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Considering potential impacts of a high-stakes test on pre-service teacher mathematical knowledge and beliefs

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Abstract. Communities expect their educators to be literate and numerate and governments and regulatory bodies take actions to provide assurances that this is the case. Australia, like other countries, has developed a test to measure pre-service teacher literacy and numeracy. Students completing an initial teacher qualification are expected to achieve the required standard before they can graduate and, in some states, register with their teacher registration body. However, the potential impacts of the test on pre-service teacher mathematical knowledge and beliefs have not been thoroughly explored. These potential impacts need to be considered as they can influence how the pre-service teacher engages with mathematics. This paper aims to start that conversation.

Résumé. Les communautés s'attendent à ce que leurs éducateurs soient *'literate and numerate'*, et les gouvernements et les organismes de réglementation prennent des mesures pour donner l'assurance que tel est le cas. L'Australie, comme d'autres pays, a développé un test pour mesurer *'literacy and numeracy'* des enseignants pré-service. Les étudiants qui terminent une qualification initiale des enseignants devraient atteindre la norme requise avant de pouvoir s'inscrire et, dans certains états, s'inscrire auprès de leur organisme d'inscription aux enseignants. Cependant, les impacts potentiels du test sur les connaissances et les croyances mathématiques des enseignants avant service n'ont pas été complètement explorés. Ces impacts potentiels doivent être considérés car ils peuvent influencer la façon dont l'enseignant pré-service s'engage dans les mathématiques. Cet article a pour but de commencer cette conversation.

1. Introduction

Since 2016, the Australian Government Department of Education and Training (DET) has provided the Literacy and Numeracy Test for Initial Teacher Education (LANTITE) to assess the skills of pre-service teachers. The LANTITE is acknowledged as focusing on personal numeracy and does not access subject-specific knowledge needed for teaching mathematics (DET, 2017). The numeracy component includes "identifying mathematical information and meaning in activities and texts … using and applying mathematical knowledge and problem solving processes … (and) … interpreting, evaluating and communicating, and representing mathematics" (DET, 2017, p. 3).

Meeting the test standard is a requirement to graduate from teacher education courses (DET, 2017) and, as a result, the test is high-stakes for pre-service teachers. High stakes tests on mathematical skills and knowledge can impact on how the pre-service teacher engages with mathematics (Meaney & Lange, 2010). This can occur through the pre-service teachers' perceptions of mathematics (Ernest, 1989; Grigutsch, Raatz, & Törner, 1998; Meaney & Lange, 2010), the pre-service teachers' mathematical knowledge (Ball, 1990), the pre-service teachers' self-efficacy and their self concept (Palmer, 2009; Parker, Marsh, Ciarrochi, Marshall, & Abduljabbar, 2014), the feelings they have towards mathematics (Bates, Latham, & Kim, 2013; Chinn, 2012), their mathematical empowerment or disempowerment (Ernest, 2002), and their beliefs about how mathematics is learned (Boaler, 2015). A single test also provides "only one aspect of the story" (Walshaw, 2011, p. 93).

2. Perceptions of mathematics

Mathematics can be perceived in many ways and these often relate to ideas around what is involved in mathematics and how it is used. Ernest (1989) considered mathematics in terms of three philosophical approaches – instrumentalist, Platonist, and problem-solving. Instrumentalist was described as viewing mathematics as rules and facts that were unrelated but utilitarian; Platonist was described as "a static but unified body of certain knowledge" (p. 100); and problem-solving was considered as a malleable and dynamic field created by people, influenced by culture, and always expanding. Grigutsch, Raatz, and Törner (1998) described a two-layered frame, where four aspects – schema, formalism, process, and application – fed into either a static view of mathematics (the two aspects of schema and formalism) or a dynamic view of mathematics (the aspect of process). The dynamic view of mathematics was proposed to be most likely to lead to application of mathematics.

Meaney and Lange (2010) found that pre-service teachers taking a mathematics test may consider it as an issue of "performance rather than competence" (p. 406) and emphasise procedural knowledge over conceptual knowledge. This could result in pre-service teachers moving towards an instrumentalist (Ernest, 1989) or static (Grigutsch et al., 1998) view of mathematics. These views of mathematics may lead to a focus on the importance of correct performances of the set rules (Ernest, 1989) or a focus on getting a correct result (Benz, 2012). Viewing 'the rules' as external and created by others who 'know' mathematics (Ernest, 1989) may also lead to a belief that there are 'maths-able' people and 'maths-un-able' people, which would effect what mathematical knowledge is seen to be, the actions the teacher should undertake, and how mathematics is learned (Ernest, 1989). Likewise, a focus on the importance of a correct result ignores the learning opportunities provided from mistakes (Boaler, 2014).

3. Mathematical knowledge

Several potential consequences related to mathematical knowledge may eventuate from pre-service teachers completing a high-stakes test addressing mathematical skills and knowledge. As stated above, if a pre-service teacher focuses on procedures in an attempt to meet the standard, this could lead to their focusing on procedures and rules as the required mathematical knowledge and use an approach that conveys these to their future students (Ernest, 1989). A narrow focus such as this would discount the understandings about mathematics and mathematical teaching such as outlined by Selling, Garcia and Ball (2016) in their mathematical work of teaching framework.

