WORKING GROUP B / GROUP DE TRAVAIL B

CIEAEM 69

Berlin (Germany)

July, 15 - 19 2017

MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

***

MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

Mathematisation: A general mathematical model as a point of departure of a didactic arrangement

Charlotta Andersson, Jane Tuominen, and Lisa Björklund Boistrup,
Stockholm University

Email: charlotta.andersson@mnd.su.se,   jane.tuominen@mnd.su.se,
lisa.bjorklund@mnd.su.se

Abstract. This paper is aligned with sub-theme 1 of CIEAEM 69 in that it concerns mathematisation as a didactic principle. The mathematisation of this paper is about drawing on general mathematical structures when students in grades 3, 8 and 9 solve equations where negative numbers may occur. The context of the paper is two research projects conducted with teachers. In this paper we present a specific didactic principle in the form of a part-whole model which can visualize relationships between numbers within equations. Through the adoption of this model we explored how this specific mathematical model may work as a mediating tool for the students when solving equations, even when negative numbers are involved. This approach is based on general mathematical structures and not on real-world contexts or empirical material. In this sense, it challenges an arithmetic teaching tradition where mathematics is introduced using specific numbers and sometimes real-world tasks, and thereafter gradually shifting the teaching to the abstract and the general. In both projects sub-studies have been carried out using semi-structured interviews and a paper-pen-test selectively combined with interviews. The findings indicate that a part-whole model, presented below, is working as a tool for students while facilitating the solving of equations. During CIEAEM 69 we would like to present results from the sub-studies.

Résumé. Ce papier est des alliés du sous-thème 1 de CIEAEM 69 dans lequel il concerne mathematisation comme un principe didactique. Le mathematisation de ce papier s'agit de comprendre des structures mathématiques générales quand les étudiants dans les classes 3, 8 et 9 résolvent des équations où les nombres négatifs peuvent se produire. Le contexte du papier est deux projets de recherche conduits avec les enseignants. Dans ce journal nous présentons un principe didactique spécifique dans la forme d'un modèle partiellement entier qui peut visualiser des rapports entre les nombres dans les équations. Par le biais de l'adoption de ce modèle nous avons exploré comment ce modèle mathématique spécifique peut travailler comme un outil de médiation pour les étudiants en résolvant des équations, même quand les nombres négatifs sont impliqués. Cette approche est basée sur les structures mathématiques générales et pas sur les contextes de monde réel ou la matière empirique. Dans ce sens, il défie une arithmétique la tradition enseignante où les mathématiques sont introduites en utilisant des nombres spécifiques et quelquefois des tâches de monde réel et en déplaçant par la suite progressivement l'enseignement au résumé et au général. Dans les deux projets les sous-études ont été réalisées en utilisant des interviews semi-structurées et une épreuve du stylo en papier sélectivement combinée avec les interviews. Les conclusions indiquent qu'un modèle partiellement entier, présenté ci-dessous, travaille comme un outil pour les étudiants en facilitant la solution d'équations. Pendant CIEAEM 69 nous voudrions présenter des résultats des sous-études.

1.   Background

Results from international tests such as TIMSS 2011 illuminate that challenges occur when Swedish students calculate subtraction tasks even without negative numbers appearing (Skolverket, 2012).

Swedish research (e.g., Kilhamn, 2011) as well as elsewhere (e.g., Ball, 1993) highlight that different challenges appear when subtraction tasks with negative numbers are present in teaching. Ball (1993) discusses the importance of bringing negative numbers into the students' context. In this respect, it appears

that it is an advantage for both students and teachers to be aware of the mathematical issues that historically have been a challenge for humanity. Students may perceive negative numbers simply as positive numbers with a subtraction sign in front of them. Moreover, negative numbers are difficult to visualize as quantity of an amount (Kilhamn, 2011).

Mathematics teaching based on an algebraic teaching tradition, and Davydov's curriculum, which was constructed and designed in Russia in the late 1950s, is based on the idea that even young students need to distinguish general mathematical structures, but not based on rules and strategies of knowing how to solve tasks. Instead it allows students to explore relationships, for example, in equations in order to find missing numbers (Davydov, 2008; Kieran, 2004; Schmittau & Morris, 2004; Slovin & Venenciano, 2008; van Oers, 2001). According to Davydov's curriculum, equations can be described as relationships with a part-whole model. This relationship is visualized by a diagram (see Figure 1).

The interest in the two projects, of which this paper is one part, is to explore whether and how the part-whole model is fruitful when solving equations also when the minuend assumes a lower value than the subtrahend (e.g., 4 – 7 = ?); in other words, when the whole assumes a lower value than one or more of its parts. To our knowledge, Davydov's curriculum and the part-whole model has previously only been explored regarding natural numbers. As far as we are concerned, it is not possible to empirically demonstrate negative numbers as quantities. Consequently, students need to handle the part-whole model abstractly and generalize the model mathematically. The diagram (shown below) can be used in order to visualize the part-whole relationships in equations.

Following Davydov's curriculum, students initially handle various equations on the basis of quantities. After a while graphical diagrams are created, and further on formulas such as: a + b = c; b + a = c; c – b = a; c – a = b (Davydov, 2008).

[…] it is only the use of the letter formulas that produces an abstraction of the mathematical relation. But the letter formulas record only the results of real or mental actions with objects, while a graphical representation […], being a visible quantity (a length), enables the children to perform real transformations whose results can be not merely imagined but also observed. (Davydov, 2008, p. 151)

A consequence of the quotation above is that mathematics tools may mediate new knowledge development, but both students and teachers need to differentiate between the tools themselves and the mathematical content that is intended for students to be aware of (Kinard & Kozulin, 2008).

2.   Methodology

Within our two respective research projects, a number of sub-studies have been conducted. The findings of these sub-studies will serve the planning of lessons where the model addressed in this paper will be adopted. The projects are conducted with researchers and teachers in collaboration. In this paper the focus is on the initial sub-studies. Semi-structured interviews were conducted with students in fifteen pairs in grades 3, 8 and 9. The interviews were audio- and video recorded and the data was transcribed and analyzed based on qualitative analysis. A further study has been performed with some students with the intention to analyze the quality of the questions for an upcoming pre-test (before designing a lesson). The study was conducted using interviews or a paper-pen test, or using a combination of these. Also a test was conducted concerning students' ability to find the missing number in two different ways: one with classical equations, and another with a corresponding relationship, visualized by the diagram concerning the part-whole model (see Figure 1).
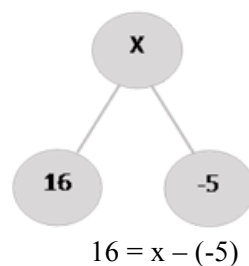

$$16 = x – (-5)$$

Figure 1. A relationship expressed through the "diagram" concerning the part-whole model, and through the corresponding equation.

3.   Tentative findings

In the pre-test, when the students (in grades 3 and 8) encountered the equation $8 – 5 = x$, and were expected to solve it and write it in its other three forms ($8 – 3 = 5$; $3 + 5 = 8$; $5 + 3 = 8$), most of the students managed to do this correctly. However, the students seldom expressed that the equations have relationships to each other, nor that the numbers in each equation have relationships to one another. Instead, they tried to rearrange the numbers in positions "not used before" in each equation. A consequence of this reasoning resulted in expressions like: $3 – 5 = 8$. Another common answer from the students in grade 8 was: $5 – 8 = 3$. However, a few students did express the relationships between the numbers in an equation. For example, when the students were supposed to find the missing number in the equation $7 – x = 2$, they rearranged the equation to $7 – 2 = x$ to make an easier calculation.

A subsequent study was conducted in order to explore whether the part-whole model (Figure 1) could give access to finding the missing numbers. Students in grade 8 managed to do this successfully in 83% of the tasks where the part-whole model was adopted, compared to 43% when solving classical equations. The relations and numbers were exactly the same in the two versions of the tasks. For example, two out of eleven students in grade 8 found the missing number solving the equation $16 = x – (-5)$ algebraically, while seven of eight students found the missing number solving the corresponding task with the part-whole model as a mediating tool. Also in grade 3 the part-whole model seemed to be helpful to the students.

In our different sub-studies, when the students described relationships between quantities from a picture without any number values, most of the students chose to attribute specific numbers to the quantities instead of describing the relationships based on general mathematical symbols.

Findings also indicate that students in grade 3 expressed "-2" (negative two) as a "minus-number," as a "take-away-number" and as a "take-away-two." Students also expressed that "it has to be a number in front of the minus two." For an equation such as: $7 + x = 5$ the students' solution was "-2". When asking the students about their solution they explained their solution as replacing the addition of negative two with subtraction of two, and formulated the equation as $7 – 2 = 5$, "you need to take away two". A corresponding finding in grade 8 indicates that these students did express that there is no difference between the two equations: $(14) + (15) = x$ and $(-14) – 15 = x$.

At the conference we will present more elaborated findings based on a deeper analysis.

4.   Discussion

Previous research by Kilhamn (2011) shows that students face challenges when the minuend assumes a lower value than the subtrahend in different tasks. The students in our sub-studies do not exhibit such corresponding struggles when solving equations through the part-whole model. Using the part-whole model does not rely on rules in the sense of procedures, a property shown also in Davydov's curriculum (Davydov, 2008; Kieran, 2004; Schmittau & Morris, 2004; Slovin & Venenciano, 2008; van Oers, 2001). Instead, the students need to analyze the relationships within the equations to find the parts and the whole and thereafter choose an appropriate strategy (with respect to their mathematical development).

According to the findings there are indications that the students need to discern several aspects regarding equations. For example, the students in both grades 3 and 8 need to discern the relationships between different forms of an equation (e.g., $x = 8 – 5$; $x + 5 = 8$). The students in grade 3 also need to discern that negative numbers exist and that it is possible to operate with them. When the students solve equations with the part-whole model as a didactic tool, they work on an abstract and a theoretical level, not connected to their everyday contexts. Still, the students in this study solved the equations, including when negative numbers were present. Our intention is to further explore whether and how the part-whole model, despite – or maybe owing to – its absence of context, is useful as a mediating tool when solving equations even when negative numbers are present.

Acknowledgements

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397.
Davydov, V. V. (2008). *Problems of developmental instruction. A theoretical and experimental psychological study*. New York: Nova Science.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator 8*(1), 139-151.

Kilhamn, C. (2011). *Making sense of negative numbers* (Doctoral dissertation). Göteborgs universitet, Göteborg.

Kinard, J. T. Sr., & Kozulin, A. (2008). *Rigorous mathematical thinking: Conceptual formation in the mathematics classroom.* England: Cambridge University Press:

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V. V. Davydov. *The Mathematics Educator 8*(1), 60-87.

Skolverket. (2012). *TIMSS 2011. Svenska grundskoleelevers kunskaper i matematik och naturvetenskap i ett internationellt perspektiv. Internationella studier.* Rapport 380. Stockholm: Skolverket.

Slovin, H., & Venenciano, L. (2008). Success in algebra. In *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273-280).

van Oers, B. (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics, 46*, 59-85.

Touchscreen devices and task design to improve plane transformation in high school classroom

Marcelo Bairral* & Alexandre Assis**
Federal Rural University of Rio de Janeiro and Institute of Education Rangel Pestana, Brazil

E-Mail: *mbairral@ufrrj.br; **profalexandreassis@hotmail.com

Abstract. Finding ways to improve learning or teaching of geometry with technological   resources is still a challenge in mathematics education. In this paper we illustrate some strategies used by students to solve tasks on GeoGebra with touchscreen and we reflect about the design of 4 tasks to explore plane transformation in geometry classrooms. The designed tasks were fruitful to make emerge concepts related with plane transformation and to help students solve them making composition among some of them. The study highlights that the decision on the nature of the task is related with the type of touchscreen devices used. This intertwined process is challenging for both teaching and the design of the research.

Résumé. Trouver de façons d'améliorer l'apprentissage de la géométrie avec de ressources technologiques est encore un défi dans l'éducation de mathématique. Dans cet article, nous voulons exemplifier des stratégies utilisées par les élèves pour résoudre des activités sur le GEOGebra touche et nous réfléchirons sur le projet de 4 activités pour exploiter les transformations dans le plan de classe de géometrie. Les acitivités projetés ont réussi, en faisant surgir des concepts relatifs aux transformations dans le plan et ont aidé les élèves a resoudre en faisant la composition entre eles. L'étude montre que la décision sur la nature de l'activité est liée a un certain type de dispositif écrain tactile utilisé. Ce processus entrelacé est un défi pour l'apprentissage et le design de recherche.

1. Introduction

As we have had a first major shift (cognitive and epistemological) and improved teaching by passing from paper and pencil environments to dynamic geometry environment (DGE) with drag and drop activities (e.g. Cabri Géomètre, Sketchpad, etc.), now we have a further shift and improvement with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, Sketchometry etc.) and to the variety of simultaneous fingers' actions they allow. The evolution of digital technology makes available different practices in the classroom, specifically related to the way users can interact with the screen: from the drag and drop actions with the mouse to the tap, drag, and flick with one or more fingers on the screen of multi-touch   devices and from the one-to-one interactions of the former to the multiple simultaneous interactions that the latter makes possible. These different technological features allow designing different tasks, which can change the cognitive processes of users and deeply modify their mathematical inquiries.

   The way we deal and interact with touchscreen devices (TD) is providing new insights and challenges in mathematics learning and instruction (Arzarello et al., 2014). For instance, rotating and other kinds of gyrating movements on screen often take place, due the freedom of handling a touchscreen device. In this paper we discuss previous results from a research project[1] that investigates aspects of geometric learning during the process of solving tasks dealing with dynamic geometric environment with touchscreen. In CIEAEM67 we illustrated strategies used by Brazilian High School students applying rotation concept to solve task on GeoGebra with touch. In CIEAEM69 we will provide reflection on how task designing can improve specific cognitive process that occur when students learning plane transformation using touch

---

devices (Subtheme 3).

2. *Interaction on screen and performing plane transformation*

We assume that touchscreen manipulation on a mobile device is not cognitively the same as mouse clicks, those we often do in dynamic geometry environment (Arzarello et al., 2014), for instance, due to the simultaneity of motion in different elements (points, sides, angles, areas etc.) from one picture (Bairral et al., 2015). This particular feature was observed by one of the students in our research. According to him, "*in a very complex figure, moving several elements at the same time can become a bit difficult*".

Mobile touchscreen devices provide more freedom in manipulation, that particular way of rotation may serve as an important function of grounding mathematical ideas in bodily form and they may also communicate spatial and relational concepts (Boncoddo et al., 2013) in the field of plane transformation. In general, users manipulate the screen using mainly one or two fingers and, sometimes, when working in pairs they also can share fingers or hands to manipulate some shape. Users also can interact with the device in three different ways: *with* the device itself (gyrating it in different positions etc.), and interact *on* or *from* the screen. In this sensorial process, motion and manipulation on screen take an important cognitive role and, in their movement into existence, in which they become objects of thought and consciousness, geometric concepts are endowed with particular determinations; they have to be actualized in sensuous multimodal and material activity (Radford, 2014).

As Ng and Sinclair (2015) pointed out, transformations do not appear explicitly in many curricula until later elementary or middle school. In Brazil, even in High School, plane transformations do not appear in current official curricula. Based on previous research (Bairral et al., 2015) we identified students who, even without previous instruction concerning rotation and reflection, applied these concepts naturally, sometimes even doing composition between them. Besides alternative kinds of rotation applied by students to solve the geometric tasks, justifications to analyze students performing rotation or other plane transformations in TD are the following (Bairral et al., 2017):

1) Rotation and other gyrating movements on screen are often applied due to the various alternatives of handling touchscreen devices (Kruger et al., 2005; Tang et al., 2010).
a) Rotation and other plane transformations have remained unaddressed in Brazilian
b) Touchscreen devices provide possibilities of gyrating movements on screen, or with the device itself, which might result in new insights on embodied cognition.
c) Rotation and other plane transformations are concepts that involve intrinsically embodied motions.

