education about the need of empowering citizens to become a risk-literate society has increased. Research in the field has proposed possible common concepts between risk and probability for Elementary and High School; however, there are still no suggestions about possibilities for Middle School. We discuss in this paper the possibilities for an innovative curriculum, which should attend to the need of simultaneously developing the concepts of probability, strategies for decision-making and risk management. Moreover, we present a task in which students have to develop strategies to decide which is the best choice between multiple routes to move injured people from a car accident to the nearest hospital. Those routes have adverse, unwelcome and hazardous circumstances as possible sources of risk. We explore the strategies that students use when confronted with verbal, visual and numerical representation of risk. We report on the insights gained from the retrospective analysis of the implementation of the task with 9 Spanish students of grade 8 (aged 13).

**Résumé.** Au cours des dernières années, l'intérêt pour l'enseignement des statistiques à propos de la nécessité d'habiliter les citoyens à devenir une société alphabétisée a augmenté. La recherche dans le domaine a propos des concepts communs possibles entre le risqué et la probabilité pour l'école primaire et secondaire ; cependant, il n'y a toujours pas de suggestions sur les possibilités de l'école intermédiaire. Nous discutons dans cet article les possibilités d'un programme novateur, que devrait participer à la nécessité de développer simultanément les concepts de probabilité, des stratégies pour la prise de décisions et la gestion des risques. De plus, nous présentons une tâche dans laquelle les élèves doivent développer des stratégies pour décider quel est le meilleur choix entre plusieurs routes pour déplacer les blesses d'un accident de voiture à l'hôpital le plus proche. Ces routes ont des effets indésirables, et des circonstances dangereuses comme sources possibles de risques. Nous explorer les stratégies que les élèves utilisent lorsqu'ils sont confrontés à des verbale, visuelle et de représentation numérique de risqué. Nous faisons rapport sur les leçons tirées de l'analyse rétrospective de la mise en œuvre de l'équipe avec 9 étudiants espagnols de la 8 année (âgé, 13).

## 4. Introduction

Over the last years, the interest for policy makers, curriculum developers and researchers in the field of stochastics about the need of empowering citizens to become a risk-literate society has increased. The Canada and USA curricular guidelines suggest empowering students through exploring alternative situations of risk management, which lead them to become confident risk-takers. Meanwhile, many other educational systems, such as Germany, UK or Spain, are looking for ways to embed risk-based decision making into the K-12 school system, through the analysis of the potentialities of decision-making when assigning probabilities (Russel, 2015). Researchers have proposed that the possible common concepts between risk and probability are proportions, conditional probabilities and expected values for Primary Education; and conditional probabilities, Bayes' formula, independence, absolute and relative risk, and distribution of probabilities and frequencies for High School (e.g. Martignon, 2016). Nevertheless, there are no suggestions about the possible curriculum for Middle School (ages 12 to 16).

With the aim of providing continuity in the transitions between Primary Education and High School, this possible curriculum should attend to the need to simultaneously develop the concepts of probability, strategies for decision-making and risk management. From a cognitive point of view, we consider the possibility of decomposing the concept of risk management into smaller pieces of knowledge to analyze the possible changes when making informed decisions in probabilistic contexts. We have called this process systematic *operational risk management* [ORM]. We define operational risk management as a decision-making process to systematically evaluate possible courses of action, identify risk and its benefits, and determine the best course of action for any given situation (Serradó, 2016).

In order to think about the possibilities of this innovative curriculum, we have designed, implemented and retrospectively analyzed a task, which included a systematic ORM.

## 5. Empowering decision-making in risky situations

In this desirable risk-literate society, citizens should have the ability to interpret risk in its many forms in order to make informed life-decisions (Gal, 2005). It would mean empowering people to think for themselves in a Kantian sense, to estimate risk and make choices in the environments they encounter. This *Bildung* may equip them with strategies for translating information about risks provided by (possible) any environment into formats they could naturally understand and deal with. Once they are confronted with adequate information formats, they can make choices and decisions, often by means of very simple heuristics or presented in percentages or even in probabilities (Martignon, 2016). Those who have been empowered by simple rules for dealing with information can translate these formats into so called natural frequencies. Once they grasp the validity or "diagnosticity" of the features in their decision environment in terms of natural frequencies, they tend to make use of simple heuristics for making comparisons, for estimation and for categorization. And, finally they can make informed decisions in any risky situations.

A second controversy emerges due to the kind of numbers that citizens are able to use when assessing risky environments. Gigerenzer (2002) informs us about one main king of statistical innumeracy when assessing risk: the miscommunication of risk. This miscommunication of risk refers to not knowing how to communicate risk. In order to overcome the misunderstanding due to the miscommunication of risk, it is desirable for learners to experience risk information in several formats, and to appreciate that the provider of information may present information in a manner that suits their own agenda, and not necessarily that of the student, understood as a future risk-literate citizen. Three formats of communication are suggested: verbal, visual and numerical.

## 6. Methodology

We present a study, exploratory in nature, with the purpose of developing a deeper awareness of the possibilities of a curriculum for Middle School, that simultaneously develops probabilistic concepts, strategies for decision-making and risk management. The participants in this study were 9 grade 8 (aged 13) students in a coastal city in south Spain. The students solved a task designed as an ORM process.

The task consisted in helping an ambulance driver to decide the best route to minimize the time to move injured people from a car accident location to a hospital. The students have to develop strategies to decide which is the best of multiple possible routes. The routes have adverse, unwelcome and hazardous circumstances, such as sources of risk, which could modify the driving time for a particular route. We explore the strategies that students use to decide the best route when they are successively confronted by informed decisions with visual, verbal and numerical communication of risk in four stages. The task was implemented collecting written and verbal data about their responses related to the strategies used to decide the best of multiple routes to drive injured people from a car accident to a nearest hospital with verbal, visual and numerical representation of risk. A retrospective analysis of the data was conducted.