Lange and Meaney (2011) proposed that a test of specific, non-subject-matter mathematics may also create un-unhelpful separation of mathematical knowledge and mathematical pedagogical knowledge. In 1987, Shulman discussed the knowledge base required for teaching and the pedagogical processes that use that knowledge base. Content knowledge was listed as an aspect of the knowledge base and comprehension of that knowledge was listed as as aspect of the pedagogical process, with the latter requiring teachers "to understand what they teach and, when possible, to understand it in several ways. They should understand how a given idea relates to other ideas within the same subject area and to ideas in other subjects as well" (p. 14). A focus on procedures would run contrary to these ideas.

Alternatively, those who met the required standard in the high-stakes test by having focused on procedures may have a procedural view of mathematics reinforced by their success. They may continue with their focus on procedures and rules during their course, to the potential detriment of their engagement with mathematics in their course and the mathematical learnings provided to the children they will teach. The teacher educators in their course could struggle to change this focus towards "the need for conceptual understanding … gained through active engagement" (Lange & Meaney, 2011, p. 444). This is supported by Chinnappan and Forrester's (2014) study of the fraction knowledge of pre-service teachers. Their findings indicated procedural knowledge could impede the development of knowledge "necessary for quality mathematics teaching" (p. 894).

4. Self-beliefs and feelings related to mathematics

Completing a high-stakes test of mathematical skills and knowledge may impact on self-beliefs. Two constructs within self-beliefs that have been linked to mathematics knowledge are self-efficacy and self-concept. Parker, Marsh, Ciarrochi, Marshall, and Abduljabbar (2014) viewed self-efficacy as a description of capabilities (I can/cannot do this) and self-concept as an evaluation against external standards (I am/am not good at this), with self-efficacy guided by previous experiences and self-concept guided by comparisons with their other skills and the skills of their peers. They proposed that both were related to mathematical achievement, including high-stakes tests. Their research confirmed the relationships between self-efficacy and self-concept with current and future achievement. Parker et al. proposed that this demonstrated the importance of providing opportunities "develop appropriately positive assessments of their competence" (p. 43). Even though a test of mathematics is assessing a subset of mathematical skills and knowledge, it could have longer term impact on the pre-service teacher self-concept and self-efficacy.

A high-stakes mathematics assessment may provide either a positive or negative impact on the pre-service teacher, depending on how they position themselves and mathematics. Appelbaum (2008) proposed that mathematics could be viewed as an object. The 'object' of mathematics would include all interactions with mathematics, as well as how others are viewed to interact with mathematics, and smaller components that make up mathematics (Appelbaum, 2009). He stated that these connections to mathematics as an object may be positive or negative. The individual pre-service teacher's positive or negative perception of mathematics may contribute to the impact of a high-stakes mathematics assessment, and then flow through to creating further positive or negative conceptions, perceptions, and connections. Therefore, identifying *how* the pre-service teacher positions mathematics and their relationships to it would be highly beneficial in terms of the impact of a high-stakes assessment.

Pre-service teachers may express a dislike of mathematics (Bates, Latham, & Kim, 2013) or an inability to do mathematics and see themselves as not being a maths-person (Palmer 2009). These may become self-beliefs (Kimball & Smith, 2013) and lead to mathematics anxiety (Palmer, 2009) or withdrawing from and not attempting mathematical tasks that they believe they will fail (Chinn, 2012). A high-stakes test that assesses mathematical skills and knowledge may exacerbate these feelings towards mathematics and lead to the supposition that "only some people can be 'math people'" (Boaler, 2013, para. 5). This is in contrast to Boaler's (2016) claim that mathematics teachers should enact the belief that all students can do mathematics.

5. Beliefs about how mathematics is learned

Procedural views of mathematics and anxiety when engaging in mathematics may lead pre-service teachers to specific beliefs about how children learn mathematics. These beliefs may result in a focus on procedures and on mathematics only occurring in the mathematics classroom. Focusing on procedures can lead to an instructor approach (Ernest, 1989), the use of memorisation and tests (Boaler, 2015), and ignorance of the learning opportunities mistakes present (Boaler, 2014). A focus on mathematics as occurring only in the mathematics classroom may empower children within a narrow mathematical domain (Ernest, 2002). This overlooks social empowerment, gained from extending mathematics beyond the classroom to use it in everyday life (Ernest, 2002), and epistemological empowerment, where individuals have "a personal sense of power over the creation and validation of (mathematical) knowledge" (Ernest, 2002, p. 1).

6. Discussion and Conclusion

It may be that the phrase attributed to the Hippocratic oath, first do no harm (Sokol, 2013), should be the focus. A high-stakes assessment may not be "an equitable and quality experience in mathematics" (Walshaw, 2010, p. 98). Finding out how the individual engages with mathematics and sees themselves in terms of mathematics needs to be considered as this will enable the impact of a high-stakes assessment to be monitored and addressed (Appelbaum, 2008; Ernest, 1989; Grigutsch et al., 1998). The points raised in this paper will hopefully start a conversation that encourages consideration of unintentional consequences of actions undertaken, consequences that may impact on the pre-service teacher completing the high-stakes test and on their future students through its potential impact on the pre-service teacher's sense of self (Walshaw, 2010) in terms of mathematics, their "mathematical identities" (Walshaw, 2010, p. 96). Teaching mathematics requires more than a knowledge of mathematics and a focus on mathematics skills and knowledge could be harmful. As Grootenboer and Marshman (2016) state, not considering attitudes, beliefs, and feelings could result in mathematics education that is "irrational, unsustainable and unjust" (p. 124). Mathematics education should make learners want to be involved and engaged with the activities - this will enable mathematics to become available to all, rather than the few (Boaler 2016).