3. *Methodological aspects of the study*

We are conducting teaching experiments with High School students (15-17 years old) at *Instituto de Educação Rangel Pestana* (Nova Iguaçu, Rio de Janeiro, Brazil). All of them had no previous experience with DGE and had no lesson concerning plane transformation. Each session was 2 hours long and in each one the students worked alone or in pairs. The analysis process was mainly based on the (1) videotapes of students working on the software, (2) written answers for each task and (3) the use of one shift in which he or she could write down and describe the function of each device icon. We observed all the students' manipulations *(Arzarello et al., 2014)* on the screen and identified the type of actions (tap, double tap, hold, drag, drag to approach, flick, free and rotate).

4. **Some tasks for improve plane transformation using GeoGebra touch**

In this section we illustrate three tasks elaborated for improve plane transformation in touchscreen device.

Task 1.1: For introduction and familiarization with Geometric Constructer device (30 minute)[2]
Use the software commands (construct, measure, etc.) to understand their functions, them draw the triangle using the commands on the iPad; write your remarks. Before exploring the software write down two observations:

---

[2] Links where to find the software and this activity:
a) with PChttp://www.auemath.aichi-edu.ac.jp/teacher/iijima/GChtml5/GChtml/server_e/gc_00026-test.htm
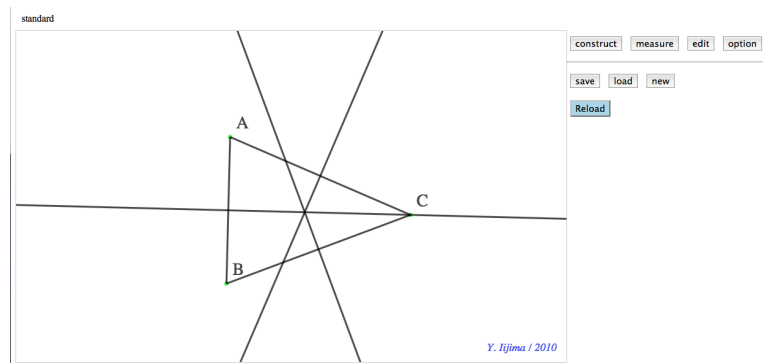b) with I-pad:2012/10/10   16:39   482434 gc_00026-test.htm

Figure 1. Screen from GC

Conceptually, in order to rotate one shape we need to determine before in each point (the center of rotation) and with the use of two fingers the decision could have not been done beforehand. This type of action was not explicit for students exploring task 1.1. We became intrigued and we are investigating new conceptual aspects for the way we deal with rotation and other gyrating movements (with two fingers in movement, one fixed finger and the other in motion etc.).

Task 2.2 (design 1): Stair task
Using only triangle rectangle and isosceles construct the following picture.

Now, write to a friend and tell him or her how you constructed the picture.

When solving task 2.2, which involved the concept of rotation and using a device with a single touch, we observed that students used their fingers – no more than two (Tang et al., 2010) – in a similar way to what students did when dealing with software Geometric Constructer in task 1.1 which did not apply the referred concept. Although the task 1.1 had been designed (without a specific geometric concept) for free exploration and to know the software, the students made a lot of interesting gyrating movements. After observing such way of manipulation, we elaborated a set of tasks (see task 4.5 below), for which students have to apply the concept of rotation and other plane transformation.

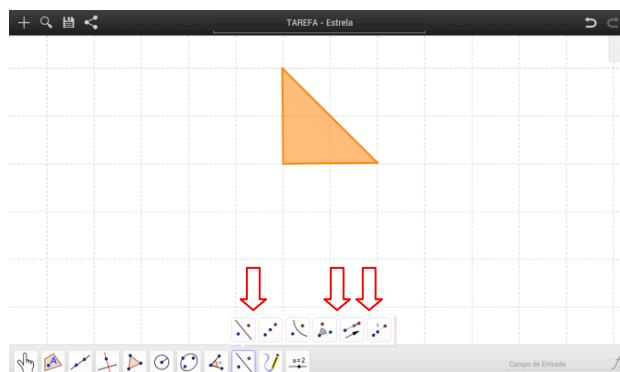*Task 4.5[3] (design 2 from task 2.2): Stair task[4]*
Open the file "Stair task". Only the following triangle will appear:
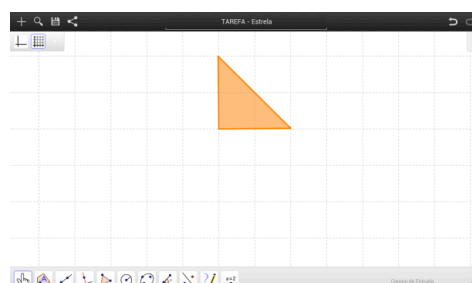
Selecting the tool will open a bar with 6 options:

---

[3] Access https://drive.google.com/file/d/0B6zQPvF8JeJcbzNsU0dMbUh2bE0/view to see the video recorded by Adriano solving task 4.5 as discussed in Bairral et al. (2015).
[4] This version restricts the use of icon.

Elaborate a strategy to construct the following picture using only the tools

The iterative task design was mainly based on two strategies: task that generated new (or reformulated) task (for example, task 2.2 became 4.5) and students' answer that inspires new task

5. *Final remarks*

The type of task has an important role in the growth of the mathematical thinking. For researchers it also bears influence on the findings. The way in how a multi-touch-screen is used allows alterations on the task design in a substantial way. The kind of task needs to be strongly interconnected with the choice of the device and its features and artifacts mediators. In current analyses, we are checking whether the students use one and the same sequence in their reasoning, or if their strategies emerge naturally and without the traditional linearity taught in Brazilian schools (reflection/symmetry → rotation → translation).

In terms of promoting new ways to discover and to think mathematically, it doesn't make sense to propose, for instance, task 2.2 using only pencil and paper. The possibility of to make different constructions, to do simultaneous movements and adjusting by touch on screen seems to be a powerful resource for changing tasks as well as the nature of the geometric understanding concerning plane transformations using TD.

6. References

Arzarello, F., Bairral, M., & Dané, C. (2014). *Moving from dragging to touchscreen: geometrical learning with geometric dynamic software. Teaching Mathematics and its Applications 33(1), 39-51.*

Assis, A. R. de (2016). *High school students working in GeoGebra and in Geometric Constructer: Hands and rotations in touchscreen.* M.Ed. thesis. Federal Rural University of Rio de Janeiro: Institute of Education.

Bairral, M., Arzarello, F., & Assis, A. (2017). Domains of manipulation in touchscreen devices and some didactic, cognitive and epistemological implications for improving geometric thinking. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and technology: a CIEAEM source book*. Cham: Springer.

Bairral, M. A., Arzarello, F., & Assis, A. (2015). High School students rotating shapes in GeoGebra with touchscreen.*Quaderni di Ricerca in Didattica: Matematica*25 (suplemento 2). Proceedings CIEAEM 67, 103-108.

Bairral, M., Assis, A. R., & Silva, B. C. da. (2015). *Mãos em ação em dispositivos touchscreen naeducação matemática*. Seropédica: Edur.

Boncoddo, R., Williams, C., Pier, E., Walkington, C., Alibali, M., Nathan, M., Dogan, M.F. &Waala, J.

(2013). Gesture as a Window to Justification and Proof. Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. M. C. S. Martinez, A. Chicago, IL, University of Illinois at Chicago: 229-236.

Kruger, R., Carpendale, S., Scott, S.D., Tang, A. (2005). Fluid Integration of Rotation and Translation. Paper presented at the Proceedings of the ACM Conference on Human Factors in Computing Systems (CHI)'05, April 2-7, 2005, Portland, Oregon, USA, pp. 601-610.

Ng, O., & Sinclair, N. (2015a). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM Mathematics, 47*(3), 421–434.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM – The international journal on Mathematics Education, 46*(3), 349-361.

Tang, A., Pahud, M., Carpendale, S., & Buxton, B. (2010). *VisTACO: Visualizing Tabletop Collaboration.* Paper presented at International Conference on Interactive Tabletops and Surfaces (ITS '10), Saarbrücken, Germany.

"Quaderni di Ricerca in Didattica (Mathematics)", n. 27, Supplemento n.2, 2017
G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

140

The role of empirical evidence in the construction process of validations of geometric conjectures

Álvaro Sebastián Bustos Rubilar and Gonzalo Zubieta Badillo
Center for Research and Advanced Studies of the National Polytechnic Institute

E-mail: abustos@cinvestav.mx, gzubieta@cinvestav.mx

Abstract. In this document, we present the experience of the implementation of a geometry activity designed under the guidelines of the ACODESA methodology and the principles of task design. This work is part of an experimentation carried out with students coursing a Master of Educational Mathematics. The students were proposed a task for them to conjecture and then, validate. Here we show the case of a student who, during the construction process of a mathematical validation, was supported by empirical evidence to understand the conjecture under discussion, even after it had been validated.

Résumé. Dans le texte qui suit, nous présenterons l'expérience de l'application d'une activité de contenu géométrique conçue suivant les principes de la méthodologie ACODESA et les principes de la conception des tâches. Dans ce document nous exposons partie d'une expérimentation menée avec des étudiants d'une maîtrise en Didactique des mathématiques. Les étudiants ont été proposés une tâche pour qu'ils conjecturent et valident. Ici, nous montrons le cas d'un étudiant qui, pendant du processus de construction d'une validation mathématique, utilisa preuves empiriques pour comprendre la conjecture qui été en cours de discussion, même après avoir été validée.

1. Background and research problem

When reviewing the first great precedent based on the deductive method, Euclid's Elements, we observed that the role the figures accompanying the demonstrations of the different propositions play is that of being an aid to understand the chain of logical deductions and not that of being visual evidence. This is detailed by Szabó (1960), who seeks to explain the historic moment in which pre-Euclidean mathematics, practical and empirical, became a deductive system based on definitions and axioms. The transformation of mathematics into a deductive system led to a decline in the importance that visualization had when related to mathematical discovery and as a tool to persuade others. In pre-Euclidean mathematics visualization played a key role in demonstrations; verifications or refutations of any assertion related to geometry consisted in making facts concrete and visible (Szabó, 1960). This prompted us to reflect upon the role that empirical evidence should have in the validation process of a conjecture, considering that validation is the process through which a student justifies and provides reasons to explain why he or she thinks that a conjecture is true or false. The research question we seek to answer in this work is: How does empirical evidence influence students during the construction process of validations of geometric conjectures?

2. Theoretical references

In this research, we consider the justification created by a student as a process we expect to evolve towards mathematical demonstration. In this process, empirical evidence plays a significant role as a medium to verify, persuade and persuade oneself, as stated by De Villiers (2010). According to De Villiers, encouraging students to follow their intuition to create a validation might help them better understand what they seek to justify. It might also help them discover knowledge or mathematical relationships, yet unknown to them, in which empirical evidence is a resource to understand both the conjecture and its validation more completely.

To distinguish how a validation created by a student evolves, we used the typology of levels and types of proof developed by Balacheff (1987), who categorized the students' procedures in two levels of proof:

pragmatic and intellectual. In the first level are the proofs that resort to action and concrete examples: naïve empiricism, crucial experience and generic example. The second level hosts the proofs supported by the formulation of mathematical properties brought into play and the relationship between them: the thought experiment and calculation on statements.

Another theoretical reference used in the research are the guidelines of scientific debate in mathematics class (Alibert & Thomas, 1991; Legrand, 1993, 2001). Scientific debate in mathematics considers that rational arguments—justifications based on the theoretical corpus of mathematics—should prevail. During the development of the debate, the teacher's role is to promote the expression of ideas and allow clarifying different points of view so that students defend their assertions, as long as they consider them to be more reasonable than those expressed and justified by their peers. The students themselves must lead the consensus of the matter under debate.

3. Method

Students of a Master of Educational Mathematics participated in the study for two sessions of two hours each. The data collection was done using the students' work sheets and two video cameras, which recorded an overall view of the classroom and specific moments. We also video recorded the dialogs produced by the students during all the task.

The task was designed following the principles of the ACODESA methodology (Hitt, 2007), which allows promoting processes of conjecture, argumentation and validation in the classroom (Hitt, 2011; Hitt, Saboya, & Cortés, 2016) through its five stages:

1. **Individual work**. The student develops the task individually using paper and pencil.
2. **Teamwork**. The students work in teams of two or three members.
3. **Scientific debate**. The students debate—as stated by Legrand (1993)—on the proposals of solution presented by each team.
4. **Self-reflection**. Each student individually reconstructs the solution to the problem using paper and pencil.
5. **Institutionalization**. The teacher introduces the institutional solution to the problem. To do so, the teacher summarizes and incorporates the contributions that helped to find the solution in the previous stages.

Besides the ACODESA methodology, we followed the recommendations by Prusak, Hershkowitz, & Schwarz (2013) to design tasks that promote argument production in the classroom: creating multiple situations and collaborative situations, involving socio-cognitive conflict, providing tools for checking hypotheses, reflecting and evaluating the created solutions.

The students were proposed a task in which they had to conjecture and justify the relationship between the areas in the triangles formed when tracing the diagonals of any parallelogram. As a tool to verify hypotheses, they were given grids as the ones shown in 'figure 1'.
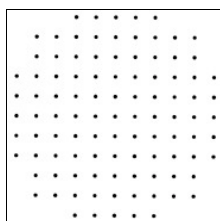

Figure 1. Tool for verifying hypotheses.

4. Result discussion

The conjectures formulated by the students in the individual work stage focused on the four triangles that have no diagonal as side (figure 2). From this, some students conjectured that the four triangles would always have the same area (figure 2a), while others stated only opposite triangles would have the same area (figure 2b). All the students justified their conjectures.
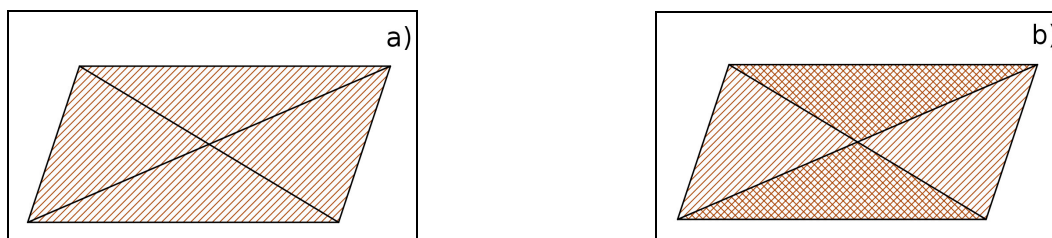
Figure 2. Representation of the conjectures in a general parallelogram; Four triangles of equal areas (a) and Opposite triangles of equal area (b).

In the teamwork stage (three members per team), the students presented to the others the arguments they used to justify their conjectures, which prompted discussions. As a result, the consensus of solution was led by those students who showed more persuasion and quality in the arguments used to validate their conjectures. Although some teams agreed and then validated that the four triangles would always have the same area, regardless the type of parallelogram, others did not fully accept the conjecture since some students stated they saw no equality of areas in the four triangles. They could not visualize how the loss of base in the adjacent triangles was compensated with the gain of height.