## 7. Results and discussion

In the first stage of the task, students had a map, as a visual communication, with 13 different unidirectional routes with hazardous situations such as many traffic lights, a school, people walking along the route and works on the road. Moreover, they were provided with verbal communication that the accident took place at night, and that the ambulance, driving with a uniform speed of 50 Km/h, and did not need to stop if a traffic light was red.

The verbal communication of the information made students feel that there were not any risks. In consequence, they reduced their strategies to the algebraic codification of these routes (8 of 9 students), computed the length of most of the routes (8 of 9 students), computed all the routes (1 of 9 students), and used proportional reasoning to compute the time needed in each route (1 of 9 students). Students used basically arithmetic procedures and heuristics, which helped them to compared and decide the best of multiple routes.

In a second stage, with the same visual communication of risk, a map, students were asked to reason about how they would change the route selected if the driver must stop at a traffic light in the case of its being red. We consider that the students misunderstood the description of risk, perceiving the risk of stopping at a red light as certain action, not as a possibility. Seven students of nine did not change their previous nomination of the route. Five of the students added to the previous heuristics by counting the number of traffic lights of every route, selecting the route with the least number of them. Four of the nine students considered when comparing the best route both the number of traffic lights and the length of each route. In consequence, they made informed decisions.

In the third stage, students were asked exclusively to analyse the map and describe possible sources of risk. Pedestrians in the road, works on the road or a college were identified by eight of the nine students as sources of risk due to hazardous situations. Nevertheless, only four of nine students considered the traffic lights as sources the risk due to slowing down the driving.

In the last and fourth stage, students were also provided with numerical information about the possible risks. Those risks were expressed as probabilities. One student did not perceive the visual and numerical risks, considering only the time in absolute terms. Six of the students changed their initial route, combining both the perception of the works on the road as a risk source and the analysis of the numerical of traffic lights and the length of the road. Two of the students used the numerical information about the probabilities, the length of the routes, and the uniform speed to estimate the possible risks and compare two or three routes. Their estimations came from a computation considering the division of length by speed and multiplying by the probability of the hazardous situation in each route. In consequence, they made well informed decisions about the route they consider less risky in terms of the possible time of the journey.

Students perceived and reacted to risk as a feeling for those hazardous situations visually communicated, such as the pedestrians or works on the road. Meanwhile, the traffic lights were not recognised as adverse or unwelcome sources of risk by all the students due to the influence of verbal information, understood in absolute terms. Those students who understood it in absolute terms made a deterministic analysis of the situation. Nevertheless, those students, who considered the uncertainty of the colour of the traffic lights, perceived the sources of risk, analysed the situation and made informed decisions. When we introduced the numerical communication of risk, we observed differences in the strategies used by the students. Those students with a faulty use of probability who understood the risk numerically communicated, developed numerical strategies to compute it and decide in accordance with their computations.

## 8. Conclusions

We have analyzed the possibilities of innovating Lower Secondary School curriculum through the introduction of a task which aimed to integrate probabilistic knowledge, strategies for decision-making and risk management. The task was designed using systematic operational risk management, considering of four stages of visual, verbal and numerical communication of adverse, unwelcome and hazardous situations as sources of risk. Students perceived and reacted to risk differently in the three formats of communication of risk. The visual communication provided students with risk feelings only. The verbal communication was not only an opportunity to undertake risk analysis; but also, a constriction due to the deterministic perception of the situation. The numerical communication of the risk needed the knowledge of probability to manage risk, compute it and decide accordingly.

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# Young children solving multiplicative reasoning problems

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**Abstract.** This paper describes a study focused on kindergarten children multiplicative reasoning. It addresses two questions: 1) how do children perform when solving multiplication, partitive and quotitive division problems? And 2) what arguments do children present to justify their resolutions? An intervention program comprising 4 sessions was conducted with 12 kindergarten children (5-6-years-old), from a state supported kindergarten, in Viseu, Portugal. Similar Pre- and Post-tests were used to identify changes on children's understanding during the intervention program. In each test children solved 28 problems (18 additive structure problems; 6 multiplicative structure problems; 4 control problems on geometry) in two different consecutive moments. The intervention comprised 4 partitive division problems, 4 multiplication problems, and 4 quotitive division problems. The problems were presented to the children by the means of a story, and material was available to represent each problem. After each resolution, each child was asked "Why do you think so?". Results suggest that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems. Results also suggest that children's multiplicative reasoning can be enhanced when they can experience multiplicative structures problem solving, being able to interact with peers and discuss their ideas, after receiving some prompts from teacher. This study also suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Educational implications of these findings will be discussed.the addresses.

**Résumé.** Cet article décrit une étude centrée dans le raisonnement multiplicatif des enfants de l'école maternelle. L'étude aborde deux questions: 1) comment les enfants résolvent des problèmes de multiplication et division, partitive et quotitive? Et 2) quels arguments les enfants présentent-ils pour justifier leurs résolutions? Un programme d'intervention comprenant 4 sessions a été organisé avec 12 enfants (5-6 ans), d'une école maternelle publique à Viseu, au Portugal. Des Pré- et Posttests similaires ont été utilisés pour identifier les changements sur la compréhension des enfants pendant le programme d'intervention. Dans chaque test, les enfants ont résolu 28 problèmes (18 problèmes de structure additive, 6 problèmes de structure multiplicative, 4 problèmes de contrôle sur la géométrie) en deux moments consécutifs différents. L'intervention comprenait 4 problèmes de division partitive, 4 problèmes de multiplication et 4 problèmes de division quotitive. Les problèmes ont été présentés aux enfants au moyen d'une histoire, et le matériel était disponible pour représenter chaque problème. Après chaque résolution, chaque enfant a été interrogé "Pourquoi tu as fait comme ça?". Les résultats suggèrent que les jeunes enfants peuvent réussir à résoudre des problèmes de multiplication et de division, en s'appuyant sur leurs connaissances informelles, en présentant des arguments qui montrent qu'ils sont en mesure d'établir le raisonnement correct lors de la résolution des tâches, en articulant correctement toutes les quantités impliquées dans les problèmes donnés. Les résultats suggèrent également que le raisonnement multiplicatif des enfants peut être amélioré lorsqu'ils peuvent résoudre des problèmes multiplicatifs, interagir avec des pairs et discuter leurs idées, après avoir reçu des instructions du professeur. Cette étude suggère encore que le raisonnement additif et multiplicatif, sous leurs formes simplistes, semble être simultanément accessible aux enfants de l'école maternelle, ce qui leur permet d'avoir un sens. Les implications éducatives de ces résultats seront discutées.