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Embodied, arts-infused, historico-cultural mathematics (out-of-doors) as a counter-narrative to hegemonic scientism and mathematisation

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Abstract. In this paper, the author explores aspects of the nature of hegemonic narratives of scientism and mathematisation, and offer a lived counter-narrative through an alternative pedagogy of mathematics. This pedagogy is one of embodiment of mathematical patterns and relationships, experienced viscerally through whole body movement and voice, using multiple senses and natural materials. It draws on historical and pluricultural traditions of mathematics, on artistic modalities, and aims to engage emotions including pleasure and delight. It is situated in the out-of-doors – in learning gardens (or parks, woods, fields, beaches) – where the living world is palpably present, and where human mechanisms of control are less prominent than in typical school buildings and classrooms. The paper begins by characterizing aspects of hegemonic scientism/ mathematisation and ways that they play out in unquestioned routines and rituals of stereotypical mathematics classrooms, particularly at the secondary level. A brief account follows of counter-hegemonic pedagogies as the author has experimented with them in a Canadian university teacher education program, in collaboration with students, mathematical artists, environmental educators and fellow researchers. Finally, suggestions are given for some further directions for intended research, with an invitation for others to collaborate or take up related research strands.

Résumé. Dans cet article, l'auteur explore les aspects de la nature des récits hégémoniques du scientisme et de la mathématisation, et offre un contre-récit à travers d'une pédagogie alternative des mathématiques. Cette pédagogie est une qui utilise la corporalisation des relations mathématiques, via des expériences viscérales par le mouvement du corps et par la vocalisation, en utilisant de multiples sens et des matériaux naturels. Il s'appuie sur les traditions historiques et pluriculturelles des mathématiques, sur les modalités artistiques et vise à susciter des émotions, y compris le plaisir. Il est situé à l'extérieur - dans des jardins (ou des parcs, des bois, des champs, des plages) - où le monde vivant est manifestement présent et où les mécanismes de contrôle humains sont moins importants que dans les écoles et les salles de classe. L'article commence par caractériser les aspects du scientisme / mathématisation hégémonique et le rôle qu'ils jouent dans des routines et des rituels de plusiers classes de mathématiques, en particulier au niveau secondaire. On décrit les pédagogies contre-hégémoniques avec lesquel l'auteur a expérimenté dans un programme de formation des enseignants universitaires canadiens, en collaboration avec des étudiants, des artistes mathématiques, des éducateurs en environnement et d'autres chercheurs. Enfin, des suggestions sont données pour de nouvelles orientations pour la recherche envisagée, avec une invitation pour que d'autres collaborent ou adoptent des domaines de recherche connexes.

When we, as critical mathematics educators, confront the excessive mathematisation of society and education, our arguments are often congruent with critiques of the scientization of contemporary societies and educational systems. In some ways, mathematisation is a pervasive aspect of scientization that uses numbers as a blunt instrument to quantify and calculate that which cannot meaningfully be counted. By a reductionist strategy of an unquestioning quantification of everything, fundamental epistemological questions may be stifled, and alternative ways of knowing quashed.

Heesoon Bai writes (in her foreword to Hyslop-Margison & Naseem 2007) that scientism comprises "an ideology that believes that science (and its rationalist foundation in modern epistemology) has an undeniable primacy over all other ways of seeing and understanding life and the world, including more humanistic, mythical, spiritual, and artistic interpretations." (p. vii). While Bai has no complaint against science per se, and is grateful for the valuable forms of knowledge science makes possible, it is the *hegemony* of positivistic science and the extension of its ways of knowing to all areas of life and society that she critiques.

We might take up our critique of mathematisation and education in a similar mode: that we are not quarreling with mathematical ways of knowing in themselves, and that we appreciate the valuable and often beautiful insights that these offer, but we have problems with the extension of mathematical models to every aspect of life and society, including those aspects that would be better or differently served by alternative epistemological approaches. It is the hegemony of quantification and its erasure of heterogeneous, potentially fruitful non-quantifiable ways of knowing and understanding that are objectionable, even dangerous.

There is a current pop culture mantra about goal-setting (in personal life, in business, etc.) that claims, for any desired result, that "it has to be measurable". That is to say, only quantifiable things count, and only what has been mathematised can be tracked, ranked, even believed. I have seen this mantra of the sacredness of 'measurability' taken up by all kinds of organizations, including NGOs, environmental and social activist groups, and at all levels of education, in many situations where quantification is entirely inappropriate. Through this neoliberal nostrum of requiring 'measurability' for everything, the hegemony of mathematisation and scientization has come to affect nearly every realm of contemporary society. It has the pernicious effect of distorting values and a sense of community, promoting spurious or dangerous rankings, and suppressing Indigenous and traditional knowledges and wisdoms, and artistic, poetic, holistic and spiritual approaches.

Hegemonic scientism and mathematisation carry with them implicit foundational assumptions for pedagogy, based on Enlightenment/ Modernist values. These values are based in Platonic and Cartesian axiom of mind-body separation, and the identification of scientific mind with a 'clean', decontextualized, sterile white laboratory, and of mathematical mind with universal, context-free esoteric knowledge, and other-worldly 'pure' mental cognition. These values, so familiar as to be invisible within science and mathematics education, are based in fear and loathing of their 'opposites': contextualized place-based knowledge, living (and dying) things, 'dirt', bodies, emotions, the senses. Without stretching the comparison very far, these reviled qualities can be seen to be associated with the Other as foil to the clean, white, male idealized scientist – so that hegemonic scientism implicitly reviles women, people of colour, Indigenous and colonized subjects, local and traditional knowledges, those living close to the land and valuing particular places that have meaning for them, and so on.