An example of this is Daniel who, in the stage of individual work, correctly conjectured and validated that only opposite triangles would have equal areas (figure 3).
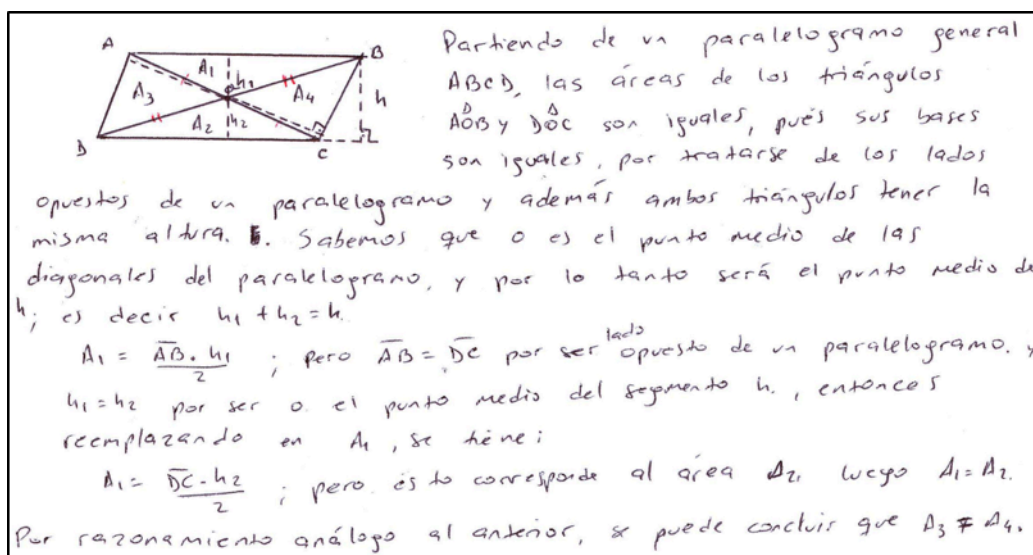
From a general parallelogram $ABCD$, the areas of triangles AOB and DOC are equal since their bases are equal, given that they are opposite sides of a parallelogram and both triangles have the same height.
We know that O is the median of the diagonals of the parallelogram; then, it is the median of h, that is
$$h_1 + h_2 = h.$$
$A_1 = \frac{AB \cdot h_1}{2}$; but $\overline{AB} = \overline{DC}$ because they are opposite sides of a parallelogram, and $h_1 = h_2$ since O is the median of segment h; then, substituting in A_1, we have that:
$A_1 = \frac{AB \cdot h_2}{2}$; but this corresponds to area A_2, then $A_1 = A_2$.
After a similar reasoning, we conclude that $A_3 = A_4$.

Figure 3. Conjecture and validation created by Daniel in the stage of individual work

During the teamwork stage, Daniel persuaded his teammates it was impossible for the four triangles to have equal areas. To do so, Daniel based his statement on empirical evidence by relying on a particular verification on the grids (figure 4). Daniel verified his conjecture by counting the points inside the triangles formed on the grid. Using this method, the student persuaded his teammates to think that only the opposite triangles had equal areas since they have the same number of points inside.
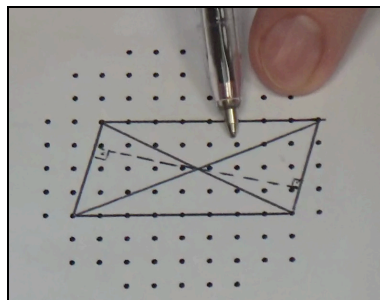
Figure 4. Empirical evidence created by Daniel.

In the debate stage, after all the students agreed on the validation to justify equal areas of all the triangles, Daniel expressed he did not see how all the areas could be equal. He said to the class that he failed to understand how the loss of length in the base was compensated by the gain of height in the same triangle.

Daniel: Then, here in this figure [rhomboid in figure 5] I don't see how what's lost from the base here [segments pointed at in figure 5] is compensated in height [height corresponding to the segments pointed at].
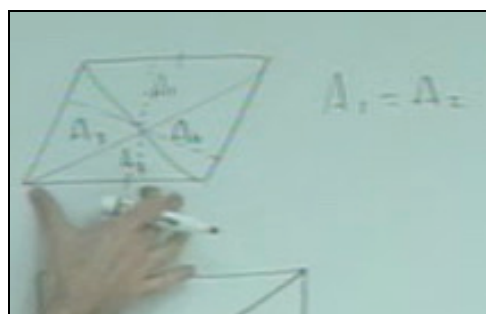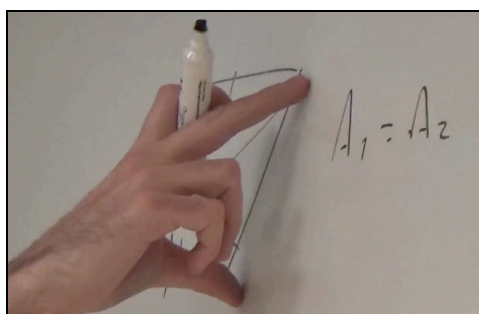

Figure 5. Segments pointed at by Daniel.

After this, another student suggested Daniel to work on another parallelogram, which helped Daniel to clarify his question. He used a rectangle as example (empirical evidence) to visualize and explain how the loss of base is compensated with the gain in height.

Daniel: When tracing the height here, let's say [figure 6a], I see that this, here, is half of the base of this triangle [figure 6b]. And then, there is the compensation, say, that what is lost in the base is gained in height.

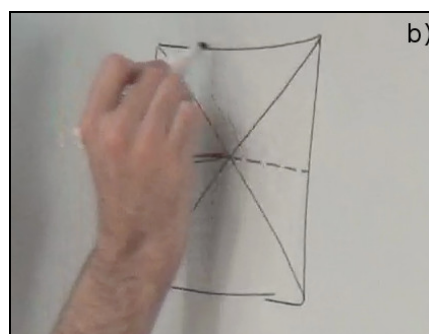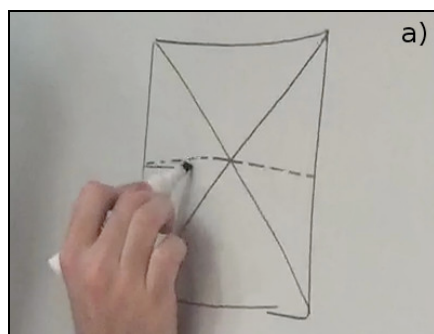**Figure 6**. The case of the heights (a) and base (b) in the rectangle.

5. Conclusions and final remarks

During the development of the task and in the first stages of the ACODESA methodology, we observed the creation of an environment for the students to debate about their arguments and agree on what they considered to be the best solution. In addition, we observed validations supported by empirical evidence

(verified on the grids) and justifications that were closer to intellectual proofs, as defined by Balacheff (1987). Some students expressed they did not understand why the four triangles had equal areas despite having understood and accepted the demonstration all the class had agreed on. However, the use of other parallelograms as examples (empirical evidence) helped them to better understand the conjecture under discussion. We observed that, for some students, accepting the conjecture—even after having validated it— did not occur after accepting or understanding the demonstration, but after verifying it in particular cases, as Daniel did. The empirical evidence used by the student helped him to understand the equality of the areas in the four triangles. Finally, when designing the task following the ACODESA methodology, an environment of social interaction was created in the stages of teamwork and debate. In these stages, the students themselves constructed the solution to the problem after using both empirical evidence and properties of mathematical relationships to create their responses.

References

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 215–230). New York: Kluwer.

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147–176.

De Villiers, M. (2010). Explanation and proof in mathematics: Philosophical and educational perspectives. *Experimentation and Proof in Mathematics*, 1–294. https://doi.org/10.1007/978-1-4419-0576-5

Hitt, F. (2007). Utilisation de calculatrices symboliques dans le cadre d'une méthode d'apprentissage collaboratif, de débat scientifique et d'auto-réflexion. *Environnements Informatisés et Ressources Numériques Pour L'apprentissage. Conception et Usages, Regards Croisés*, 65–88.

Hitt, F. (2011). Construction of mathematical knowledge using graphic calculators (CAS) in the mathematics classroom. *International Journal of Mathematical Education in Science and Technology*, *42*(6), 723–735.

Hitt, F., Saboya, M., & Cortés, C. (2016). Rupture or continuity: The arithmetico-algebraic thinking as an alternative in a modelling process in a paper and pencil and technology environment. *Educational Studies in Mathematics*.

Legrand, M. (1993). Debat scientifique en cours de mathematiques et specificite de l'analyse. *Repères-IREM*, *10*, 123–159. Retrieved from http://www.univ-irem.fr/exemple/reperes/articles/10_article_68.pdf

Legrand, M. (2001). Scientific Debate in Mathematics Courses. In *The Teaching and Learning of Mathematics at University Level* (pp. 127–135).

Prusak, N., Hershkowitz, R., & Schwarz, B. (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education*, *15*(3), 266–285.

Szabó, Á. (1960). The transformations of mathematics into deductive science and the beginnings of its foundations on definitions and axioms (Part one). *Scripta Mathematica*, *27*(1), 113–139.

L'enseignement de la statistique en France et au Brésil

Marlene Alves Dias*, Nielce Meneguelo Lobo da Costa* et Samira Fayes Kfouri da Silva**
* UNIAN, São Paulo, Brésil, ** Unopar, Paraná, Brésil

E-mail: maralvesdias@gmail.com, nielce.lobo@gmail.com; samira.kfouri@unopar.br

Abstract. In this article, we present a comparative study on the teaching of statistics in Brazil and in France, on the middle school to high school transition, focusing on the analysis of the work carried out in middle school on the first notions of descriptive statistics and their representations, considering them as precursors for the introduction of inferential statistics in high school programs in France and university programs in Brazil. Based on theoretical constructions of the Anthropological Theory of Didactic (ATD), and more specifically, using the hierarchy of levels of co-determination, we show the existence of different habitats for the introduction and development of statistics in the curricula of the two countries.

Résumé. Cet article présente une étude comparative entre le Brésil et la France sur la transition collège-lycée en statistique. Il se concentre sur l'analyse du travail mené dans le collège sur les premières notions de statistique descriptive et ses représentations, en les considérant comme des précurseurs pour l'introduction de la statistique inférentielle dans les programmes de lycée en France et universitaires au Brésil. Basé sur des constructions théoriques de la théorie anthropologique du didactique, et plus spécifiquement, sur la hiérarchie des niveaux de co-détermination, il montre l'existence d'habitats différents pour l'introduction et le développement des statistiques dans les curricula des deux pays.

1. Introduction

Nous présentons ici, une étude comparative qui se situe dans un projet plus vaste d'étude de la transition de l'enseignement élémentaire a l'enseignement supérieur en mathématiques en France et au Brésil. Des nombreuses recherches sur la transition secondaire-supérieur ont été déjà développées, comme en témoignent les récentes synthèses (Artigue et al., 2007) ou (Gueudet, 2008). Ces synthèses montrent la diversité des contextes dans lesquels ces travaux ont été menés et la nécessité de bien comprendre l'influence de ces caractéristiques contextuelles pour penser l'action didactique. Par ailleurs, les études comparatives qui se sont développées dans la dernière décennie ont bien mis en évidence l'intérêt de telles comparaisons pour identifier et comprendre les effets de ces caractéristiques contextuelles et culturelles (voir, par exemple Clarke et al., 2007 et Leung et al., 2006).

Dans l'étude rapportée ici, nous nous intéressons plus particulièrement au domaine de la statistique au niveau lycée (étudiants de 15 à 17ans) et à ses précurseurs développés dans l'enseignement de la statistique au collège (étudiants de 11 à 14ans). Ce choix est motivé par le rôle joué par ce domaine de l'enseignement élémentaire à l'enseignement supérieur mises en évidence par les recherches didactiques (Gattuso & Vermette, 2013) et par les orientations curriculaires au Brésil et le programmes en France. Nous utilisons le contraste entre les contextes français et brésiliens pour une meilleure compréhension des problèmes de transition dans ce domaine dans les deux pays et pour penser le développement des ressources éducatives susceptibles d'aider à surmonter les difficultés rencontrées.

D'un point de vue théorique, cette étude s'appuie sur la théorie anthropologique du didactique (TAD dans la suite) développée par Chevallard (1992, 2002), et plus particulièrement l'accent est mis sur la notion de praxéologie et l'hiérarchie des niveaux de co-détermination.

2. Méthodologie

En articulant le cadre théorique et les objectifs, la méthodologie du projet combine plusieurs approches: (1) une approche institutionnelle centrée sur la transition entre le collège e le lycée, exploitant les documents curriculaires et les outils d'évaluation à l'échelle régionale et nationale, (2) une approche des relations personnelles aux statistiques développées par les étudiants, (3) une approche des continuités et discontinuités entre les pratiques d'enseignement dans les institutions dans les deux pays. Les comparaisons sont donc à la fois internes à chaque pays et croisées entre les deux pays.

Par le biais de ces différentes approches, notre intention est d'identifier et d'analyser les similitudes et différences entre les deux contextes, et les effets de la transition entre le collège et le lycée sur le contenu étudié, en prenant en compte les conditions et contraintes intervenant aux différents niveaux de la hiérarchie de co-détermination et à leurs interactions.

Dans cette contribution, nous nous limitons à une dimension de cette recherche, celle concernant l'analyse des relations institutionnelles aux notions de la statistique dans les deux pays. Pour cela, au-delà des paramètres et/ou programmes de l'enseignement du collège et du lycée, nous utilisons deux sources de données: pour le Brésil, l'évaluation annuelle des étudiants du collège et lycée de l'etat de São Paulo – SARESP (évaluation régionale) et l'évaluation qui assure la sélection des étudiants à l'entrée de l'université (l'évaluation nationale – ENEM), pour la France, le baccalauréat qui donne accès à l'enseignement supérieur. Dans les deux cas, les données ont été recueillies et analysées sur les cinq dernières années pour permettre de repérer des régularités mais aussi mettre en évidence d'éventuelles évolutions. Cette analyse sur le long terme, connaissant l'influence des évaluations sur l'enseignement, devrait aussi nous donner une idée plus précise de l'activité statistique développée par des étudiants dans la résolution tâches de telles évaluations, et nous permettre de distinguer entre tâches routinières et tâches nécessitant adaptation et créativité, ce qui n'est pas sans influence sur la compréhension des questions de transition (Castela, 2008).

3. Les systèmes éducatifs français et brésiliens

Nous présentons brièvement ci-après les deux systèmes éducatifs et la façon dont la transition collège - lycée y est organisée (caractéristiques situées aux niveaux supérieurs de l'échelle de co-détermination).

Au Brésil, la structure globale de l'éducation comporte un enseignement fondamental, avec deux étapes (5 puis 4 ans) et un enseignement moyen (3 ans) correspondant au lycée français, mais sans filières spécifiques. L'enseignement fondamental et l'enseignement moyen sont obligatoires. Il existe des paramètres nationaux qui définissent des orientations pour l'enseignement, mais pas de programme national, les élèves peuvent suivre leurs cours pendant la journée ou le soir et, dans ce dernier cas, ils ont moins d'heures d'étude. Par ailleurs, la formation des enseignants varie fortement d'un état à un autre. L'entrée à l'université est actuellement basée sur l'examen national ENEM, mais il existe aussi des évaluations sélectives appelées «vestibular», organisées par les universités elles-mêmes. Beaucoup de jeunes fréquentent des cours spéciaux privés pour préparer ces examens.

En France, la structure globale est similaire, avec 5 années d'école primaire, quatre années de l'enseignement secondaire et trois années de lycée. L'éducation est obligatoire jusqu'à l'âge de 16 ans. En entrant au lycée, il y a une séparation entre l'enseignement général, technologique et professionnel. Dans l'enseignement général auquel nous nous intéressons plus particulièrement ici, l'enseignement se différencie aussi du fait des options en seconde, mais surtout des séries en première, et des enseignements de spécialité en terminale. Trois séries existent en première: littéraire (L), sciences économiques et sociales (ES), sciences (S), et pour la série S trois spécialités en terminale : sciences mathématiques, sciences physiques et sciences de la vie et de la terre. Les programmes de mathématiques diffèrent de la classe selon la série choisie ainsi que les horaires. Il y a une évaluation nationale à la fin du secondaire, le baccalauréat, qui donne accès à l'enseignement supérieur. Le taux de réussite est d'environ 85%.