## 1. Framework

Children possess informal knowledge relevant for the learning of mathematical concepts. The mathematical ideas children acquire in kindergarten constitute de basis of future mathematical learning. Thus, the development of the mathematical skills in early age is crucial to the success for future learning (NCTM, 2008). Children can use their informal knowledge to analyze and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996). But they also can know quite a lot about multiplicative reasoning when they start school (Nunes & Bryant, 2010b).

Piagetian theory supported the idea that children first quantify additive relations and can only quantify multiplicative relations much later (see Piaget, 1952). In spite of his undoubted contribution to research, more recently research has been giving evidence of a different position. Thompson (1994), Vergnaud (1983) and Nunes and Bryant (2010a) support the idea that additive and multiplicative reasoning have different origins. Vergnaud (1983), in his theory of conceptual fields, distinguishes the field of additive structures and the field of multiplicative structures, considering them as sets of problems involving operations of the additive or the multiplicative type. Vergnaud (1983) argues that "multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects" (p.128). Nunes and Bryant (2010a) also consider that additive and multiplicative reasoning have different origins, arguing that "Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant." (p.11).

Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: "Joe bought 5 sweets. Each sweet costs 3p. How much did he spend?" Or "Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?" are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes & Bryant, 2010a).

Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school. For that they use the schema of one-to-many correspondence. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993), reported high percentages of success when observing kindergarten children solving multiplicative reasoning problems involving correspondence 2:1, 3:1 and 4:1. Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: "In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?", 60% of the 1$^{st}$-graders and above 80% of the children of the other grades succeeded. When children were asked to solve a division problem, such as: "There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?", the levels of success for 1$^{st}$-graders was 80% and above that for the other graders (2$^{nd}$ to 4$^{th}$-graders).

In Portugal, there is still not much information about kindergarten children understanding of multiplicative reasoning, relying on their informal knowledge.

## 2. Methods

An intervention program was conducted with 12 kindergarten children (5-6-years-old), from a state supported kindergarten in Viseu, Portugal. Pre- and Post-tests were used to identify changes on children's understanding during the intervention program.

### *2.1 The intervention program*

In the intervention program, the participants were divided into three groups of four elements each, having each the same age and pre-test results conditions. The intervention took place in the pre-test following week and lasts for 3 weeks. Four sessions were planned, organized by level of difficulty, equal to all the groups. In each session children solved 3 problems, and the same kind of problems was explored twice a week. Each group had the opportunity to discuss and solve the same type of problem 4 times, in a total of 12 problems. The tasks presented to the children, during the intervention comprised 4 partitive division problems (for example, "Sara has 10 candies to give to 5 children. She is doing it fairly. How many candies is each child receiving?"), 4 multiplication problems (for example, "Bill has 3 boxes with pencils. Each box has 4 pencils. How many pencils does Bill have in total?"), and 4 quotitive division problems (for example, "The teacher Anna has 12 children in her group. She wants to seat the children in groups in the tables. Each group must have 4 children. How many tables does teacher Anna need?"). The problems presented to the children in the intervention program were similar to those of the multiplicative structure problems given in the tests. All the problems were presented to the group of children by the means of a story, and material was available to represent them. After each answer, each child was asked "Why do you think so?" in order to reach a better understanding of his/her reasoning. No judgments were conducted, and group discussion was stimulated. Video recorder and field notes were used in data collection.

### *2.2 Pre- and Post-tests*

Individual interviews were used in the Pre- and Post-tests, in which children solved a battery of 28 problems (18 additive structure problems; 6 multiplicative structure problems; 4 control problems) in two different moments. The problems presented to the children were selected and adapted from the Vergnaud's classification (see Vergnaud, 1982, 1983).

The problems of both tests were similar. The additive structure problems presented to children in the tests comprised: i) composition of two measures, (for example, "Mary has 8 dolls but only 2 are in the box. How many dolls are outside the box?"); ii) transformation liking two measures, with the starting and element of transformation omitted, (2 for addition, 2 for subtraction), (for example, "There are 5 frogs in the lake. Some more join the group. Now there are 8 frogs. How many frogs came to join the group?"); iii) static relation linking two measures, (2 involving "more than", 2 for "less than"), (for example, "Anna has 4 puppies. John has 2 more than Anna. How many puppies does John have?"). The multiplicative structure problems in the tests comprised: iv) Isomorphism of Measures, selecting the problems of Multiplication, Partitive Division, and Quotitive Division. The control problems included only geometry tasks (geometric regularities and shape with tangram).

The problems were presented to the children by the means of a story, and material was available to represent the problems. After each resolution, each child was asked "Why do you think so?" in order to reach a better understanding of his/her reasoning. Data was registered in video and researcher's field notes.

## 3. Final remarks

This study explores the effects of a short intervention program focused on multiplicative reasoning on young children solving additive and multiplicative structure problems. Results will be presented in the conference. The intervention was effective as children improved their understanding of multiplicative reasoning problems. Multiplication problems revealed to be easier for children than division ones. Also children's arguments revealed improvements. Young children provided arguments and explanations that sustain the idea that their successful resolutions were not obtained randomly, as they were supported by valid or partially valid explanations.

Previous research on kindergarten children solving multiplicative reasoning problems (see Carpenter, et al., 1993) reports levels of success, but does not refer children's explanations or arguments to give a better insight of children's way of thinking. Also Nunes et al. (2005) reports remarkable success levels when 1st-graders solve multiplication and division problems, but give no reference to their explanations. The study reported here gives evidence that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems.