Environmental educators in particular have drawn attention to the colonizing, racist, sexist, anti-ecological pedagogies founded upon these assumptions, and the problematic they pose for a more holistic, anti-oppressive education for sustainability (see, for a few examples, Williams 2008; Hauk 2011; Bonnett 2013). They make the case for education outside the walls of the traditional classroom and in more natural settings; for the use of other 'geometries of liberation' (from Hauk) that allow ways of thinking beyond the Cartesian grid; for inclusion of the whole person (body, mind, emotions, senses, spirituality) in learning; for appreciation of patterns in the complex, living, growing world; and for a sense of continuity with our ancestors, our cultures, and the living world.

If we take these ideas seriously, there is an urgency in revisiting many of the normally-unquestioned assumptions and practices around so-called traditional mathematics education pedagogy. Why do we so often hold classes in rather barren, sterile classrooms, with learners sitting static and silent in rows of desks facing the teacher/lecturer? Why are emotions, sensory ways of learning, aesthetic pleasures, bodily movement and social interactions banned from so many mathematics classrooms, and increasingly so as learners become more grown up and mathematically sophisticated? Why are 'wrong answers' feared? Why does so much of assessment ride on individual, timed written tests? So much of what we have learned to take for granted is a reflection of Platonic/Cartesian values systems, and a scientistic domination of education by quantification and ranking of students' academic grades.

For the past seven years, a group of us at the University of British Columbia in Vancouver, Canada, has been experimenting with more integrative ways of teaching and learning mathematics and other school subjects, in the setting of the Orchard Garden, a student-led teaching and learning garden on campus. Our group has endeavoured to work in truly collaborative ways across disciplines and generations, to develop ways to teach and learn mathematics in a school garden (or other outdoor places), with the garden as co-teacher, and in dialogue with the history of mathematics and mathematical arts.

Our experiential mathematics pedagogies include: measuring sky and earth with our bodies, to understand seasonal cycles and growing things through angle, distance and estimation; using six-month pinhole cameras to chart the path of the sun through the sky from summer to winter solstice; building mathematically-interesting artistic objects (e.g. Platonic solids invariant-volume windsocks) and structures (e.g. hyperboloid arbourways) at both small and large scale, in collaboration with mathematical artists-in-residence, to understand these forms in fully-embodied ways; drawing-to-learn to inquire into the nature of angle and line in living and human-made forms; growing and foraging natural fibre materials and exploring geometries through making small and large-scale twine, spun fibres, braids and weavings; and developing narrative, poetry and mini-operas through close collaborative observation of geometries and ecologies in the garden. I will offer examples (and potentially a later hands-on workshop) to introduce these pedagogical experiments and the rationale for using them to establish a counter-narrative to oppressive scientism and mathematisation.

These pedagogical initiatives are only the early starting points for ideas and praxis of embodied, arts-infused, historico-cultural mathematics in the out-of-doors. An important part of the ongoing work in this area will be the discussion and possibly future collaborations and new directions that may come from these beginnings.

Figure 1. Students and conference participants building hyperboloid arbourway with mathematical artist-in-residence George Hart, UBC Orchard Garden. (Photo by author)