4. Statistique dans les systèmes éducatifs français et brésiliens

La 'figure 1' résume ce qui concerne la statistique développée au deuxième étape de l'enseignement fondamental (étudiants de 11 à 14 ans) et à l'enseignement moyen (étudiants de 15 à 17 ans) dans les paramètres nationaux au Brésil. La statistique est introduite au premier cycle de l'enseignement fondamental en tant qu'outil pour la collecte et l'organisation des données dans des tableaux et des graphiques et l'étude des relations entre des événements, ce qui rend possible les prévisions sur l'observation de la fréquence de leur apparition. Dans le deuxième étape la statistique fait partie du domaine « traitement de l'information » en tant que complément au travail déjà initier et est utilisée comme outil dans l'enseignement pour établir des liens entre les mathématiques et d'autres domaines de contenu et les thèmes transversaux (cf. figure 1). Le

niveau d'enseignement des notions à développer est de la responsabilité de l'enseignant qui doit considérer le développement et les intérêts des étudiants de chaque classe. Les capacités attendues sont la construction et l'analyse des différents processus de résolution de situations-problèmes et trouver des solutions pour construire des arguments plausibles.

Niveau et contenu	Domaines	Capacités attendues
Enseignement fondamental : Premières années: comprendre la collecte et l'organisation des données dans des tableaux et des graphiques, pour établir des relations entre les événements. Dernières années: revisiter les connaissances développées dans le premières années, formuler des questions pertinentes pour un ensemble d'informations, développer des conjectures, communiquer de façon convaincante l'information, interpréter des diagrammes et des organigrammes. **Enseignement moyen :** Comprendre la relation entre aperçu statistique, représentation graphique et les données primitives ; exercer la critique dans la discussion des résultats ; construire des arguments rationnels basés sur l'information et les commentaires.	**Traitement des donnés** Les notions de statistique descriptive et ses représentations.	Reconnaître et utiliser, sous forme orale et écrite, les symboles, les codes et la nomenclature du langage scientifique. Lire, articuler et interpréter les symboles et les codes dans différentes langages et systèmes de représentations. Identifier dans une situation problème donnée les informations ou des variables pertinentes et élaborer des stratégies de résolution possibles. Reconnaître, utiliser, interpréter et proposer des modèles pour des situations problématiques, des phénomènes et des systèmes naturels et technologiques.

Figure 1. Paramètres brésiliennes pour le développement des notions de la statistique au deuxième etape de l'enseignement fondamental et à l'enseignement moyen.

Les notions de statistique descriptive développées au enseigment fondamental sont reprises dans l'enseignement moyen au Brésil. Elles s'inscrivent alors en tant qu'objets mathématiques dans le domaine du traitement de l'information et sont reliées à de nouveaux objets de mathématiques et des autres sciences.

En France, la statistique est introduite au collège et son développement est regulier jusqu'au fin de lycée, c'est-à-dire, la statistique descriptive est développée au collège et la statistique inferentielle est introduite au lycée. La situation est donc de ce point de vue très différent à celle du Brésil comme nous pouvons remarquer dans la 'figure 2'. On a donc un habitat pour la statistique très différent de celui du Brésil où nous ne développons que la statístique descriptive. Dans la 'figure 2', nous ne présentons que les donnés pour le quatrième et le seconde.

Série et contenus por le quatrième au collège, série, filière et contenu pour le seconde au lycée	Domaines	Capacités attendues
Quatrième Effectives cumulées, fréquences cumulées ; moyennes ponderées ; iniciations à l'usage des tableurs-grapheurs ; valeur approchée de la moyenne d'une série statistique regroupée en intervales. **Seconde** Resume numérique par une ou plusieurs mesures de tendance centrale(moyenne,	**Collège (quatrième)** Statistique Descriptive; **Lycée (seconde)** Statistique; Inferentielle	**Collège** S'engager dans une démarche de résolution de problèmes; utiliser des outils mathématiques pour résoudre des problèmes concrets; appréhender différents systèmes de représentations; tenir compte d'éléments divers pour modifier son jugement; utiliser l'oral et l'écrit pour explicter des démarches, argumenter des raisonnements. **Seconde** Utiliser les propriétés de linéarité de la moyenne d'une série statistique;

médiane, classe modale, moyenne élargiée) et une mesure de dispersion. Définition de la distribuition des fréquences d'une série prenant um petit nombre de valeurs et dela fréquence d'um événement. Simulation et fluctuation d'échantillonage.		Calculer la moyenne d'une série à partir des moyennes de sous-groupes et de distribution de fréquences; concevoir et mettre em oeuvre des simulations simples à partir d'échantillonsde chiffres au hasard.

Figure 2. Orientations françaises pour le développement de la statistique au collège et au lycée.

Pour le lycée, il y a encore les filières pour les deux annés succédant à la classe de seconde, à savoir : série économique et sociale (ES), série littéraire (L) et série scientifique (S). Pour les deux annés des différents séries il y a un programme pour l'enseignement de la statistique. Nous avons remarqué que le programme brésilien du lycée et plus proche de celui de la classe littéraire.

5. Conclusion

La comparaison France-Brésil montre donc des habitats différents pour l'enseignement de la statistique dans les deux pays et des relations différentes aussi entre le collège et le lycée. Pour comprendre ces différences et leur impact sur la transition entre le collège et le lycée, il nous semble nécessaire de revenir aux conditions et contraintes qui ont façonné ces choix curriculaires et leur évolution. Le nombre de pages réduit de cette contribution ne nous permet pas de développer ici cette analyse, pas plus qu'il ne nous permet de rentrer, en nous appuyant sur les données recueillies dans les détails de l'analyse praxéologique. Nous présenterons, si cette contribution est retenue, une synthèse des résultats obtenus selon ces deux dimensions au colloque.

Références

Artigue, M., Batanero, C., Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, pp. 1011-1049. Greenwich, CT: Information Age Publishing Inc. and NCTM.

Castela, C. (2008). Travailler avec, travailler sur la notion de praxéologie mathématique pour décrire les besoins d'apprentissage ignorés par les institutions d'enseignement *Recherches en Didactique des Mathématiques*, 28(2), 135-182.

Chevallard, Y. (2002). Organiser l'étude 3. Ecologie & régulation. In Dorier, J.-L. et al. (Eds.), *Actes de la 11ème Ecole d'Eté de Didactique des mathématiques*. Grenoble: La Pensée Sauvage.

Chevallard, Y., (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, 12(1), 73-112.

Clarke, D., Emanuelsson, J., Jablonka, E. & Mok, I. A. C. (2006). *Making connections: comparing mathematics classrooms around the world*. Rotterdam: Sense Publishers.

Gattuso,C. &Vermet, S. (2013). *L'enseignement de statistique et probabilités au Canadá et en Italie. Accès en 30 janvier, 2017, de* http://publications-sfds.math.cnrs.fr/index.php/StatEns

Gueudet, G (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics,* 67(3), 237-254.

Leung, F. K. S., Graf, K.-D. & Lopez-Real, F. J. (Eds.) (2006). *Mathematics education in different cultural traditions – a comparative study of East Asia and the West: the 13th ICMI study*. New York: Springer.

Algèbre élémentaire et les rapports personnels d'un groupe d'étudiants de l'État de São Paulo - Brésil

Marlene A. Dias*, Valdir B. Santos Júnior**, Miriam R. Guadagnini* et Renato S. Ignácio***
* UNIAN, São Paulo, Brésil, ** UFPE, Recife, Brésil, *** UFCG, Cuité, Brésil

E-mail: maralvesdias@gmail.com, valdir.bezerra@gmail.com, miriamguadagnini@hotmail.com, renatosignacio@gmail.com

Abstract. This paper presents the research that aims at identifying both existing institutional relationships and students' expected real personal relationships and understanding the difficulties for students who complete basic education of the teaching-learning process of algebra. Notions associated to the Anthropological Theory of Didactics by Chevallard are considered for the central theoretical framework. The first results show confinement within the arithmetical frame by the students, even those who have started higher education.

Résumé. Cet article présente les résultats d'une recherche qui vise à identifier les rapports institutionnels existants et les rapports personnels réels et comprendre les diffficultés des étudiants terminant l'éducation de base par rapport à l'enseignement et l'apprentissage de l'algèbre. Les notions associées à la Théorie Anthropologique du Didactique développée par Chevallard sont considérées pour le cadre théorique central. Les premiers résultats montrent le confinement des étudiants dans le cadre arithmétique, même ceux ayant commencé l'enseignement supérieur.

1. Le contexte de la recherche

Nous présentons dans ce travail une recherche sur l'enseignement et l'apprentissage de l'algèbre dans l'éducation basique au Brésil. Nous considérons pour cela la transition entre les trois étapes scolaires, comprenant l'enseignement basique obligatoire au Brésil à savoir: l'école élémentaire (les élèves de 6 à 10 ans), le collège (les étudiants de 11 à 14 ans) et le lycée (les étudiants de 15 à 17 ans).

La problématique de cette recherche est apparue en classe de quatrième année de l'école primaire avec des élèves de neuf et dix ans. L'un des chercheurs a posé une question aux élèves qui de son point de vue représente un grand « défi » pour les étudiants de troisième année de l'école primaire. Ce chercheur, dont la condition est d'être enseignant dans cette classe, a prétexté que les élèves n'avaient pas les moyens nécessaires pour résoudre le problème, car ils n'avaient pas les connaissances requises concernant les notions et les techniques algébriques pour celui-ci. Toutefois, il a été surpris par les réponses de ses élèves ainsi que par l'utilisation des techniques de résolution.

À partir de cette expérience nous avons décidé de mener une recherche basée sur la question suivante: Quelles sont les connaissances, les techniques et les stratégies utilisées par les élèves pour résoudre les problèmes algébriques proposés?

Ainsi, nous avons émis l'objectif suivant : identifier les rapports institutionnels existants, les rapports personnels attendus ainsi que les marques de ces derniers sur les rapports personnels réels des étudiants afin de mieux comprendre les difficultés rencontrées, en particulier, celles des élèves terminant l'enseignement secondaire comprenant l'apprentissage de l'algèbre.

Nous croyons également que l'étude de l'évolution historique de l'algèbre peut révéler des éléments permettant de comprendre les difficultés et les défis auxquels sont confrontés les éducateurs et les chercheurs dans l'enseignement et l'apprentissage de ce domaine des mathématiques. Pour mieux comprendre ces difficultés nous avons étudié l'évolution historique de l'algèbre, selon Robinet (1989) et Radford (1991).

Cette étude nous a permis de remarquer que les difficultés rencontrées dans l'histoire s'approchent beaucoup des difficultés de nos étudiants. Cela nous a amené à réfléchir sur les difficultés rencontrées par les

étudiants de l'enseignement supérieur lorsqu'ils ont besoin d'appliquer leurs connaissances de l'algèbre élémentaire et que celles-ci ne sont pas disponibles. Cela conduit à une grande perte d'intérêt pour les cours dont les mathématiques sont un outil important pour le développement.

Ainsi, nous avons choisi la théorie anthropologique du didactique pour développer la recherche et les notions de cadre et changement de cadre selon la définition de Douady (1992) et des niveaux de connaissances attendues des élèves selon la définition de Robert (1998).

2.   Cadre théorique de la recherche

La recherche est basée sur des éléments de la Théorie Anthropologique du Didactique (TAD) de Chevallard (1992, 1998). La principale raison pour nous de prendre la décision d'utiliser la théorie mentionnée ci-dessus est parce qu'elle place l'étude de l'activité des mathématiques dans l'ensemble des activités humaines et des institutions sociales (Chevallard, 1998).

Comme la théorie situe l'activité d'étude des mathématiques au sein des institutions sociales, il semble important de considérer les notions de rapports institutionnels et personnels définies par Chevallard (1998). Pour définir ces rapports, Chevallard (1992) introduit la notion d'objet qui est définie comme toute entité, matériel ou immatériel, qui existe pour au moins un individu, ce qui l'a amené à considérer que tout est objet. Un autre élément clé de la théorie est la notion d'institution, qui selon l'auteur, sont des dispositifs sociaux qui permettent et imposent différentes positions des personnes qui peuvent occuper différentes positions dans l'institution.

Alors pour Chevallard (1992) l'univers cognitif d'un individu particulier, c'est-à-dire, l'ensemble de ses rapports personnels à la connaissance est un autre élément qui compose la structure de la TAD, puisque Chevallard (1998) définit que toutes les interactions possibles, que ce soit la manipulation, l'utilisation, etc. d'un objet particulier, correspond donc à un rapport personnel avec cet objet. Cette notion nous aide à l'identification des rapports personnels réels des étudiants, de 10 à 18 ans, en ce qui concerne le domaine de l'algèbre, par la confrontation avec les rapports institutionnels existants.

Pour Chevallard (1992, 1998) la position qu'un objet donné occupe dans une institution est ce qui détermine le rapport institutionnel de l'objet avec l'institution analysée. Ainsi, nous avons analysé les documents officiels et les manuels, pour les manuels nous avons choisi ceux qui sont utilisés actuellement dans les classes de l'éducation de base (étudiants de 6 à 17 ans) à São Paulo. Nous remarquons que, dans les écoles de São Paulo, les manuels peuvent être considérés comme des rapports institutionnels existants, parce que, en général, les enseignants suivent ces documents. Déjà, les documents, qui guident le système d'enseignement de l'État de São Paulo, sont considérés comme des rapports institutionnels attendus parce que, en général, dans ces documents ne sont présentés que des directives générales sans des exemples de la façon de les travailler. Ceux-ci sont développés dans deux autres document appelés "cahier de l'enseignant et cahier de l'élève".

Nous soulignons également que dans la TAD, toutes les activités humaines sont organisées par praxéologie, ce qui rend cette notion un autre principe structurant de la théorie. Une praxéologie consiste en: types de tâches et techniques qui forment le bloc du savoir-faire et le discours technologique-théorique didactique qui forment le bloc du savoir. Les praxéologies ici sont identifiées à travers l'analyse des manuels, parce qu'à partir des types de tâches et des techniques développés dans ces manuels, nous avons identifié les embryons de technologie et nous avons pu considérer la théorie qui les justifient.

Outre la TAD nous avons utilisé comme cadre théorique de référence les idées de Douady (1992), en particulier les notions de cadre et de changement de cadre, car à partir de ces notions sera possible d'identifier la nécessité de changement du cadre arithmétique au cadre de l'algèbre dans l'analyse des tâches proposées dans les documents analysés.

Une autre théorie du support, qui nous avons utilisé, est celle de Robert (1998), plus particulièrement la notion de niveaux de connaissances attendus des étudiants, à savoir: le niveau technique, le niveau mobilisable et le niveau disponible. Notant qu'il n'y a pas de hiérarchie entre eux et ce qui est voulu est que les élèves atteignent toujours le niveau disponible pour les concepts et notions d'algèbre développées pendant leur scolarité. Nous remarquons encore que la différence entre le niveau mobilisable et le niveau disponible est l'explicitation des connaissances à utiliser dans les tâches où le niveau de connaissance attendu est le niveau mobilisable et la non explicitation des connaissances nécessaires pour résoudre les tâches où le niveau de connaissance attendu est le niveau disponible. Le choix de l'analyse des types de tâches proposées en utilisant le niveau de connaissances attendues des étudiants est un outil nous permettant de reconnaitre si les types de tâches proposées dépassent la répétition des techniques algébriques sans être possible d'appliquer

ces techniques pour résoudre les tâches qui impliquent que l'étudiant lui-même trouve quelle est la technique la plus appropriée, en particulier quand il est question de situations du quotidien.

Compte tenu du scénario présenté des éléments théoriques que nous avons utilisés dans notre recherche, nous abordons par la suite la méthodologie utilisée pour la recherche.