This study suggests that children's multiplicative reasoning can be enhanced when they can experience problem solving being able to interact with peers and discuss their ideas, after receiving some prompts from teacher. In agreement with Soutinho's (2016) ideas, this study suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Thus, perhaps kindergarten mathematics should include more of these experiences in order to develop children's mathematisation. Educational implications will be discussed.

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# Teaching shifts using as tool mathematical modeling

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**Abstract.** An important challenge for students in the secondary school is to be able to move 'comfortably' in different contexts, transferring mathematical knowledge during transitions between these contexts. In this presentation, we explore possibilities of knowledge transfer during transitions[13] between micro-contexts —as they are determined by digital or tangible material— of mathematical practices. Focusing on the idea of shorter path, students of 7th grade are involved in horizontal and vertical mathematization through realistic situations and algebraic and geometrical representations.

**Résumé.** Un défi important pour les étudiants dans l'école secondaire est d'être capable de bouger 'confortablement' dans les contextes différents, en transférant la connaissance mathématique pendant les transitions entre ces contextes. Dans cette présentation nous explorons des possibilités de transfert de connaissance pendant les transitions entre les micro-contextes — puisqu'ils sont déterminés par la matière numérique ou tangible — des pratiques mathématiques. En se concentrant sur l'idée de sentier plus court, les étudiants de 7ème qualité sont impliqués dans mathematization horizontal et vertical par les situations réalistes et les représentations algébriques et géométriques.

## 1. Introduction

We discuss part of a research project that was conducted in order to explore issues of knowledge transfer between micro-contexts of mathematical practices as they are defined by digital tools and tangible materials. The focus of the part we present here concerns the big scientific idea: «The nature selects the most economic mode of development, based on shorter paths and the consumption of minimal energy together with maximizing the work produced». The kick-off activity was based on the 'car road problem'[14]. The tool used for modeling was the software of dynamic geometry, Geometer's Sketchpad, with arrangements of Plexiglas and pins as shown in fig. 3 as hands-on materials. The didactical performance was based on the pedagogical approach: «Following students' intentions and expectations» (Tsitsos & Stathopoulou, 2011).

## 2. Some theoretical points

Mathematization in formal teaching concerns the mathematical modeling process with starting point situations or objects - real or imaginary- that make sense for students. So mathematization is a bridge through the educational community of the classroom between students' knowledge and capabilities, and mathematical structures socially, historically and culturally defined. We can see the mathematization[15] as a

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[13] About Transitions between contexts of mathematical practices see: de Abreu, Bishop, & Presmeg (2006).

[14] The motorway problem: The problem concerns the finding of the shortest path that connects four cities located at the four vertices of a rectangle.

[15] The Freudenthal (1991) and Treffers (1987) distinguish mathematisation in horizontally and vertically. The horizontal mathematization concerns the transition from the real world to the world of symbols. It constitutes a metaphor, a shift of semantic structures. The vertical mathematization concerns the reorganization of these concepts of the mathematical framework and τhe connection of magnitudes involved.

transition process – an initiation or entrance into new worlds, one world of human experience which is subjective and changeable, and another, 'objective world of knowledge', produced by science (Husserl, 1970; Patocka, 2008). The mathematization in this sense is a tool of pedagogical action; students have been taught how to find and work with authentic situations and real-life problems (Rosa & Orey, 2010). Bassanezi (2002, p. 208) describes this process as "ethno-modeling".

The main research pillars of the research presented here are both the activity theory approach and the dynamic dimension of knowledge transfer. The activity theory, based inter alia on the Cole work (1996), Engeström, (1999), influenced by Vygotsky (1978) and the Russian tradition, falls on sociocultural approaches with central idea that learning activity is a collective action mediated by cultural symbols, words and tools (Cole, 1996).

The lesson is conceived as an activity system (Engeström & Cole, 1997, p. 304) that both mediates between subject and object, and integrates the community in which the subject belongs, including the rules they are governed by, and the division of labor. 'Dynamic transfer' (Schwartz, Varma, & Martin, 2008) refers to possibilities for prior knowledge to create new concepts, in contrast with 'similarity transfer': In dynamic transfer, people coordinate situations in order to deal with new concepts; the coordination process is performed in this research with the contribution of different frameworks, and mostly via the digital tool of Dynamic Geometry (Geometer's Sketchpad). Students were focusing on the concept of covariant quantities, in an effort to extend previous work by James Kaput (1996), who conjectured that the difficulties in understanding the concept of regression might have to do with a lack of connection between the representations (numerical, algebraic, analytical, tabular, graphical) and the physical or virtual experience of student; Kaput suspected that the graphical representation of covariance might better precede the algebraic and analytic representation.

## 3. Methodological issues- the method of the research

The methodology followed is that of action research; specifically, we adopted the model of Kemmis and McTaggart (1988), which consists of a series of actions where the researcher repeats the following cycle of steps: plan, act, observe and reflect upon. The actions were organized into ten (micro) cycles, consisting of activities that belong to the category of Design Experiments (Cobb et al, 2003). Both the collection and analysis of data were based on Grounded Theory (Classer and Strauss, 1967), where the task of the researcher is to understand what happens during the research procedure, and to focus on how the people involved manage their roles; analysis in this approach aims to define categories through which the researcher can highlight a theory implied by the data. The basic analysis unit is the activity.

<u>The setting of the research</u>

The research was conducted in a typical 8th grade class with 24 students. It was developed in six teaching hours of curriculum. The students worked divided into six groups of four people.

## 4. Design and implementation of the research

The main pillar of the teaching objective was to develop a teaching situation where the students would be able to capture aspects of the abstract mathematical concept of covariance, without explicit direction, while the research objective was to investigate issues in students' transitions between the mirco-contexts of mathematical practices.