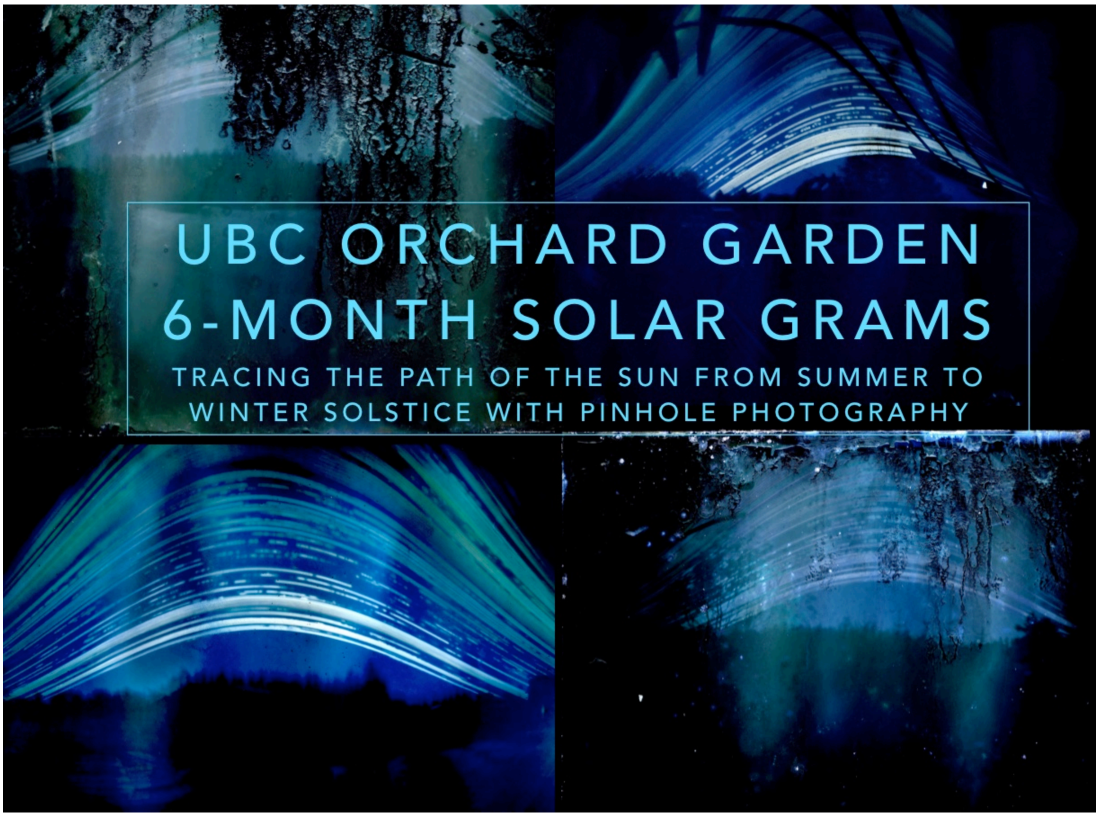

Figure 2. Student six-month pinhole camera images, made with guidance from mathematical artist-in-residence Nick Sayers. (Photo by author).

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How to deal with the modelling of epidemics? Some ideas and examples to be implemented in the classroom!

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Abstract. The general objective of this work is to deal with different approaches for the representation of an epidemic considering the state of the individuals in the population, Susceptible, Infected or Recovered (SIR), which generate models that students can explore with the computer using the contents acquired in mathematics subjects. Specifically, the purposes of the tasks designed focus on the identification and analyses of the variables and parameters involved in an epidemic through three modelling methodologies: i) Simple systems of ordinary differential equations reflecting the SIR representation; ii) Systems of difference equations and matrix representations by means of discretization of the time variable in the SIR formulation; and iii) Computational simulations from the use of the agent-based model "Virus" already implemented in the NetLogo platform. Exploration and comparison of the dynamics of Ebola and AIDS epidemics show the students the potential of modelling in this context and allow them to link with real applications. The assessment given by students of this activity was positive showing their interest in the topic.

Résumé. L'objectif général de ce travail est de faire face à différentes approches pour la représentation d'une épidémie compte tenu de l'état des individus dans la population, Sensible, Infecté ou Récupéré (SIR), qui génèrent des modèles que les élèves peuvent explorer avec l'ordinateur en utilisant les contenus Acquis en matières mathématiques. Plus précisément, les objectifs des tâches ont été axés sur l'identification et l'analyse des variables et paramètres impliqués dans une épidémie à travers trois méthodologies de modélisation: i) Systèmes simples d'équations différentielles ordinaires reflétant la représentation SIR; Ii) Systèmes d'équations de différence et représentations de matrice au moyen de la discrétisation de la variable de temps dans la formulation de SIR; Et iii) Des simulations informatiques issues de l'utilisation du modèle "Virus" basé sur l'agent déjà implanté dans la plate-forme NetLogo. L'exploration et la comparaison de la dynamique des éboles d'Ebola et du SIDA montrent aux étudiants le potentiel de la modélisation dans ce contexte et leur permettent de se lier à des applications réelles. L'évaluation donnée par les étudiants de cette activité a été positive montrant leur intérêt pour le sujet.

1. Introduction

In biology in general, and in epidemiology in particular, the real systems are extremely complex, so the models for studying them must inevitably include simplified idealizations. Thus, it is indispensable to know how these models work, the assumptions they make, the possibilities they offer and their limitations, in order to be critical and rigorous with their applications.

The first application of mathematics to epidemiology can be found in 1760, when D. Bernoulli published a treatise on smallpox, an infectious disease caused by the variola virus. In 1926, A.G. McKendrick, who studied medicine at the University of Glasgow but also studied mathematics, published an article on the "Applications of mathematics to medical problems". He introduced a continuous-time mathematical model for epidemics that took into account infection and recovery aspects. W.O. Kermack began to collaborate with McKendrick on the mathematical modelling of epidemics and they published together a series of "Contributions to the mathematical theory of epidemics" (Bacaër, 2011). It was considered that a population of size N (large enough) was made up of diverse groups of persons represented by three variables: $S(t)$, the portion of the population that was susceptible to infection; $I(t)$, the portion of the same population that was currently infected; and $R(t)$, the remaining portion of the population that had recovered from infection (although the R sometimes could mean removed, not recovered, and if the disease was fatal then this third state meant death). Hence, a mechanistic and deterministic representation of a dynamic of an epidemic, assuming homogeneous mixing of the contacts (interactions between individuals are instantaneous) and conservation of the total population, was produced, and a SIR model was generated, which is still the building block for most of the more complex models used nowadays in epidemiology. In the review of mathematical modelling of infectious disease dynamics of Siettos & Russo (2013) the reader can find an interesting presentation of diverse methodologies with references to successful real applications.

In recent decades emerging and re-emerging epidemics such as AIDS cause death to millions of people each year. Modelling is one of the tools used by international health institutions to tackle those epidemics, playing an important part in efforts that focus on predicting, assessing, and controlling outbreaks. In the summer of 2014, Ebola was spreading in Africa. The Centre for Disease Control constructed a modelling tool called *EbolaResponse* to provide estimates of the potential number of future cases, tracking patients through the following states: susceptible to disease, infected, incubating, infectious, and recovered (https://www.cdc.gov/vhf/ebola/index.html), releasing one type of SIR type model to the public.