3. Méthodologie de la recherche

Comme déjà indiqué dans le cadre théorique, il s'agit d'une recherche qualitative basée sur la technique de la recherche documentaire selon Lüdke et André (1986), car nous l'avons commencé par l'étude des documents officiels pour identifier les rapports institutionnels et les rapports personnels attendus.

Les documents analysés pour l'identification des rapports institutionnels, en ce qui concerne les notions du domaine de l'algèbre, étaient: les cahiers de professeur pour les années scolaires équivalents aux cinquièmes et quatrièmes années au collège (étudiants de 12 et 13 ans), qui correspond aux rapports institutionnels existants. Les lignes directrices curriculaires sur le contenu à développer avec les étudiants de cette étape scolaire de l'État de São Paulo sont considéré comme les rapports institutionnels attendus et le rapport pédagogique des écoles de cet état sont les rapports personnels attendus. Nous notons que le rapport pédagogique est publié chaque année après l'évaluation à grande échelle nommé Système d'évaluation du rendement scolaire de l'État de São Paulo (SARESP).

La partie expérimentale de la recherche, qui nous a permis d'analyser les difficultés des étudiants, correspond à un test diagnostic qui a été appliqué à un groupe de cinquante-six élèves de l'éducation de base, âgés entre 10 et 18 ans, répartis comme suit: vingt-six étudiants avec l'âge de 10 ans, vingt-cinq à l'âge de 15 ans et cinq étudiants de 18 ans d'âge, tous dans les phases de transitivité pour les étapes éducatives brésiliennes, c'est à dire, le passage de l'école élémentaire au collège, du collège au lycée et du lycée à l'université.

Les tâches proposées dans le test de diagnostic ont été identifiés grâce à l'analyse des rapports institutionnels et des rapports personnels attendus des étudiants. Pour ces tâches nous avons analysé les techniques, les technologies, les théories, les cadres et les changements et les niveaux de connaissances attendus des étudiants, pour comprendre les rapports institutionnels existants et les marques de ces derniers sur les rapports personnels des étudiants.

4. Résultats des analyses

Dans l'analyse des rapports institutionnels existants, qui ont été analysées par l'intermédiaire de cahier de l'enseignant, nous avons observé que les praxéologies développées dans le matériel de classe sont présentés au moyen d'exemples, dont beaucoup d'entre eux peuvent être résolus en n'utilisant que l'arithmétique, en laissant peu de place pour le développement de l'algèbre, en particulier, lorsque nous considérons les tâches contextualisées.

Dans l'analyse des manuels, nous avons aussi remarqué que les types de tâches proposées favorisent l'arithmétique, même si les exemples sont développés au moyen d'équations et de systèmes d'équations, ce qui indique la nécessité d'un travail qui montre l'importance de l'algèbre en tant qu'outil pour résoudre les tâches pour lesquelles les étudiants ont déjà d'autres techniques pour les exécuter.

L'analyse du test de diagnostic (en annexe), qui nous a permis de commencer notre étude sur l'identification des véritables rapports personnelles des étudiants, appliqué a un groupe de cinquante-six étudiants entre 10 et 18 ans, tend à montrer que, peu importe la phase de transitivité, car ils utilisent tous les techniques, qui ne nécessitent pas de l'algèbre, pour résoudre les tâches du test. La plus grande différence est le nombre d'étudiants capables de résoudre les tâches proposées, parce que les étudiants du «collège» et du «lycée» ont plus de compétences arithmétiques que ceux de l'école élémentaire.

References

Chevallard,Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques 12(1),* 73-112.

Chevallard, Y. (1998). *Analyse des pratiques enseignantes et didactique des mathématiques*: L'approche anthropologique. Recuperado em 17 de setembro de 2014 de http://yves.chevallard.free.fr/

Lüdke, M., & André, M.E.D.A. (1986). *Pesquisa em educação:* abordagens qualitativas. São Paulo: EPU.

Radford, L. (1991). Diophante et l'algèbre pré-symbolique. *Bulletin AMQ* 63(4).

Robert, A. (1998). Outils d'analyse des contenus mathématiques à enseigner au lycée à l'université. *Recherches en didactique des Mathématiques* 18(2), 139-190.

Robinet, J. (1989). *La genèse du calcul algébrique (une esquisse).* Paris : IREM Paris 7.

São Paulo. (2009). *Caderno do Professor*: Matemática, Ensino fundamental. São Paulo: Secretaria de Educação.

Annexe: Questions du test diagnostic

UNIVERSITÉ ANHANGUERA
DOCTORAT EN ÉDUCATION MATHÉMATIQUE

1. Une usine produit des poussettes pour bébés et des tricycles. Aujourd'hui, les travailleurs ont produit 11 unités et pour les assembler, ils ont utilisé 40 roues. Combien des tricycles ont été produits?

2. Trouvez deux nombres dont la somme est de 20 et le produit entre eux est 96.

3. 54 oranges ont été réparties entre Kátia, André et Cláudia et on sait qu'André a reçu deux fois plus que Kátia et Claudia a reçu le triple de ce qu'a reçu André. Combien d'oranges chacun a reçu?

Expressing and justifying pattern generalization algebraically

Jenny Fred and Lisa Björklund Boistrup,
Stockholm University
Email: jenny.fred@mnd.su.se, lisa.bjorklund@mnd.su.se

Abstract. The main objective in this paper is on learning more about younger students' emergence of the ability to express and justify pattern generalization algebraically, particularly in relation to what aspects students need to discern to be able to express and justify pattern generalization algebraically. This forms a point of departure for discussing the meaning of making algebraic generalizations in the early grades. The findings constitute a foundation for a project on classroom teaching and learning in mathematics, carried out as a collaboration between researchers and teachers.
Résumé. L'objectif principal dans ce document est d'apprendre plus sur les façons d'expérimenter la généralisation des schémas par les élèves plus jeunes et ce à propos de quels aspects les étudiants doivent discerner pour pouvoir exprimer et justifier les généralisations de formes algébriquement. Ceci comme un point de départ dans les discussions concernant la signification de faire des généralisations algébriques dans les premières années. Les résultats constituent une base pour un projet de développement de cours en collaboration entre chercheurs et enseignants.

1. Background

Researchers (e.g., Greer, 2008; Usiskin, 1988) advocate alternative approaches to the teaching of algebra, since the literature reveals that teaching today often focuses on learning a number of procedures rather than creating the conditions for enabling students to develop abilities such as reasoning algebraically, making algebraic generalizations, and using algebraic representations. Furthermore, Usiskin (1988) and Greer (2008) highlight how teaching which does not go beyond the practicing of manipulative skills, instead of developing understanding, can prevent students from using algebra as a powerful tool for solving mathematical problems. Included here are processes like describing and analysing relationships, characterizing and understanding mathematical structures and ideas (e.g., Davydov, 2008; Kaput, 2007; Kieran, 2006; 2004; Radford, 2010, 2014). In relation to enabling younger students to develop algebraic understanding, Mason (1996), Radford (2006), and Warren (2006) all suggest the use of mathematical patterns as an introduction**.**

2. Generalizations in relation to mathematical patterns

In research regarding mathematical patterns and generalizations, there are different descriptions of the meaning of making generalizations. Radford (2006), for example, highlights how generalization is about different layers of consciousness; to perceive the pattern's mathematical structure; to perceive the commonality of the pattern; to generalize a local commonality to all the parts of the sequence; as well as being able to express the general. In a more recent article, Radford (2011) stresses the ability to generalize in relation to being able to perceive both the pattern's spatial and numerical regularity, where the spatial structure is about, for example, how matches may be positioned in patterns (see figure 1). This entails distinguishing how both the numerical and the spatial structures belong together, including what is equal and what separates them, and then to abstract this commonality into all elements of the sequence (Radford, 2011). In relation to patterns and generalization, Mason, Burton, and Stacey (2010) highlight how students need to be able to discern an underlying general structure to be able to express a generality algebraically. Mulligan and Mitchelmore (2009), in turn, address pupils' ability to structure based on the pattern's vertical, horizontal and spatial structures.

Radford (2006) and Venenciano and Dougherty (2014) highlight the different strategies students use when making pattern generalization. Radford separated different generalization strategies in relation to how advanced they are. First of all, he made a distinction between the so-called "naive induction" versus generalizations. In the "naive induction" strategy, the students use a "trial and error strategy," which can be described as a guessing strategy and thus, according to Radford (2006), is not a generalization strategy at all. A generalization strategy is about discerning and using a general commonality of a pattern (Radford, 2006). The strategies that are counted as generalization strategies consist of both arithmetic and algebraic ones. The difference between those, according to Radford (2006), is that an arithmetic strategy does not make it

possible to predict any term in a pattern as an algebraic strategy could. In other words, a generalization like "It constantly increases with two matches" is seen as an arithmetic generalization, since it only supports the prediction of the "next" positions in the sequence and does not make it possible to predict any term in the pattern.

The algebraic strategy is divided into three different strategies: factual, contextual, and symbolic. They are all categorized as algebraic since the students using these strategies are expressing a commonality that can be applied to all terms in the pattern, and thus used to predict the number of elements in any term in the pattern (Radford, 2006). Here there is not only an increase in the number of elements between the terms perceived, but the number of elements of each term is, rather, related to the position of the term in a pattern sequence (such as "the n:th term") and to all elements in the visible pattern (Radford, 2006).

The difference between those strategies is about *how* the generalization is expressed. In a factual generalization, the indeterminacy remains unnamed, and the "generality rests on *actions* performed on numbers; *actions* are made up of words, gestures and perceptual activity" (Radford, 2006, p.16). The generalization is here based on actions in relation to facts on *a* local term, 'If it's *term 1*, I *did one row*', and is then put in relation to the other terms in the sequence '*term 2*, it's *two*', *term 3*, it´s *three*'.

Contextual and symbolic generalizations address a more mathematical level of generalization. In the contextual generalization, on the contrary to a factual generalization, the indeterminate is "made linguistically explicit: it is *named*" (Radford, 2006, p. 16). The generalization is, in other words, symbolized by words 'you *double* the *terms number*'. The difference between a contextual generalization and a symbolic generalization is that a symbolic one is based on algebraic symbols, such as '$2 \cdot x$' instead of words 'you *double* the *terms number*'.

Venenciano and Dougherty (2014) highlight another kind of strategy as algebraic. It is a measuring strategy where, for example, two squares are used as a measurement unit (see figure 2) and this puts the number of measurement units in relation to where the term is positioned in the pattern sequence.

> From a measurement approach […] one may view the unit of measure as a composite of the two squares, that which is iterated with each successive figure. This […] approach enables one to apply the notion of defining a unit and consider a scale factor to solve the problem.
>
> (Venenciano & Dougherty, 2014, p. 23)

It is argued that the teaching of algebra should give the students the opportunity to use algebra as a tool for characterizing and understanding mathematical structures (e.g., Greer, 2008; Usiskin, 1988). Additionally, a focus on making generalizations in relation to mathematical patterns is advocated in the early grades (e.g., Radford, 2006, 2011). Distinctions in relation to different types of algebraic generalisations (e.g., Radford, 2006), opens up for a broader understanding in relation to generalizations. What is lacking is descriptions of aspects which students simultaneously need to discern and take into consideration in order to be able to express and justify pattern generalization algebraically. Hence, the main object of this paper is neither about what an algebraic generalization is, nor which strategies students may use. The aim of this paper is to describe the emergence of the ability to express and justify pattern generalization algebraically. The research questions for this paper are: "What are students' qualitatively different ways of seeing pattern generalization?" and "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?"

3. Theoretical framework

Variation theory (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) has been used as a theoretical framework in this study. Learning in a Variation theoretical perspective is considered to arise in the relationship between the one who is learning and what is to be learned (Marton & Booth, 1997, see also Marton, 2015). Variation theory provides theoretical tools for the analysis of the conditions of qualitatively different ways of seeing specific knowledge, and what aspects that are critical to discern in order to be able to see this knowledge in a more powerful way. Variation in relation to a Variation theoretical perspective refers to a meaningful, conscious, directed and systematic variation of content. Critical aspects are aspects that the students need to discern to be able to develop this specific knowledge (Marton, 2015). In this paper we are exploring students' quality different ways of seeing pattern generalization. In a Variation theoretical perspective 'ways of seeing' are seen in relation to what aspects the students are discerning and focusing upon in relation to a demarcated knowledge (Marton, 2015).

4. Methodology considerations

This study is included in a more extensive practice-based research project, in which Learning study (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) is used as a research approach. This paper does not present the final results of the Learning study, but rather the analysis of semi-structured interviews, conducted initially in the study.

The semi-structured interviews

One of the first steps in a Learning study is the mapping of the students' current perceptions of a specific knowledge. In this research project semi-structured interviews were chosen as a mapping tool to grasp the students' qualitatively different ways of seeing pattern generalization and to identify aspects that students need to discern to be able to express and justify pattern generalization algebraically.

The semi-structured interviews were performed with eight of the students from the overall project. The students were 9-10 years old, and both girls and boys were interviewed. The idea was that this selection of students would cover much of the diversity that existed within the group (Marton & Booth, 1997). The students were selected in relation to their previous results in mathematics and were supposed to represent students with different performances in the subject of mathematics. The selected students were divided into pairs and were then, in the interview situation, presented with three different pattern tasks which they were asked to solve together. While the students were working with the tasks, the interviewer asked question such as "Can you tell me how you´re thinking?" and "Can you show me how you are looking at it (pointing at the pattern) when you´re saying this?". The aim was trying to explore the students' ways of seeing pattern generalization in the process of solving tasks where making pattern generalization were required. The idea was not about how pattern generalization may be defined by the students and thus was the interviewer not supposed to ask any direct questions about pattern generalization per se.

Analysis

The data in this paper consists of transcriptions of the interviews. In the analysis, Variation theoretical tools were used (critical aspects and variation of content), in order to try to distinguish qualitative dimensions of the variations in different ways of seeing pattern generalization and in relation to identifying critical aspects of the ability to express and justify pattern generalization algebraically. In the analysis, there was an interplay between the data and previous research (e.g., Radford 2010). The process of the analysis was as follows:

1. *Reading of compiled interviews*. The transcribed interviews were compiled in a running document without markings for which student said what. The document was then read several times without making any markings on the document. The aim was to try to understand what different students were saying in relation to what other students were saying.

2. *Analysis of what the students talked about*. The transcripts were read again, this time with the intention of marking those excerpts where the students talked about pattern generalization. The excerpts of the transcriptions where the students did not talk about pattern generalization were identified and removed.

3. *Analysis of how the students were talking about pattern generalization*. The excerpts where the students talked about pattern generalization were repeatedly read through a so-called comparative reading (Marton, 1995). The aim was to distinguish between the dimensions of variations of students' ways of seeing pattern generalization that were realized through students' expressions.

4. *Categorization of the students' ways of seeing pattern generalization*. Different excerpts of the students' expressions were marked with the aim to identifying qualitatively different ways of seeing pattern generalization. Those excerpts were analysed, in relation to what the students emphasized and what they seemed to discern and focus upon in relation to pattern generalization.

5. *Identifying critical aspects regarding the ability to express and justify a pattern generalization algebraically*. In the identifying process the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does this generalization work to predict any figure in the pattern?" (Radford, 2006)

5. *Findings*

The findings consist of two parts. Part one answers the research question "What are the students' qualitatively different ways of seeing pattern generalization?". It consists of four categories. This result is in relation to point 1-4 in the analysis.

Part two answers the research question "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?". It consists of identified critical aspects regarding the ability to express and justify a pattern generalization algebraically. This result is in relation to point 5 in the analysis.

Part one - Students' qualitatively different ways of seeing pattern generalization

In the following, there are descriptions of the categories which contain student's expressions. Each category is summarized in relation to which aspects of pattern generalization that the students seemed to discern and focus upon.