A short description of the 10 micro-cycles[16] that were developed during our interventions follows. Students designed possible routes with the shortest distance connecting all four cities (Figure 1 and 2), comparing their measurements while experimenting with the real model (Figure 3), adjusting the virtual model in order to match to the actual; they explored the shortest distance on the interactive environments depicted on the pricing table (Figure 4), and tested possible analytical correlations of the involved magnitudes (Figure 5). Figure 6 is an example of how a student had the chance to explore one hypothesis on his own, by re-designing figure 5.

These student actions moved them towards finding algebraic relationships between the magnitudes

[16] As micro-cycle it is considered a task delimited in time resulting from a community of practice constituted by students and teacher and it has an autonomy.

involved in the problem. Then, based on the students' question, "How could the computer automatically find the shortest path?", we turned their attention to the investigation of the problem in designing parametric figures conforming to geometric conditions (Figure 7). This was the chance to pose a new question: "In the case where the cities are not arranged in a rectangle, how might we calculate the shortest path?". This question led to a geometric generalization (Figure 8), and in this way contributed to further emerging questions and the modification of the real model, and hence, to the empirical determination of the geometric solution. Through the formation of the films and the comparison of the position of the nodes in both the real model and the visual one, students posed additional questions, for example: "How does the film know that this is the shortest path?"Around this phenomenon a discussion was developed about the fact that nature seems to always choose the most economical way to act, selecting symmetry to accomplish this.

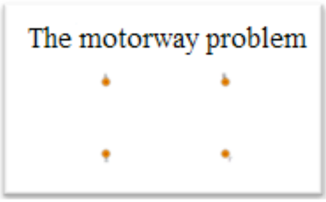
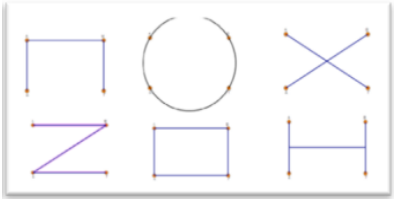



*Figure 3.*



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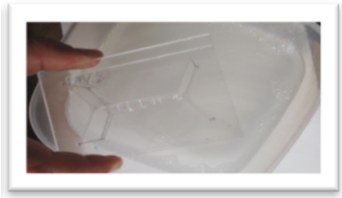

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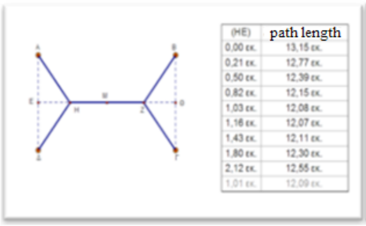


*Figure 4.*

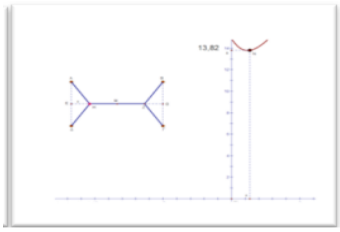

*Figure 5.*

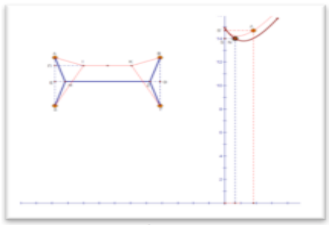

*Figure 6.*

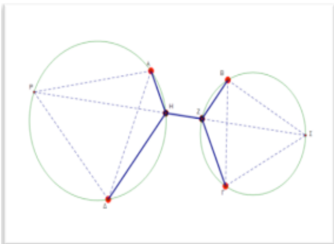

*Figure 7.*

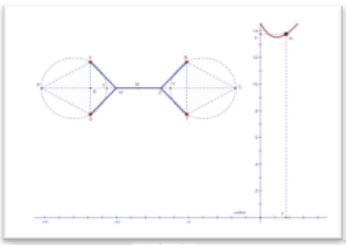

*Figure 8.*

## 5. Discussion

The whole procedure constituted a coordination of different cognitive areas and representations that combined different techniques and methods, aiming to emphasize for the students that mathematics does not function in a piecemeal or fragmented way, and is characterized by continuity, structure, affinity, and in connection with other sciences. The action research cycles made evident how the challenging conceptual points that appeared were faced on the one hand by a repositioning of the problem in the Dynamic Geometry framework, and on the other hand by modeling the situation of the problem. Opportunities to make conjectures, refutations and cognitive connections made it possible for students to experience both the repositioning and the modeling. The notion of transitions to different contexts in this research is connected with knowledge transfer. The transfer is considered as the coordination of all these contexts. The crucial point of the transitions is the real experiment (experiment in real situations). In each micro-cycle, we were able to recognize as interconnected both horizontal and vertical mathematization, and even that some very different epistemological approaches and different types of representations were involved. What is also included in our research but cannot be discussed here, because of the limited space, is the study of structures that appeared through the 10 micro-cycles in other thematic areas, e.g., 'How might we represent magnitudes that covary?', 'Which is the role of the minimum on the graph?', 'What does a geometrical construction do?'. We believe that these kinds of questions emerge because of the kind of instructional approach that was enacted within this action research. Both the learning outcomes of the activity and the findings of the research create 'un-mapping regions' of the generalization and abstraction of mathematical concepts related to the digital tools and the tangible materials. Students learn more than the mathematics: they also learn along with the mathematics a belief in mathematics as opening up questions and exciting directions for exploration. Researchers find new pathways for un-mapping both the classroom learning environment and the content they originally thought was being taught.

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# Favoriser la dévolution de la mathématisation horizontale aux élèves engagés dans une activité de modélisation

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**Résumé.** Dans nos travaux, nous étudions les conditions et les contraintes permettant de favoriser la dévolution de la mathématisation horizontale aux élèves de collège et de lycée engagés dans une activité de modélisation. Notre objectif est d'aider les élèves à comprendre la nécessité de faire des choix pour envisager un traitement mathématique d'un problème posé dans un domaine non mathématique. Nous présentons dans cette communication les éléments d'analyse *a priori* d'un problème posé dans un contexte relevant des sciences du vivant, expérimenté en 2016 dans une cinquantaine de classes de l'enseignement secondaire français, ainsi que les tous premiers éléments de notre analyse *a posteriori*. Les premières analyses des travaux d'élèves, en appui sur nos indicateurs de dévolution de la mathématisation horizontale, montrent la faisabilité et la fécondité potentielles d'une telle activité mathématique.