Agent-based models (ABMs, or also called individual-based models) are computational and stochastic models to simulate the actions and interactions of autonomous agents (individuals) in order to evaluate the effects on the system as a whole (Railsback & Grimm, 2012). In an epidemiological context, ABMs are also being effectively used (Siettos & Russo, 2013).

Continuing in the line of combining different modelling approaches in teaching and learning in a life science context (Ginovart, 2014), a set of modelling activities to work with students in the classroom has been designed in order to deal with epidemics. The general objective of this work is to deal with diverse approaches for the representation of a SIR epidemic, generating diverse models that undergraduate students would be able to explore in the computer lab using the contents acquired in mathematics subjects. Specifically, the purposes of the tasks designed focused on the identification and analyses of the different variables and parameters involved in an epidemic, by means of three modelling methodologies: i) Simple systems of ordinary differential equations based on the SIR model, some of which will be solved by hand and others with the help of mathematical software; ii) The discretization of the time variable in the formulation of the SIR model that will generate a system of difference equations, with their corresponding matrix formalization, obtaining an approximation to the problem from fairly simple calculations that can be assisted by appropriate software; and iii) The use of an ABM to generate a set of individual-based simulations of the behaviour of a population developing in a spatial domain with infected people.

The outcomes of these models were analysed and discussed by the students, comparing their advantages and disadvantages for the representation of real systems.

2. Material and methods

The participants in this study were a group of forty third-year students of a Bachelor's degree in the field of Biosystems Engineering at the Universitat Politècnica de Catalunya (Barcelona, Spain). The prior coursework for these students was related to the following compulsory subjects: Mathematics I and II, Physics I and II, Chemistry I and II, General Biology, Microbiology, and Statistics, among others. This previous preparation guarantees a good knowledge of some biological concepts and basic mathematical and statistical concepts, as well as some control of computer tools for calculation and resolution. The use of Maple in the previous mathematics subjects provides the students with sufficient skills for the resolution of the ordinary differential equations of a SIR model and the calculations involved in the matrix representation of this model.

NetLogo is a free access multi-agent programmable modelling environment, which has a library with simulators ready to be used, with an extensive documentation about their main features and how to use them. Among them there is the ABM called "Virus" (Wilensky, 1998), the simulator chosen and employed for this activity. "Virus" simulates the transmission and perpetuation of a virus in a human population (https://ccl.northwestern.edu/netlogo/models/Virus). Figure 1 shows a screenshot of the friendly interface of this simulator with the input parameters and outcomes generated.

Students' responses regarding analyses and modelling of the dynamics epidemics with the distinct methodologies were collected via outputs of the mathematical software Maple, screenshots of NetLogo, and questionnaires, as well as face-to face dialogues during the sessions in the computer lab. The students' perceptions of the set of tasks conducted were explicitly questioned and collected at the end.

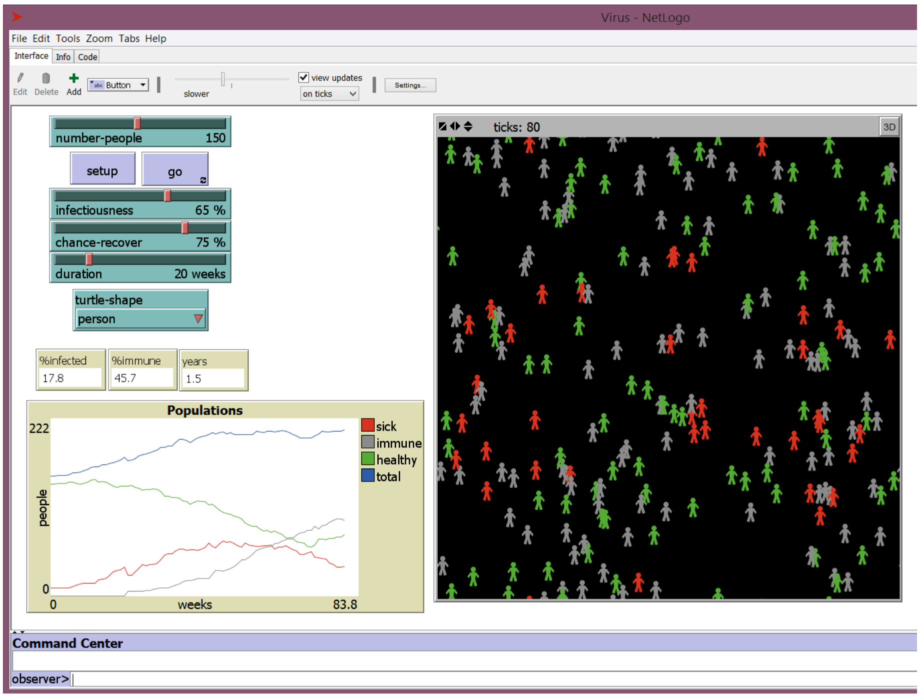

Figure 1. Screenshot of the interface of the "Virus" simulator in NetLogo platform.

3. Results and discussion

The results accomplished with these three distinct modelling approaches and the possibilities offered for each type of model to characterize the various dynamics of the population were assessed by the students, firstly in a mathematical context solving equations with the help of Maple software (Figure 2), and secondly, in a computational framework carrying out individual-based simulations with "Virus" to inspect the behaviour of a population developing in a spatial domain with healthy people (Susceptible), sick people (Infected), and immune people (Recovered) as Figure 1 shows. Students were trained to see how different values for the parameters of the model might approximate the dynamics of real-life viruses.