...as some kind of grouping structures
In this category, the students emphasize the grouping of quantities in the sense of using a structure as a strategy to see how the pattern is built. Students, for example, undertake groupings based on the number of elements in a term (see figure 1): "... Term 1 has three (matches) and (pointing to term 2) has six (matches) ...".
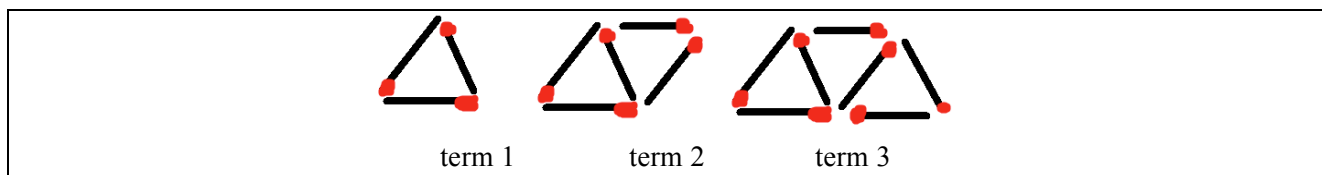

term 1   term 2   term 3

Figure 1. Matches

The students additionally grouped by adding together the number of elements in the visible terms in the pattern (Pattern 1): "... if all the matches up to term 3 is fifteen, then if you take this three, plus this three (the student is talking about the terms 1-3), it is term 6, then it is thirty matches, fifteen plus fifteen is thirty (here the student is talking about the number of matches in terms 1-3)". A characteristic of this category is that the grouping is used rather as a statement, not to predict the number of elements in a specific term.
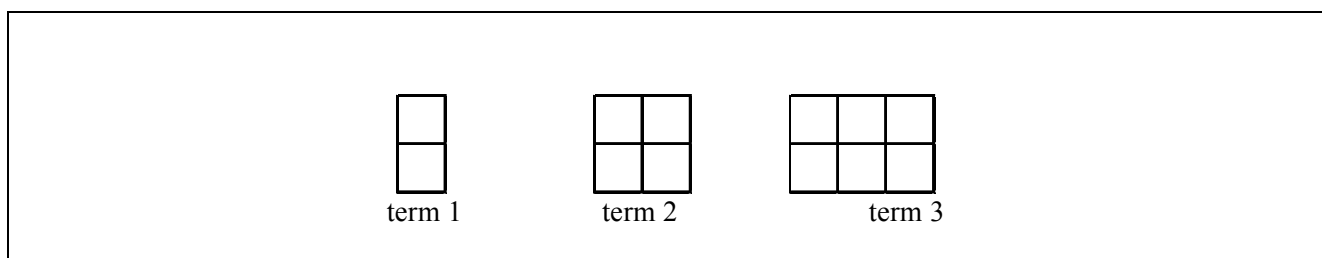

term 1   term 2   term 3

Figure 2. Squares

The students in this category seemed to discern and focus upon the following aspect: that there is a structure to follow that involves grouping objects.

...as additive constant structures
In this category, the students emphasize the adjacent terms in the sequence and the number of elements or units by which the pattern is growing. Students calculate the difference between two terms in a given sequence and distinguish this difference as being the same between all the terms. They conclude that the growth of the pattern is according to an additive structure. Based on the pattern in figure 2, a student expresses how to create the next term in the sequence: "... always add two (squares)." Other students use the column of two squares in the pattern as an integral unit, which they use as a rate of growth of the pattern: "... you only add one of those (pointing to a column of two squares)." The students see the growth of the pattern as "jumps" in the addition table: "... here are two, here are four, six, and the next eight and then it's ten."

The students in this category seemed to discern and focus upon the following aspect: the additive structure of the pattern and what constitutes the so called expansion unit (mathematical) of the pattern.

...as one dimensional relational structures

In this category the students emphasize one dimension of the pattern generalization; the number of elements or units in relation to the position of the term. The following student uses a column of two squares as a unit in relation to figure 2: "When it is term 1, I make one line (shows as a column) when it is term 2 it is two, if it is term 3 it is three columns". Another student expresses the connection between the term and the number of units (pattern 2): "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns ".

The students in this category seemed to discern and focus upon the following aspect: the relationship between the position of the term and its units.
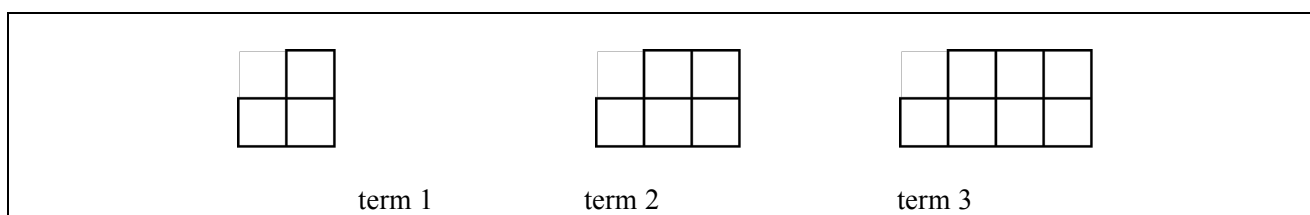

term 1   term 2   term 3

Figure 3. Squares in other way

...as two dimensional relational structures
In this category the students emphasize two dimensions of the pattern generalization; the relationship between the position of the term and the number of its elements or units and use it to predict a non-visible term in the pattern sequence. The following student expresses it, in relation to figure 2, like this: "Look, number 1 it is two, number 2 then it is four, number 3 is six, as it doubles everything… then you have to double forty-eight". In one of the tasks the students are supposed to determine the number of squares in term 46 (figure 3): "... is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)... then you can add forty and forty, its eighty and three plus three is six, then its forty-six and then we take one (the one that the student suggested should be put away) then there will be forty-seven ... no, it is eighty-seven."
The students in this category seemed to discern and focus upon the following aspects: what is the relationship between the figures number and its units and use this relationship to predict any figure in the pattern and what constitutes the constant in the pattern.

Part two - Critical aspects regarding being able to express and justify pattern generalization algebraically

The critical aspects were interpreted in relation to the categories' descriptions and which aspects the students seemed to discern and focus upon. In the identifying process, the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does the aspect enable the student to predict any figure in the pattern?" (Radford, 2006).
The categories, *as some kind of grouping structures* and *as additive constant structures,* do not encompass critical aspects in relation to *algebraic* pattern generalization. The aspect *that there is a structure to follow that involves grouping objects* is not an algebraic aspect in terms of making it possible to predict the number of elements in any term in the pattern. The structure in this case concerns 'only' the grouping of quantities. The aspects *the additive structure of the pattern* and *what constitutes the expansion unit (mathematical) of the pattern* are of general character. However, this kind of generalization only works to predict the adjacent terms, since one cannot say the number of squares of any term in the pattern, such as the thousandth.
It is primarily the categories *as one dimensional relational structure* and *as two or more dimensional relational structures* that we see as encompassing aspects of algebraic character in terms of making it possible to predict the number of elements in any term in the pattern. The aspects we identified as critical aspects in relation to be able to *express and justify pattern generalization algebraically* are:

- to discern the relationship between the term's position and its units
- to discern the relationship between the term's position and its units and to use this relationship to predict any term in the pattern
- to discern what constitutes the constant in the pattern.

Here is a short description of why we consider these aspects to be critical. By discerning *the relationship between the figures number and its units* it is possible for the students to use this relationship to predict the number of squares of any term in the pattern. "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns." In other words, the discerned relationship is the same throughout the whole pattern, and it doesn´t matter if you are talking about term 5 or if you are talking about term 1 000. When the students discern *the relationship between the term's position* and *its units and use this relationship to predict any term in the pattern*, the students both discern the relationship "number 1 it is two" and transform this relationship "as it doubles everything" so it can be used to predict any term in the pattern. Regarding the aspect what *constitutes the constant in the pattern*, the students discern that this unit, the constant, is the same through the whole pattern "is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)?". In other words, the constant is containing the same number of units in any term if the constant is one unit in term 1 it is also one unit in term 1 000.

6. Concluding discussion

The aim of this paper was to describe the emergence of the ability to express and justify pattern generalization algebraically. In the following we will put the categorization and identified aspects of this paper in relation to other research, and mainly Radford's distinction between arithmetic and algebraic generalization strategies. The difference is, according to Radford (2006), that an arithmetic strategy does not make it possible to predict any term in a pattern, which would otherwise be the case with an algebraic strategy. The main contribution of this paper, in relation to Radford's categories, lies in the specification of critical aspects regarding what students need to discern in their learning of how to express and justify pattern generalization algebraically.

In the category *pattern generalization as additive constant structures* the students seem to discern *the additive structure of the pattern* and/or *what constitutes the expansion unit (i.e., mathematical unit) of the pattern*. Students calculate the difference between two terms in a given sequence and identify this difference as being the same between all the terms, concluding that the growth of the pattern is according to an additive structure. There might be a qualitative difference between the expressions "... always add two (squares)," where the students talk about the growth of the pattern, and "... you only add one of those (pointing to a column of two squares)," where the students use the columns of two squares in the pattern as an integral unit. This latter can be put in relation to the position of the term, which can be used to predict any term in the pattern (Moss & London McNab, 2011; Radford, 2006), since it can be seen as the beginning of using an expansion unit as a measurement unit (Venenciano & Dougherty, 2014). However, if the generalization stops at discerning only *the additive structure of the pattern* and/or *what constitutes the so called expansion unit (mathematical) of the pattern,* this is not enough to express and justify pattern generalization algebraically.

In relation to students in the early grades, we want to highlight how factual generalization and contextual strategies (Radford, 2006) can be seen as a starting point regarding developing an understanding of the meaning of algebraic notations. In other words, we consider Radford´s factual strategies and contextual strategies as indicating the emergence of being able to make symbolic generalizations. The difference between those strategies is about *how* the generalization is expressed. In the category *as one dimensional relational structures* a student expresses the following: "When it is *term 1*, I make *one line* (shows as a column) when it is *term 2* it is *two*, if it is *term 3* it is *three columns*." In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a factual strategy, although the indeterminacy is unnamed, and the generalization here is symbolized by actions. We would equate this, in relation to our findings on critical aspects, as the student is discerning *the relationship between the term's position and its units.*

Contextual generalizations address a more mathematical level of generalization, the indeterminate is "made linguistically explicit: it is *named*" (Radford, 2006, p. 16). In the expression "Look, number 1 it is two, number 2 then it is four, number 3 is six, as *it doubles everything…* then *you have to double* forty-eight", the generalization is symbolized by words '*as it doubles everything'*. In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a contextual generalization, although the indeterminacy is named, and the generalization here is symbolized by words. We would equate

this, in relation to the findings on critical aspects, as the student is discerning *the relationship between the term's position and its units and to use this relationship to predict any term in the pattern.* Finally, our point is that this kind of generalization can later be transformed into a symbolic generalization, while drawing on the students' more informal way of describing it.

Acknowledgements

References

Davydov, V. V. (2008). *Problems of developmental instruction: A theoretical and experimental psychological study*. New York: Nova Science.

Greer, B. (2008). Algebra for all? *Montana Mathematics Enthusiast, 5*(2/3), 423-428.

Kaput, J. J. (2008). What is algebra?: What is algebraic thinking? In J. J. Kaput, D. W. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 5-17). Hillsdale, NJ: Erlbaum & the National Council of Teachers of Mathematics.

Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 11-50). Rotterdam: Sense.

Marton, F. 1995: Cognosco ergo sum: Reflections on reflections. *Nordisk Pedagogik, 15*, 165-180.

Marton, F. (2015). *Necessery conditions of learning*. New York: Routledge.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum.

Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton, & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-40). Mahwah, NJ: Lawrence Erlbaum.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht: Kluwer.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd. ed.). Harlow: Prentice Hall.

Moss, J., & London McNab, S. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and covariation. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 277–301). Heidelberg: Springer.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, *21*(2), 33-49.

Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of "critical aspects". *Scandinavian Journal of Educational Research*, *60*(3), 323-336.

Pang, M. F., & Lo, M. L. (2012). Learning study: Helping teachers to use theory, develop professionally, and produce new knowledge to be shared. *Instructional Science, 40*(3), 589-606.

Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, North American Chapter (Vol. 1)* (pp. 2-21). Mérida: Universidad Pedagógica Nacional.

Radford, L. (2010). Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective. *Proceedings of CERME 6, January 28th-February 1st 2009*, Lyon, France.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 303-322). Berlin: Springer-Verlag

Radford, L. (2014). Role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*, 405–422.

Rivera, F. (2010). Visual templates in pattern generalization. *Educational Studies in Mathematics, 73*, 297–328.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford (Ed.), *Ideas of algebra: K-12* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Warren, E. (2005). Young children´s ability to generalise the pattern rule for growing patterns. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for*

Psychology of Mathematics Education (Vol. 4) (pp. 305–312). Melbourne: PME.

Venenciano, L., & Dougherty, B. (2014). Addressing priorities for elementary school mathematics. *For the Learning of Mathematics, 34*(1), 18-24.

Wernberg, A. (2009). *Lärandets objekt: vad elever förväntas lära sig, vad görs möjligt för dem att lära och vad de faktiskt lär sig under lektionerna* [The object of learning: what students are expected to learn, what is made possible for them to learn, and what they actually learn during lessons] (PhD thesis). Umeå, Kristianstad: Umeå universitet, Högskolan Kristianstad.

Production and evolution of functional-spontaneous representations through the communication process

Samantha Quiroz and Fernando Hitt
Département des Mathématiques, GRUTAEM, UQAM

E-mail: samanthaq.rivera@gmail.com; hitt.fernando@uqam.ca

Abstract. The mathematical modeling process starts with the proposition of non-routine tasks where students use or construct mathematical models for their solution. During this process, students develop functional-spontaneous representations that emerge naturally while they solve the task. The class organization has an important role in the development and evolution of these representations. The research aim is studying the factors influencing the communication process during mathematical modeling activities. Using a qualitative methodology, it is described a non-routine task related with the co-variation between variables with high school students. During the class, we used a methodology that promotes a scientific debate and self-reflection named ACODESA. Results show that the individual characteristics of each student are factors that can promote or limit the learning process in a teamwork organization.

Résumé. Le processus de modelage mathématique commence avec la proposition de tâches non-de routine où les étudiants utilisent ou construisent des modèles mathématiques pour leur solution. Pendant ce processus, les étudiants développent des représentations fonctionnelles et spontanées qui émergent naturellement pendant qu'ils résolvent la tâche. L'organisation de classe a un rôle important dans le développement et l'évolution de ces représentations. Le but de recherche étudie les facteurs influençant le processus de communication pendant les activités de modelage mathématiques. En utilisant une méthodologie qualitative il est décrit une tâche non-de routine rattachée avec la co-variation entre les variables avec les étudiants de lycée. Pendant la classe nous avons utilisé une méthodologie qui promeut une discussion scientifique et une réflexion de soi appelée ACODESA. Les résultats montrent que les caractéristiques individuelles de chaque étudiant sont des facteurs qui peuvent promouvoir ou limiter le processus d'apprentissage dans une organisation de travail d'équipe.

1. Introduction

The information that exists in the actual society is created and disseminating faster than in the past. The maximization of the information had impacted the education and its goals. Schools' purpose is to educate students with the criteria to understand what they read and produce, students with competences for solving real problems using innovative ideas. The traditional curriculums no longer use traditional practices as the memorization or repetition. Those old practices had not shown any benefit in the learning process.

Now, mathematics are tools that can be used to solve real-life problems and this is a way schools could promote teaching and learning. According to that, several countries had included strategies that emphasized mathematics and applications in order to promote a diversify thinking among students. One of those strategies is mathematical modeling.