**Abstract.** In our research, we study the conditions and constraints to favor the devolution of horizontal mathematization to secondary school pupils engaged in a modeling activity. Our goal is to help students understand the need to make choices to consider mathematical treatment of a problem in a non-mathematical field. We present in this paper the elements of *a priori* analysis of a problem posed in a life sciences context, experimented in 2016 in fifty classes of French secondary education, as well as the very first elements of our analysis *a posteriori*. The first analyzes of student work, based on our indicators of devolution of horizontal mathematization, show the potential feasibility and fertility of such a mathematical activity.

## 9. Introduction

Nous présentons dans cette communication des éléments d'une recherche en cours sur la possibilité de faire la dévolution (au sens de Brousseau 1998) aux élèves du secondaire de la mathématisation *horizontale*. Nous suivons en cela Treffers (1978) qui distingue deux types de mathématisation en jeu dans une activité de modélisation en mathématiques : la mathématisation *horizontale* qui relève de la modélisation mathématique d'un fragment de réalité non mathématique et la mathématisation *verticale* qui relève du traitement mathématique d'un problème mathématique.

Dans ce texte, nous nous intéressons à la mathématisation horizontale la première étape de la modélisation qui consiste à opérer des choix permettant d'envisager un traitement mathématique du problème. Cette étape du processus de modélisation n'est généralement peu voire pas travaillée dans les classes, alors qu'elle est essentielle dans le travail de modélisation des chercheurs travaillant dans le domaine des mathématiques appliquées aux sciences du vivant.

La question didactique que nous mettons à l'étude est celle de la possibilité d'implémenter en classe des situations didactiques permettant de faire vivre cette étape aux élèves tout en prenant en compte les conditions et les contraintes qui pèsent sur les enseignants. Ce questionnement est né des observations conduites depuis plus de dix ans au sein du groupe ResCo (Résolution Collaborative de problèmes) de l'IREM de Montpellier, qui vise à mettre les élèves en position de chercheur (Yvain et Gardes, 2014)

Pour conduire cette étude, nous nous sommes placée dans le cadre de situations de résolution de problèmes posés dans un contexte relevant des sciences du vivant et pouvant se résoudre via une modélisation mathématique.

L'objectif de notre recherche est le développement d'une ingénierie didactique permettant de favoriser ce processus de dévolution. Pour cela, conformément à la méthodologie générale de l'ingénierie didactique

(Artigue, 1988), nous conduisons une étude épistémologique visant à identifier précisément ce que l'on va transposer des savoirs savants d'une part, des pratiques de références des experts du domaine d'autre part. Cette étape de notre recherche est en cours : plusieurs entretiens avec des chercheurs en mathématiques appliquées aux sciences du vivant et des chercheurs en biologie utilisant des mathématiques dans leurs travaux sont en cours d'analyse (Yvain, à paraître). Dans ce qui suit, nous décrivons tout d'abord brièvement les modalités de travail du groupe Resco. Nous présentons ensuite les critères de choix, l'énoncé et l'analyse *a priori* d'une situation mathématique propice à permettre le développement d'une situation didactique pouvant être utilisée dans le cadre du dispositif ResCo. Pour finir, nous présentons les premiers éléments de notre analyse *a posteriori*.

## 10. Un dispositif et des situations pour favoriser la dévolution de la mathématisation horizontale

Le groupe ResCo existe depuis le début des années 2000. La résolution collaborative repose sur des échanges entre des classes, par groupe de trois, qui travaillent sur le même problème de recherche, pendant cinq semaines. Toutes les classes du secondaire de la 6ème à la Terminale[17] sont concernées.

Ce dispositif comporte l'élaboration chaque année d'une nouvelle situation, un stage de formation de deux jours pour former et accompagner les enseignants qui souhaitent engager leurs classes, et la mise en œuvre dans les classes engagées pendant cinq semaines consécutives.

Les problèmes élaborés par le groupe puis proposés pour une session de résolution collaborative sont posés en dehors du cadre mathématique ; ils sont choisis de sorte que plusieurs modèles mathématiques sont envisageables. Néanmoins, étant donné que la mathématisation de problèmes réels (tels que l'on peut les rencontrer au niveau de la recherche) est généralement beaucoup trop complexe pour être prise en charge au niveau de l'enseignement secondaire, les situations proposées ne sont pas directement issues de la réalité mais relèvent ou évoquent de façon réaliste des situations du monde réel. C'est la raison pour laquelle elles sont qualifiées de «fictions réalistes» (Aldon et al., 2014). Nous renvoyons à l'atelier de S. Modeste pour plus de détails quant aux spécificités de ce dispositif didactique.

Lors de la première séance, les élèves découvrent le problème et préparent des questions qu'ils adresseront aux deux classes avec lesquelles leur classe est associée. L'objectif est de faire émerger un questionnement sur les différents choix possibles permettant un traitement mathématique du problème. La place de cette phase de questions dans le dispositif vise à déclencher le processus de mathématisation horizontale. Lors de la phase des questions, les élèves commencent à choisir des fragments de la réalité fictionnelle sur lesquels ils se questionnent. Dans la phase des réponses, ils commencent à faire des choix de mathématisation horizontale.

Nous illustrons ce qui précède dans le cas de la situation mise en œuvre 2016.