The use of the ABM "Virus", already implemented in the library of NetLogo, facilitated the development of this activity because the level of skills in programming was not an obstacle. In addition, the documentation provided by this model (Wilensky, 1998) introduced the students to the relevant effects that the parameter values implicated have on the evolution of epidemics, on the dynamics of the S, R and I subgroups of people. The noteworthy parameters to distinguish between different kinds of the epidemics tested and managed by the students were: "Infectiousness" that determines how great the chance is that virus transmission will occur when an infected person and susceptible person meet, "Duration" that determines the time before an infected person either dies or recovers, and "Chance-recover" that controls the likelihood that an infection will end in recovery/immunity (or death if it is zero). For instance, the famous Ebola virus has a very short duration, a very high infectiousness value, and an extremely low recovery rate, but the HIV virus, which causes AIDS, has an extremely long duration, an extremely low recovery rate, but an extremely low infectiousness value. Taking into account these parameters, the students were able to take the role of a public health agent to propose and test, with the help of this simulator, actions to combat a virus with the characteristics described for the Ebola virus or for HIV virus.

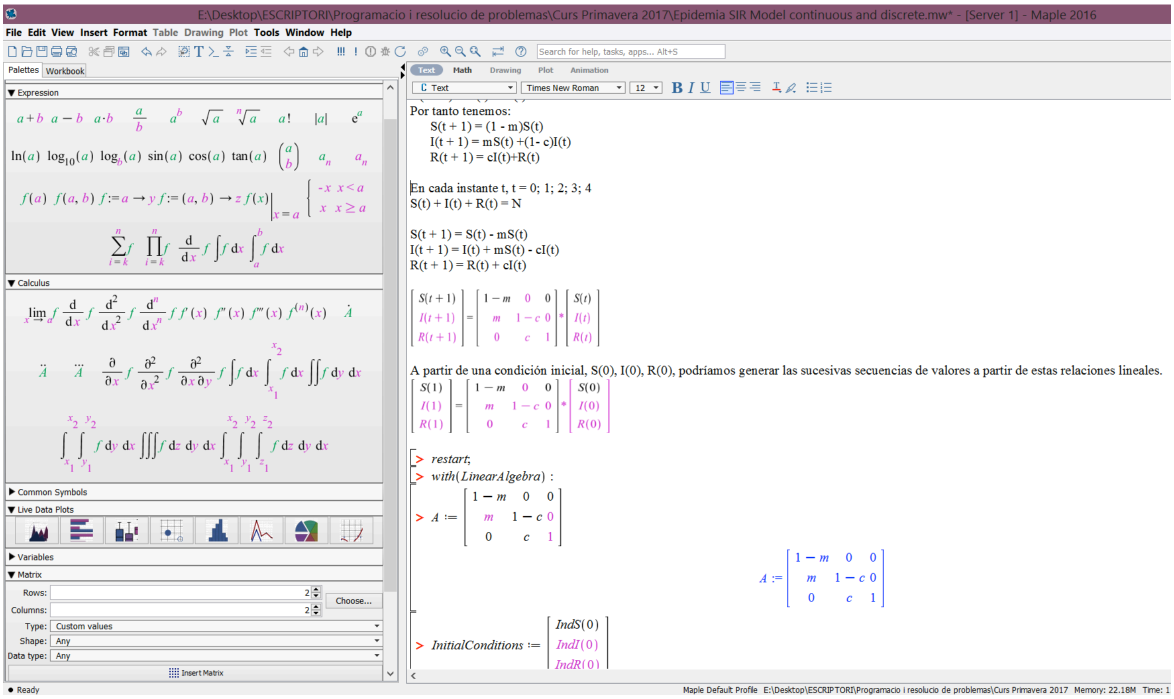

Figure 2. Screenshot of the calculations performed with Maple to resolve a SIR model.

The assessment given by students of this activity was positive and showed their interest and enthusiasm in a topic relevant to their biological studies. These three perspectives of modelling epidemics enriched the process of connecting mathematics with the investigation of life and real systems in our society, and contributed to building up reflexive knowledge on this issue in the classroom.

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Optimisation as a didactic principle

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Optimisation problems are classic problems in mathematics and the real world. Since the 1980s the landscape of solving optimisation problems has fundamentally changed in the era of high dimensional computing capacities as can be used today. Numerical approaches cap analytical ones since then. This shift recasts currently processes in industry as well as modelling of nature, climate change and so forth. In order to allow students to understand how mathematics and specifically optimisation is used and needed today to solve complex application problems, such as landing a spaceship on the moon, controlling robots to place objects precisely or to run a smart farm, mathematicians and mathematics educators need to work together. Inviting mathematics classes from schools to the university to learn about this, is one way of making this knowledge and these new approaches accessible to students and teachers. Principles of this approach and how these can be made accessible to students are discussed in this paper.

Mathematical Modelling (e.g., Blum, Galbraith, Henn, & Niss 2007, Stillman, Blum, & Salett Biembengut 2015) has been discussed in mathematics education as an important component of mathematisation for a long time. Realistic Mathematics Education (e.g., de Lange 1996, Treffers 1987) has conceptualised and examined mathematisation as a didactic principle for nearly half a century now, based on fundamental thoughts of Freudenthal (Freudenthal 1973). Introducing students to 'mathematizing unmathematical matters' (ibid., p. 133) was and is a key concern of this approach. Meanwhile the 'mathematisation-of-the-world', e.g. in modern Information Technology and other high-end technologies, has extended modeling and included simulation in engeneering and industry. High performance computing made this possible, but the widespread trial and error approaches that followed had high costs as an implication. Limiting financial resources in the industrial and economic world led to yet another turn and gave in recent years mathematicians back a stronger voice and more prominent roles in industry. Optimisation – as a mathematizing principle – became an indispensable component, being more rapid, fruitful and efficient in solving problems than mere simulation.