Mathematical modeling is defined as the cyclic process where a teacher proposes non-routine tasks based in real context and where students use or develop a mathematical model that solve the problem (Niss, Blum & Galbraith, 2007, Rodríguez & Quiroz, 2015). There are three main characteristics in mathematical modeling: the first one is the teachers' role, as the designer of problems that need to be related to the students' interest. Those problems need to be clear and with instructions easy to understand, but demanding a complex solution. Those tasks usually accept several ways of solving, and also accept diverse right solutions.

Besides the role as a designer, teachers guide the mathematical modeling process without saying the right answer or the correct way to solve the situation (Hitt & González, 2015).

The second characteristic of the mathematical modeling process, is the student's role. Students are the main actors that develop solutions of the problem. According with diSessa et al. (1991), in the first approach to the problem, students produce representations that are spontaneous and non-institutional. Institutional Representations (IR) are the kind of representations, which are usually accepted and used by the actors of the teaching system: books, computer screens and teachers.

According with Hitt (2003, 2006), the spontaneous representations are cognitive structures that emerge when the student tries to understand and solve a non-routine task. In more recent studies, those representations have been named Functional-Spontaneous Representations (FSR). While solving the task, students need to make a refinement of their FSR through a communication process. The importance of communication is the third characteristic of the mathematical modeling process. During mathematical modeling teacher must promote the work in teams and also a group debate. The evolution of the FSR is related with the transformation and coherent integration of the external representations associated with the FSR in this process of communication in the mathematical classroom (Leontiev, 1975).

Our research is based in the study of Hitt & González (2015) about ACODESA methodology (Collaborative learning, scientific debate and self-reflection). This metodology promotes the evolution of representations during the solve of non routine tasks and promotes the diversify thinking in students. ACODESA methodology distinguish five stages:

- Individual work where the students facing a non-routine task and construct FSR and produce external representations (verbal and diagrams).
- Teamwork, where students work in teams in order to solve the same task. They make refinements of external representations linked to FSR through a process of argumentation and validation.
- Debate scientific: The entire class discusses different forms of representations to solve the task at hand.
- Self-reflection: Individually, students solve the same activity in home. It allows students to reconstruct what was made in groups.
- Institutionalization: teacher introduces the topic taking into account the students' results and using IR.

Through the use of those theoretical elements, the objective of the research is:

- Describe how the communication process can promote the evolution and refinement of Functional-Spontaneous Representations in a mathematical modeling process.

2. Methodology

The research is based in a qualitative paradigm, specifically in a case study. The sample was conforming by high school students between 14-15 years old. They were in the 9th grade at the moment at the moment of the research. There were chosen three teams of four students each. Those teams had shown different ways of work and to communicate in previous sessions. All sessions were video recorded. The non-routine task was chosen from a set of five activities designed to promote learning of covariation between variables. The activity chosen is the first of the set, and as the student first approach, it is demanded to make a first representation through a design or a diagram where they described the phenomena that is studied. Besides, it is demanded to write an explication individually using words.

During the second moment of the activity, students are organized in teamwork in order to compare their ideas and express a diagram as a social construction of the phenomena. When all teams had designed the diagram, they explain their work to the whole group and make comparisons of the solving. Finally, each team decides if they change the diagram or not.

3. Expected conclusions

Research results showed that students develop different FSR when they initially solve the task. Each student produced an initial diagram where they explain the phenomena that were analyzed. In the diagrams are shown several mathematical concepts as: angle, hypotenuse, distance, sides, parallel lines, and perpendicular lines. ACODESA methodology allowed the implementation of the mathematical modelling cycle during the lesson. Using ACODESA methodology, the first stage, individual work, showed a diversity of procedures to solve the problem. During the others stages of the ACODESA (teamwork, debate in whole group), the students FSR changed, but the changes were different in each team. The communication process of each

team influenced in the FSR evolution into IR. Some teams arrived to IR during the teamwork and some others needed to wait until the whole group discussion. The analyze recognized that some of the teams work in a homogeneous way, it means that the students promoted the participation of all the team members and the ideas were listened carefully and with clarity. Nevertheless, in others teams, there was a student that played the role of leader, and his/her ideas were the ideas that the others students followed. The other member's ideas were ignored or simply, not accepted. Another important factor was related with the discussion process. In some teams, this process were open and all students can explain their ideas. In those teams, the change in the FSR was bigger and the representations became almost IR. In the other hand, the teams were the FSR were similar between the students, the discussion process was poor and without reflection.

As a preliminary conclusion, the study showed that the production and evolution of FSR could be affected depending to the team where the student is involved. Because of that, teachers need to take into consideration the students in each team, and also the promotion of methodologies where can be used different forms of organization as teams, group and individual work. ACODESA may be an interesting way to promote those types of organization and also combine the use of mathematical modelling.

References

diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior*, 10, pp.117–160.

Hitt, F. (2003). Le caractère fonctionnel des représentations [The functional nature of the representations]. Annales de Didactique et des Sciences Cognitives, 8, 255–271.

Hitt, F. (2006). Students' functional representations and conceptions in the construction of mathematical concepts. An example: The concept of limit. Annales de Didactique et des Sciences Cognitives, 11, 253–268.

Hitt, F. y González-Martín, A. (2015). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educational Studies in Mathematics, 88*(1), pp. 201-219.

Leontiev, A. A. (1975) Sign and Activity, *Journal of Russian & East European Psychology*, *44*(3), 17-29

Niss, M., Blum, W., y Galbraith, P. (2007). Introduction. *Modelling and Applications in Mathematics Education, The 14th ICMI Study*, *10*(1), 3–32.

Rodríguez, R. y Quiroz, S. (2015). El papel de la tecnología en el proceso de modelación matemática para la enseñanza de las ecuaciones diferenciales. *Revista Latinoamericana de Investigación en Matemática Educativa, 19*(1), pp. 99-124, doi: 10.12802/relime.13.1914.

Inquiring the role of visual-representations in inclusive educational activities concerning fractions

Elisabetta Robotti
Università degli Studi di Torino – University of Tourin

E-mail: elisabetta.robotti@unito.it

Abstract. Since in Italy there aren't special classes for students with special needs, inclusive educational activities play an essential role in math education. This research focuses on MLD students (students with mathematical learning disabilities or difficulties) and, more in general, dyscalculic students or students with low achievement in math. In order to design inclusive educational activities, this research takes into account both some results of cognitive science and of math education. More in details, the research aims to interpret some research results of cognitive science concerning the MLD students' learning of fractions to define educational hypothesis upon which it can design inclusive educational activities to support teaching and learning of fractions in primary school.

Résumé. Comme en Italie il n'y a pas de cours spéciaux pour les étudiants ayant des besoins spéciaux, les activités éducatives inclusives jouent un rôle essentiel dans l'enseignement des mathématiques. Cette recherche se concentre sur les étudiants ayant des difficultés d'apprentissage en mathématique (appelés, dans la littérature anglaise, MLD : Mathematical Learning difficulties or disabilities) et, plus généralement, des étudiants ayant des trouble d'apprentissage en particulier étudiants dyscalculiques. Afin de concevoir des activités éducatives inclusives, cette recherche tient compte à la fois des résultats des sciences cognitives et des didactiques des mathématiques. Plus en détail, la recherche vise à interpréter certains résultats de recherche des sciences cognitive concernant l'apprentissage des fractions par des élèves ayant des troubles d'apprentissage, pour définir des hypothèses éducatives sur lesquelles on peut concevoir des activités éducatives inclusives pour soutenir l'enseignement et l'apprentissage des fractions à l'école primaire.

3. Introduction

Even if there isn't consensus on definition and identification of MLD students and the inclusivity (Ianes *et al.*, 2013) is not a construct used consistently across different fields (education, society...) or in different countries, in this research work, we considered as MLD students, dyscalculic students, students with difficulties in math and students with low achievement in math. Since in Italy there aren't special classes for students with special needs, we consider "inclusive educational activities" those developed in the context of the class, which meet the needs of all students of the class. Moreover, because in Italy the percentage of children with learning difficulties has increased in the last years, from 0.7% in 2010/2011 to 2,1% in 2014/2015 (among these, the 16% are diagnosed dyscalculic in primary school and the 25% are diagnosed dyscalculic in Upper school), the "inclusive educational activities" are strongly needed in order to design effective teaching and learning of mathematics above all, in primary school. In particular, this research focuses on teaching and learning of fractions. Two main reasons guide my choice: the first is that fractions play an important role in theories of numerical development. As matter of fact, according to Siegler, "Algebra proficiency is more closely related to conceptual knowledge of fractions than to conceptual knowledge of whole numbers" (Siegler, R. S., *et al*., 2013, p.6). The second one is because fractions are one of the main difficulties detected at the national (INVALSI) and international level (OECD-PISA).

As in math education (Mulligan *et al*., 2013; Fandillo, P. 2007), cognitive psychology have also been very active in investigating the phenomena of (difficulties in) understanding mathematics, included fractions,

even if the different interested fields of research have not yet reached sufficiently common grounds for conducting scientific and interdisciplinary studies. In this paper, I consider some results from research in cognitive psychology about the role of representations in understanding fractions (Marzocco et al. 2014), in order to set up important design decisions during the processing of an educational experiment built around learning fractions in primary school (Robotti et al., 2015). In particular, these results help me to define research hypothesis, in order to design inclusive educational activities about fractions, which face to the needs of MLD students of the classes. To this aim, this research considers UDL (Universal Learning Design, http://www.udlcenter.org/) framework, based on cognitive neurosciences, for designing learning experiences that work across a large spectrum of learners and for making flexible the design of curriculum in order to meet the students' diversity in the same class (Robotti, 2016).

4. Some results of cognitive science research about the learning of fractions with MLD students and their interpretation in mathematics education

In this session, I consider some research results related to cognitive science in order to interpret them through the lens of math education and define research hypothesis with the aim to design inclusive educational activities about fractions.

Cognitive neuroscience shown that accurate representation of fraction magnitudes emerges as crucial both to conceptual understanding of fractions (as part of a whole) and to the arithmetic of fractions (Siegler, R. S., *et al.*, 2013). Moreover, children of 6- and 7-year- olds use 1/2 as a reference point when matching non-verbal representations of fractions. When asked which of two partially filled rectangles match a third, are more accurate when the two options are on opposite sides of 1/2. For example, when matching 3/8, participants are more accurate when the options are sets equivalent to 3/8 and 5/8 than ones equivalent to 3/8 and 1/8. These researches underline that symbolic fraction knowledge develops later than non-symbolic knowledge, but the fraction 1/2 again is prominent in early understanding.

What cognitive research says about MLD students? At this regard, we refer the cognitive science research developed by Mazzocco and colleagues in 2013. This research considered three kinds of students: MLD students (considered dyscalculic students), students with low achievement in math (LA) and students with typically achievement (TA). The research first seems to confirm that children with MLD, relative to their LA and TA peers, were less accurate on symbolic magnitude comparison tasks involving pairs of fractions. Also MLD children have an improvement over the school time (from the 4th to the 8th grade - 9/10 years to 13/14 years), like the other groups, even if the rate of improvement grows up more slowly than that of their schoolmates.

From an educational point of view, we can infer that, even for MLD students, there could be improvement both in the appropriation of meaning of fraction (here considered as part of a whole) and of its arithmetic manipulation.

As introduced before, the fraction "one-half" plays a very central role in processing fraction magnitude. Therefore, the Marzocco's premise in her study design and analysis was that magnitude comparisons of visual-representations of "one-half" are easier to correctly resolve than are fractions items that do not include a visual-representation of one-half. The researchers evaluated rate of growth on the two types of items (one-half, non-half) as a function of MLD status. The results show that, at study entry (Grade 4), children in the TA group had higher rates of accuracy on the one-half items than their LA or MLD peers, that this pattern also emerged for the non-half items, and that children with LA had higher rates of accuracy than their MLD peers on the one-half and non-half items. Students with LA or TA reach and maintain ceiling performance on one-half items over time, whereas children with MLD do not. For non-half items, the TA group is growing significantly faster than the MLD group.

From an educational point of view, this result could lead to the hypothesis that the assessment of accuracy and of the use of an effective strategies that concern ½ at the conclusion of the 4th and 5th (i.e. towards the 10-11 years) could be a relatively efficient way to identify children who may have learning difficulties on fractions and that, therefore, need a further educational support about fractions.

Moreover, Marzocco and colleagues evaluated both rate of growth and accuracy rate about effects of item format (format of representation) on one-half items. Having established that children with MLD (and, at Grade 4, also children with LA) have difficulty comparing fractions, and that even one-half items pose a challenge for children with MLD, the researchers examined whether performance is facilitated (or hindered) by any of the representational formats, across the TA, LA, and MLD groups. The representations considered are: visual representations as part of the whole, symbolic representation as Arabic numbers and incongruent
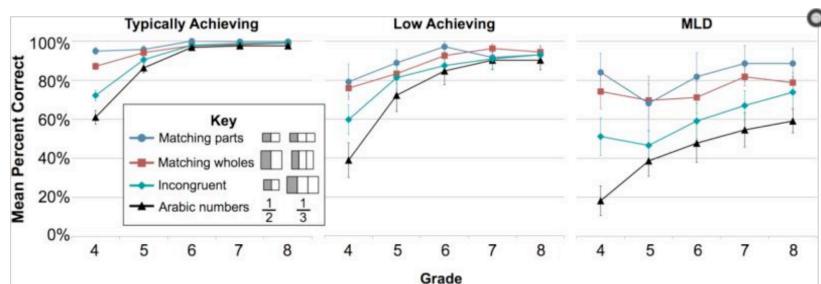
visual representations (see 'figure 1')

Figure 1. How performance is facilitated (or hindered) by the representational formats (Marzocco et al., 2014, p.12).

When visual models included matching "wholes", the LA group grew faster than the TA group, consistent with the notion of a general "catching up" after Grade 4. Children with MLD showed a faster rate of growth than the TA group, for the Arabic number format; Children with MLD grew faster than the LA or TA group on the spatially misleading format, presumably because of their markedly low initial performance levels on these formats. Rates of growth did not differ between the MLD and LA group, on the Arabic number notations, although accuracy rates did.

We observe that all the groups have performances more correct with visual representations rather than with Arabic representation. Nevertheless, if for TA group the two kinds of representation allow students to the same success after Grade 6 and for LA group after Grade 8, for MLD group the different representations of fraction never allow students to the same success. As matter of fact, we observe that visual representations allow MLD students to compare fractions in better way (with more success and accuracy) than Arabic representations.

From an educational point of view, this result could lead to the hypothesis that MLD students, as LA and also TA students, can benefit from the use of visual models to support effectively learning on fractions and solve problems involving fractions.

However, we can observe that in the MLD group, the performances on the inconsistent visual representations have always a smaller percentage of correctness (during the different school grades) than the congruent visual representations. This suggests that MLD students use visual model exclusively by referring to the perceptual strategies rather than to the meaning of part/whole.

Therefore, from an educational point of view, the teacher needs to pay attention to the use of these representations: the visual representation doesn't should be used through purely perceptual aspects but by strengthening ties with the meaning of part/whole or part of a unit of measure (as we will can see in 'figure 2'). The educational hypothesis that can be defined at this regard, is that teaching should bring out the character of "necessity" that have the solution strategies not purely perceptual. This allows MLD students to overcome the idea that perceptive strategies can be always the most effective, when comparison between fractions is required by visual representations.
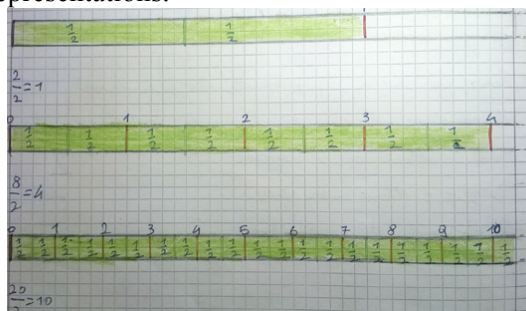

Figure 2. Comparison of "one-half" drawn on three strips of squared paper having different units of measure (30 squares in the first one, 10 in the second one and 4 in the third one).