## 11. Un exemple de mise en œuvre: la croissance des arbres

### *3.1 La situation choisie – motivations – éléments d'analyse a priori*

Pour favoriser un travail de mathématisation horizontale, nous avons élaboré une situation permettant:

- d'amener les élèves à se questionner sur le système à modéliser - $C_0$

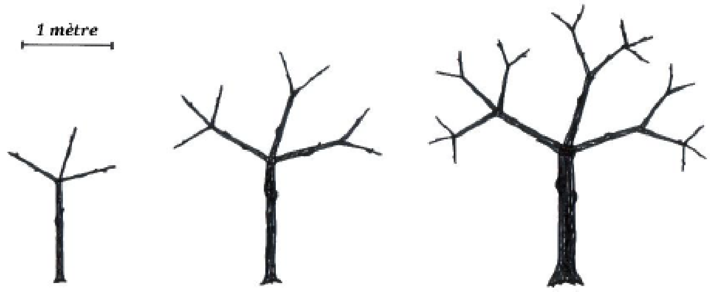
De faire prendre conscience aux élèves:

- de la nécessité de recourir à une modélisation pour répondre au problème - $C_1$ :
- de la nécessité de faire des choix lors de la modélisation - $C_2$
- de l'importance de la question posée dans la construction du modèle - $C_3$
- que le travail de modélisation nécessite un travail mathématique au sein du modèle choisi pour apporter des réponses aux questions posées - $C_4$

La situation proposée aux élèves est la suivante : il s'agit de prédire la croissance d'un arbre à partir d'informations sur ses premières années de croissance données par des schémas avec une échelle donnée. En voici l'énoncé.

[17] (De 11 à 18 ans)

**L'arbre :**
Des botanistes du Jardin des Plantes ont rapporté un arbre exotique inconnu, dont on vient de découvrir l'espèce. Pour étudier cette nouvelle espèce, les botanistes ont réalisé les croquis de l'arbre chaque année depuis 2013.




Les botanistes veulent construire une serre pour protéger l'arbre. Ils estiment qu'il aura atteint sa maturité en 2023. Pour les aider, prévoyez comment sera l'arbre en 2023.

*Figure 1.* ***Énoncé de la fiction réaliste « l'arbre » (échelle modifiée)***

Cette proposition, conçue comme une transposition d'un problème de modélisation de la croissance des végétaux en sciences de la vie (Varenne, 2007) répond à nos critères dans la mesure où

- L'échelle de temps lent de croissance de l'arbre et la trop grande complexité du système motivent le besoin de prévision et la nécessité de produire un modèle.
- Plusieurs hypothèses sur la croissance de l'arbre peuvent être faites ce qui donne lieu à différentes modélisations. Il est donc nécessaire de faire des choix lors du processus de modélisation.
- Les choix dépendront des données du problème (schémas de l'arbre) et des outils mathématiques à disposition des élèves (ou la conception qu'ils en ont).

Pour favoriser la dévolution aux élèves de la mathématisation horizontale, les valeurs des variables didactiques choisies et leur motivation sont les suivantes :

- Des schémas en 2D (et non pas en 3D) pour proposer un cadre suffisamment réaliste tout en permettant une activité de modélisation de la 6ème à la terminale.
- Le nombre de schémas : nous en avons proposé trois.
- La forme de l'arbre (symétrique versus asymétrique): nous avons choisi une croissance asymétrique pour favoriser le questionnement des élèves autour de la prévision de la croissance de l'arbre.
- Le nombre de nouvelles branches apparaissant chaque année : nous avons choisi de faire apparaitre deux ou trois nouvelles branches pour amener rapidement les élèves à faire des choix au niveau de la croissance de l'arbre.
- Les longueurs des troncs et branches : elles ont été choisies pour questionner un éventuel choix d'un modèle de croissance régulière.
- La donnée d'une échelle : afin de permettre des mesures et une prise d'information instrumentée sur les dessins

Pour la phase de l'élaboration des questions, nous avons défini les trois indicateurs de la dévolution aux élèves de la mathématisation horizontale suivants :

La question

- porte sur l'identification de grandeurs pertinentes pour permettre un traitement mathématique ($Q_{C0}$)
- montre la recherche d'un modèle permettant de traiter la situation proposée ($Q_{C1}$)
- porte sur la pertinence d'éléments de contexte à prendre en compte ($Q_{C2}$)

Pour la phase de l'élaboration des réponses, nous avons défini les cinq indicateurs de la dévolution aux élèves de la mathématisation horizontale suivants :

La réponse montre :

- la recherche d'un modèle permettant de traiter la situation proposée ($R_{C0}$)
- des choix de grandeurs pertinentes pour permettre un traitement mathématique ($R_{C1}$)
- des choix d'éléments de contexte ($R_{C2}$)
- l'analyse par les élèves de la pertinence de question reçue au regard de la question posée ($R_{C3}$)
- un premier travail mathématique pour répondre à la question reçue. ($R_{C4}$)

*3.2 Premiers éléments d'analyse a posteriori*

Les données recueillies lors de l'observation de l'intégralité du dispositif dans certaines classes engagées dans la résolution collaborative en 2016 sont en cours d'analyse à la lumière du questionnement sur la dévolution de la mathématisation horizontale aux élèves. Nous donnons ici des exemples (issus des travaux d'élèves recueillis sur la plateforme d'échanges du dispositif ResCo) qui permettent d'illustrer la manière dont on va utiliser les indicateurs de la dévolution de la mathématisation aux élèves.