3. Conclusion

Research in cognitive science suggest that MLD students show a "limited knowledge of "one half," until 8th Grade. From an educational point of view, this means that the time needed for dyscalculic children for

processing of this fraction as an effective tool in order to process the other fractions must be greater.

Moreover, cognitive science suggest that visual model leads to better performances than the symbolic one since the 9/10 years, and this gap is expected to decline from the 10/11 years except for MLD students. This means, from an educational point of view, that visual representation is an effective approach to fractions for all students but, for MLD students, remains the most effective strategy for a longer time. Moreover, in the use of visual model to compare fractions, MLD students prefer the perceptual strategy, which isn't the most effective strategy above all with inconsistent visual representations. This means that teaching, mediated by visual representations, should support the construction of meanings (for example the meaning part/whole) making sure that the perceptual aspect doesn't dominate on the development conceptual (for example, showing that the unity fraction depends on the chosen unit of measure). To this aim, teaching may make evident the need of more "sophisticated strategies". Based on these hypotheses, it was designed and implemented an inclusive educational sequence about fractions for primary school described in Robotti *et al* (2015).

References

Fandiño Pinilla, M. I. (2007). Fractions: conceptual and didactic aspects. *Acta Didactica Universitatis Comenianae*, *7*, 23–45.

Ianes, D. & Demo, H. (2013). What can be learned from the Italian experience? Methods for improving inclusion. *La Nouvelle Revue de l'Adaptatione de la Scolarisation*, 61, 125-138.

Mazzocco, M., Myersc, G. F., Lewisb, K. E., Hanichd, L. B. and Murphy, M. (2013). Limited knowledge of fraction representations differentiates middle school students with mathematics learning disability (dyscalculia) vs. low mathematics achievement. *Journal of Experimental Child Psychology*, 115(2), 371-387. Doi: 10.1016

Mulligan, J. T. & Mitchelmore, M.C. (2013). Early awareness of mathematical pattern and structure. In L. English & J. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning* (pp. 29-46). Dordrecht: Springer Science-Business Media.

Robotti, E., Antonini, S., & Baccaglini-Frank, A. (2015). Coming to see fractions on the number line. Krainer, K., & Vondrová, N., (Eds.). *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education*, 1975–1981.

Robotti, E., (in press). How the representations take on a key role in an inclusive educational sequence concerning fraction. Institute of Education, Dublin City University (Ed.). *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education.*

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: the new frontier for theories of numerical development. *Trends in cognitive sciences*, *17*(1), 13-19.

Students' awareness regarding vector "subtraction" through a dialog with the teacher

Ulises Salinas-Hernández, Isaias Miranda and Luis Moreno-Armella
Cinvestav-IPN, México; Instituto Politécnico Nacional, México; Cinvestav-IPN, México

E-mail: asalinas@cinvestav.mx; imirandav@ipn.mx; lmorenoarmella@gmail.com

Abstract. This article presents an interpretative microanalysis of the production process of meanings that students and an expert teacher carry out together in a physics class regarding vector subtraction. This is a qualitative study supported by the theory of objectification, which defines learning—objectification—as awareness. Data collection was done through video recordings of the lessons taught by the physics high-school teacher. The results show both the two different representations made by the teacher to report vector subtraction and the student's difficulty to integrate those two representations.

Résumé. Cet article présente une microanalyse interprétative du processus de production des significations que les étudiants et un enseignant expert exercent ensemble dans une classe de physique concernant la soustraction des vecteurs. Il s'agit d'une étude qualitative soutenue par la théorie de l'objectivation, qui définit l'apprentissage-objectivation - comme conscience. La collecte de données a été effectuée par des enregistrements vidéo des leçons enseignées par le professeur de lycée de physique. Les résultats montrent à la fois les deux représentations différentes faites par l'enseignant pour signaler la soustraction du vecteur et la difficulté de l'élève à intégrer ces deux représentations.

5. Background and research problem

The analysis of school practices has led to the need for stressing the role that history and culture play in the development of a subject's education. Among the diverse sociocultural research approaches in mathematics education is the theory of objectification (TO) (Radford, 2014a; 2016). From a semiotic approach, the TO focuses on teaching-learning problems in terms that are different from the ones in the individualistic educational theories revolving around the student. Therefore, in this article we seek to answer the following question: How are the meanings regarding vector "subtraction" produced in a space of joint action of students and an expert teacher?

6. Theoretical framework

This research is supported by the TO (Radford, 2014a) that conceptualizes teaching-learning in terms of a joint activity of students and teachers. Then, the concept of activity or labor is the key conceptual category of the TO (Radford, 2014a). The notion of knowledge in the TO is based on the dialectical materialism; it is not something individuals possess, acquire or construct, but the mere possibility of ways of doing and thinking [on systems of ideas] (Radford, 2014b). Hence, "The only manner in which knowledge can acquire cultural determinations is through specific *activities* [italics in the original]" (Radford, 2014b, p. 7). Learning—objectification—is then defined as the awareness of the [scientific] systems of ideas, that is, the ways of expression, action, and reflection, historical and culturally constituted. However, meanings in the classroom are produced through a social and bodily (language, gestures) process that is symbolically mediated and carried out in a space of joint action, which "is a space of relations and embodied reciprocated tunings occurring in the concrete space of interaction." (Radford & Roth, 2011, p. 231).

3. Method

The results of this article are part of a wider ongoing research that analyzes the practice of two teachers: one

expert and one novice. This is a qualitative study performed through a case study. The pilot study was done by video recording the lessons of two physics (expert and novice) teachers in original teaching configurations. This article reports results of the expert teacher's lessons. We video recorded 10 of the teacher's sessions (16 hours) during which he taught Newtonian dynamics. To do so, two cameras were used and field notes were written down. For the objectives of this article, a portion of a lesson in which the teacher talked about vector subtraction was selected and excerpts of the discussion were transcribed.

4. Analysis of the interaction between students and teacher when producing meaning

Here, we present excerpts of the process during which the students and the teacher produce the meanings regarding vector subtraction, in a graphical environment particularly. The data analysis focused on the way in which the teacher promotes participation and awareness on the concept discussed. The analysis is structured in two sections: 1) includes three excerpts dealing with vector subtraction as "sum of additive inverses"; 2) contains two excerpts which graphically show the subtraction of two vectors.

4.1 Vector subtraction by the sum of additive inverse of a vector

The first excerpt begins when the teacher explained the subtraction of natural numbers 5–3 using the analogy of a leaping frog on a number line in which the result [2] is the point the frog reaches after having jumped 5 [units] forward and 3 [units] back.

Teacher: The result [*of 5–3*] seen as a vector, seen as an arrow, would be—starting from the origin—where I reached. Then, I can think about 2 in two ways: as a point or as a small arrow that goes from zero to two. Now picture that the frog can leap in two dimensions. I want to take (2,2) from (3,1) [*writes: (3,1)–(2,2)*]. This means that [*the frog*] leaps 3 along the *x* axis and 1 along the *y* axis.

In the excerpt above, this is the moment when the activity starts, giving rise to concrete determinations related to vector subtraction. Before the teacher presented the situation of vector subtraction, knowledge was a mere possibility to the students. First, the teacher is observed to point at the result [2] on the number line as geometric object: a vector of magnitude 2 and a direction "that goes from zero to two." Then, this is when the concept of vector acquires a concrete determination. Although the teacher introduces the notion of vector subtraction, he does so abruptly and sets to work on a mathematical object that has been barely represented. The teacher makes an analogy between the subtraction of integer numbers and a vector subtraction. A moment later, the teacher graphically explains what happens when subtracting (2,2) and states that (2,2) would really be (–2,–2) even though he does not explain why "one (2,2) is really the other (–2,–2)". Then, he writes on the board: (3,1) – (2,2) = (3,1)+(-2,-2)= (1,-1). It can be said that the teacher defines the vectors as: A = (3,1), B = (2,2) and –B = (–2, –2), as the additive inverse of B and continues with the explanation.

Teacher: This is the leap (–2,–2) [*see figure 1a*] and the result [*of the subtraction*] goes from the origin to where [*(–2,–2)*] reaches [*see figure 1b*]. Then, what the first vector [*(3,1)*] had, then I actually had the second vector, which was this one [*draws vector (2,2), starting at (3,1); see figure 1c*], but I flipped it to the other side with the minus sign [to obtain (–2,–2); *he makes a gesture, see figure 1d*] and added the inverse. (…).

S1: Teacher, can't we make it just like in a Cartesian plane? (…)

Teacher: Why don't we see what you propose? (…).

When introducing the "vector subtraction", the teacher is, in reality, making the operation A + (–B). He even sets to work with –B from the beginning and clarifies that he is really adding inverse of B when he says: "then, I had actually the second vector, which was this one, but I flipped it to the other side with the minus sign and added the inverse" (see figures 1c and 1d).

On the other hand, when trying to graphically address the meaning of the vector subtraction, the teacher is observed to place the vector (–2,–2) starting at the point (3,1) and not at the origin of the Cartesian plane [Figure 1a]. This is a common way of adding vectors [graphically], which is used in courses and is known as "*head to tail addition method*" [figure 1b]. S1 is not familiar with this method and expresses the question. In that moment, there is no explanation as to why the vector sum is made in such way.
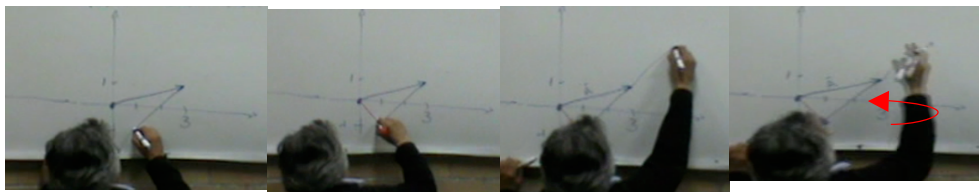
Figure 1. Graphic visualization process of the vector subtraction in four moments: from left to right (1a, 1b, 1c, and 1d).

S1: I was saying we should do it by coordinates [*marks points (3,1) and (–2,–2) in a system of coordinate axes*] (…) and then we get that it is (1,–1), but the result is a line, so I didn't know if that was correct. [*The result*] is something really twisted [*see figure 2a*]. (…).

Teacher: Then, the point that you placed here [*he means (–2,–2)*] is a point, but it is a vector, too. I mean, I can take the arrow that goes from zero to this point. (…) now the problem is how do I add this [*(3,1)*] to this [*(–2,–2)*], how do I add these two vectors? (…) Didn't you see [*referring to a previous simulation*] that we could grab this vector [*(–2,–2)*], this arrow, and take it to the edge of this other one [*makes a gesture to simulate he moves vector (–2,–2); see figure 2b*] without changing the length or the direction? So, if I move it [*referring to (–2,–2)*] it is this one [*see figure 2c; S1 says: "Oh!"*]. And the sum, which was it? It was the vector that, starting from the origin, reached the end of the second arrow. That means what you [*addressing S1*] propose is essentially the same.
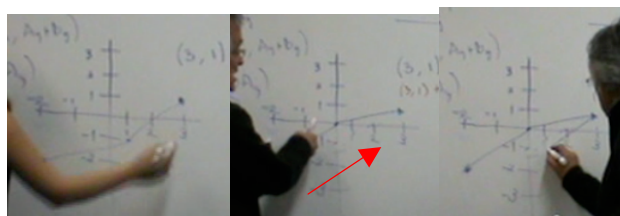

Figure 2. From left to right: process of vector subtraction by S1 (2a) and by the teacher (2b and 2c).

When placing vector (–2,–2) at the origin of the Cartesian plane, S1 is not aware that the differences is the teacher is following the graphical method to sum the vectors while she is using the arithmetic method (placing both vectors starting from the same origin). Still, S1 fails to graphically visualize the result. A moment later, the teacher introduces another representation for vector subtraction in which the additive inverse of B is no longer used.

4.2 Graphical subtraction of two vectors

Teacher: Now, I'm going to make the next [*different*] representation; let's see if it's useful for you. (…) What would this subtraction mean? [*(3,1) – (2,2)*] (…) [*it means:*] What do we have to add to (2,2) to get (3,1)? (…) I'm going to put any two vectors [*writes the vectors on the board; see figure 3a*]. (…) I put them starting from the origin [*as the student had done before*]. What does A – B mean? (…) how much is B missing to become A. However, what B is missing to become A is just the vector that starts at the end of B and reaches A [*draws the vector; see figure 3b*]. Because if I add this [*points at A – B*] to this [*points at B*], I get this [*points at A*]. Then, this is A minus B [*see figure 3c*]. Because it is the vector that added to B results in A. (…)
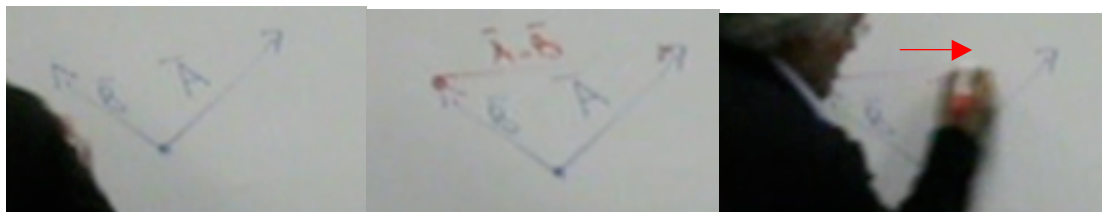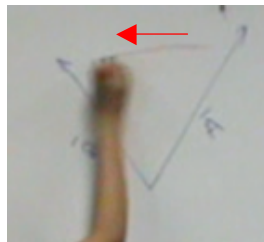
Figure 3. From left to right: representation of vectors A and B that will be subtracted (3a), representation of the subtraction A – B (3b), graphical representation of A – B (3c), and response by E1 to A – B (3d).

Unlike in the previous representation for vector subtraction, the teacher now (graphically) defines the vector subtraction as a process in which not only does he place both vectors starting from the same origin (see figure 3a) but also defines the result as: "the vector that starts at the end of B and reaches A." This work is far to be clear to the student. After the matter is discussed with the class and there seems to be an understanding, the teacher asks student S1 again about the result of A – B.

Teacher: Let's see, this is A and this, B [*he writes the same vectors in figure 3a on the board*], who is A minus B? Including line and direction.

S1: Okay, A minus B goes this way [*see figure 3d*].

The response by S1 shows how difficult it is for her to be aware of the process of subtracting two vectors (in two dimensions). It must be said that S1's response was considered since the teacher addressed mainly to her. However, the rest of the students were present to take part in the discussion regarding the meanings of the task.

5. Conclusions

The aim of this article was to analyze the way in which meanings regarding the subtraction of two vectors are produced in a task involving a teacher and students. Then, we observe from the results how meanings are produced in an original teaching-learning situation. In this situation, the activity prompts a dialog (including language, gestures, and signs) between the teacher and the students, so that the mathematical objects acquire definite determinations. On the other hand, we observed the complexity that awareness of mathematical objects poises and that the interaction between students and teacher is fundamental to that awareness. However, it must be stressed that although the task allows mathematical objects to acquire concrete determinations, it is hard for the students to be aware of such objects. One of the difficulties we observed was the teacher's change of register. This is an indication of the care a teacher should have when working with a concept in different representation registers.

References

Radford, L. (2014a). De la teoría de la objetivación. *Revista Latinoamericana de Etnomatemática*, *7*(2). 132-150.

Radford, L. (2014b). On teachers and students: An ethical cultural-historical perspective. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.) *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Plenary Conference), Vancouver (PME), Vol. 1, pp. 1-20.

Radford, L. (2016). The theory of objectification and its place among sociocultural research in mathematics education. *International Journal for Research in Mathematics Education*, *6*(2), 187-206.

Radford, L., & Roth, W. M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics*, *77*(2-3), 227-245.