| **Questions type $Q_{C0}$**<br>la question porte sur l'identification de grandeurs pertinentes pour permettre un traitement mathématique | Réponses associées | Indicateurs |
|---|---|---|
| Quelle est la circonférence des branches de l'arbre en novembre 2013-2014 et 2015 ? | on mesure le diamètre sur le dessin, et on trouve 1mm la 1ère année,<br>on convertit en grandeur réelle grâce à l'échelle : 1 m est représenté par 2,8 cm, donc 1 mm représente 3,57 cm. On multiplie par pi pour obtenir la circonférence, ce qui donne :<br>en 2013 : 11,21 cm<br>en 2014 : 22,43 cm<br>en 2015 : 33,64 cm | $R_{C4}$ |
| Le nombre de branches augmente-t-il chaque année? | 2 ou 3 branches de plus par branche, mais on n'a pas trouvé de modèle qui définisse le nombre de branches supplémentaires à chaque fois. | $R_{C0}$ |
| Y a-t-il un rapport entre les angles formés par les branches ? | Les angles ont l'air de varier entre 60 et 80° | $R_{C4}$ $R_{C1}$ |
| Quand on compte les branches, doit-on compter toutes les branches (avec celles du milieu) ou seulement les nouvelles branches ? | On pense qu'il ne faut que les nouvelles branches. | $R_{C1}$ |
| Combien d'étapes d'évolution jusqu'en 2023 ? | 2015-2016 2016-2017 2017-2018 2018-2019<br>2019-2020 2020-2021<br>2021-2022 2022-2023<br>On a compté qu'il y avait 8 étapes | $R_{C4}$ |

*Figure 2.* ***Exemples de questions recueillies relevant de l'indicateur QC0, réponses et indicateurs associées.***

| **Questions type $Q_{C1}$**<br>La question montre la recherche d'un modèle pour traiter la situation | Réponses associées | Indicateurs |
|---|---|---|
| Est-ce que l'arbre grandit proportionnellement ? | Non, l'arbre ne grandit pas proportionnellement car nous avons fait des mesures :<br>Années Hauteur max. de l'arbre (en cm, sur le schéma)<br>2013 : 4,7<br>2014 : 7,1 | $R_{C4}$ |

| | | |
|---|---|---|
| | 2015 : 8,6<br>De 2013 à 2014, il a grandi de 2,4 cm (sur le schéma) et seulement de 1,5 ensuite de 2014 à 2015. | |
| Doit-on travailler en 2D ? ou en 3D ? | Plutôt en 3d pour plus de réalisme | $R_{C0}$ |
| Les branches grandissent au cours du temps ainsi que le tronc, peut-on estimer la pousse de chaque branche ? | Toutes les branches d'une même génération grandissent de la même façon/même proportion. Et les anciennes générations poussent moins vite que les nouvelles | $R_{C0}$ |
| Le nombre de branches évolue-t-il comme la suite de Fibonacci ? | Non il suffit de regarder le nombre de branches sur les trois premiers dessins pour se rendre compte que le nombre de branches à une année n'est pas la somme du nombre de branche des deux précédentes. | $R_{C4}$ |

*Figure 3.* **Exemples de questions recueillies relevant de l'indicateur $Q_{C1}$, réponses et indicateurs associées.**

| **Question type $Q_{C2}$**<br>La question porte sur la pertinence d'éléments de contexte | Réponses associées | Indicateurs |
|---|---|---|
| Taille-t-on l'arbre? | Il faut supposer que non pour se simplifier la tâche | $R_{C0}$ - $R_{C2}$ |
| La présence de la serre va-t-elle avoir une influence sur la croissance? | Nous ne pensons pas que la serre influencera la croissance. | $R_{C1}$ |
| Quels sont les apports d'eau dont requiert l'arbre afin d'optimiser sa croissance? | Nous pensons que cela n'a pas d'importance. | $R_{C3}$ |
| Le matériau de la serre ou sa couleur influencent-ils la pousse de l'arbre (favorisation de la photosynthèse, maintien d'une température optimale,...). | La couleur n'aura pas d'influence si elle laisse passer la lumière naturelle, pareil pour le matériau tant que la température intérieure *correspond à celle du milieu d'origine.* | $R_{C2}$ |
| L'environnement du Jardin des Plantes est-il semblable à l'environnement initial de l'arbre ? | Nous ne pouvons pas savoir. | $R_{C3}$ |

*Figure 4.* **Exemples de questions recueillies relevant de l'indicateur $Q_{C2}$ réponses et indicateurs associées**

Nous donnons également quelques exemples ci-dessous de questions-réponses ne relevant pas de nos indicateurs de dévolution de la mathématisation horizontale :

| Questions | Réponses |
|---|---|
| Qu'est-ce qu'un botaniste ? | C'est un spécialiste des végétaux. |
| Que veut dire le mot serre ? | C'est un abri pour arbres en verre |
| Est-ce qu'il y a des animaux sur l'arbre ? Si oui, vont-ils faire des nids ? | Dans la serre il n'y aura pas d'animaux ni d'insecte. |

*Figure 5.* **Exemples de questions - réponses ne relevant pas de nos indicateurs de dévolution**

Nos premières analyses confortent notre hypothèse selon laquelle le dispositif de questions-réponses est de nature à favoriser la dévolution aux élèves de la mathématisation horizontale.

## 12. Conclusions et perspectives

En nous appuyant sur les indicateurs de dévolution de la mathématisation horizontale, nos premiers résultats montrent qu'il est possible de laisser les élèves prendre en charge la mathématisation horizontale dans une activité de modélisation sous deux conditions au moins :

- L'élaboration d'une situation spécifique nécessitant de faire des choix pour engager un processus de modélisation ;
- La mise en place d'un dispositif didactique proposant un temps de travail dédié à la mathématisation horizontale

Ces conditions sont en relation étroite avec notre objectif de transposition de pratiques des chercheurs dont le travail relève de la modélisation. Les indicateurs de la dévolution que nous avons retenus s'appuient en effet sur les premiers résultats des analyses en cours d'entretiens effectués auprès de chercheurs engagés dans des recherches en mathématiques appliquées aux sciences du vivant en vue de mieux cerner leurs pratiques dans un processus de modélisation (Yvain, à paraître).

Comme de nombreux travaux le montrent le rôle de l'enseignant est essentiel pour que la richesse potentielle d'un dispositif didactique soit actualisée dans la classe. Aussi, parallèlement à notre étude présentée dans cette communication, nous nous questionnons sur les conditions et les contraintes prendre en compte pour envisager la dévolution aux enseignants (via la formation) de l'enjeu de dévolution de la mathématisation horizontale aux élèves.

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