The threefold approach of Modeling-Simulation-Optimisation (MSO) (see Wets 1976), as used meanwhile in mathematics applications in Engineering, Information Technology as well as Natural, Economic and Social Sciences, is to our knowledge not widely discussed in mathematics education so far, neither has this approach been actively made visible to students in schools and their teachers. In contrast, traditional views of mathematics as an abstract discipline seem still to be prominent in schooling. New groundbreaking mathematical methods and ideas have hence not been popularized so far. This seems odd as challenging problems as how to use our limited natural ressource on Earth sensibly or building smart farms seem to be important global issues. Why not also approach these challenges together with students in mathematical ways? Finding more sophisticated solutions of a wide range of discrete, continuous or stochastic problems are of a global and societal interest. Even though progressive developments in mathematics require more complex cycles than even the MSO approach offers, making essential elements of such a cycle accessible to students is possible. We will indicate how on the following pages, based on our research and experience in conceptualising optimisation, linked to application methods and widespread cooperation with industry in the last decade, as well as our recent outreach initiatives to schools, offering math fairs to students and teachers at the university.

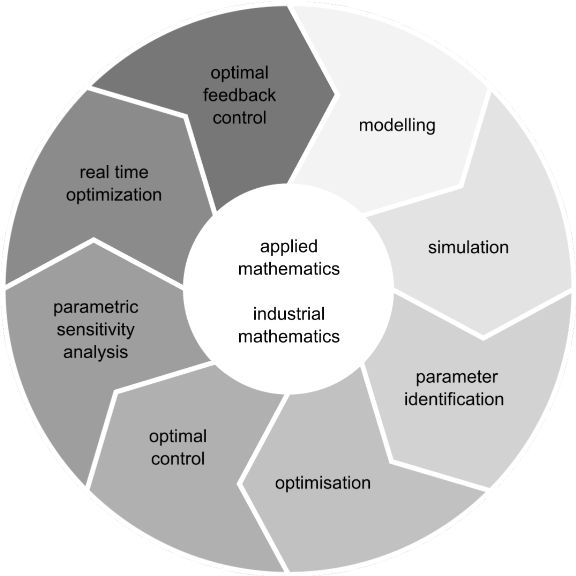

Figure 1. Extended MSO Cycle

As the cycle indicates mathematisation of 'unmathematical matters' is complex, modeling and simulation are only two parts of the whole process. Allowing students to focus on other key elements in this process we chose four components for our recently organised math fair at the University of Bremen: 1. Parameter identification, 2. Nonlinear optimisation, 3. Optimal control, 4. Optimal feedback control.

1. Parameter identification is the focus of the first station. The relevance and meaning of parameters is introduced in the context of the longterm human interest in astronomy. It is then applied to a LEGO Mindstorms vehicle, whose hardware and software parameters are set by the students so that it follows a given path. Students investigate and experience how parameters like the distance of the wheels, the speed of the car etc. determine if the vehicle can follow the path or not. This allows students to practically understand the relevance of parameters and their significance in optimisation problems. 2. In the context of a skiing problem – how to find the lowest point in a valley while avoiding trees using only local information – the theme of nonlinear optimisation is introduced. Mathematically the given problem is an optimisation problem with constraints. The mathematical conceptualisation of the skiing problem is essential at this point and introduces students to fundamental ideas of numerical solutions. The software WORHP Lab, developed by the working group Optimisation and Optimal Control at the University of Bremen, then allows the students to model, visualize and solve the given constraint problem. 3. Optimal control is experienced and thought through at the third station, where a parking manoeuver of an autonomous car is discussed. The students then use WORHP Lab to calculate the optimal trajectory for an industrial robot and experience how balancing a table tennis ball is impossible manually while perfectly easy using WORHP Lab. Sending the results to the real robot the students understand how mathematisation results in time dependent optimisation. Last but not least students get introduced to problems of feedback control at station 4. Given a dog's problem of traversing a river with a current in a most direct way, students get introduced to central ideas of feedback control. This allows students to successfully maneuver and land on the moon in a flight simulator. Besides playing, conceptualising and mathematising the situation supports students to understand how and why feedback control is a key element of optimisation.

Having students work in teams, providing hands on and theoretical activities at the same time, as well as letting them first only experience one station, but demanding a presentation of their station and the mathematisation in this context, allows students and teachers to get a sound first sense of optimisation. Both, teachers and us as a university team, are impressed how optimisation as a didactic principle can be successfully performed in this way. Further research and reflection is necessary to better understand this didactical optimisation experiment. We are looking forward to this and are planning next steps, together as mathematicians and mathematics educators, in sync with schools and collaboratively with international colleagues.

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Questioning the use of secondary school mathematics

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Abstract. This contribution presents a literature survey to defend the provocative claim that, regarding the everyday applicability of the contents taught, secondary school mathematics is widely useless. Although mathematical qualification in lower secondary school is officially considered a prerequisite for the full participation of learners in later vocational and private occupations, the individual usefulness of secondary school mathematics is questionable. Present examples for the application of these mathematical contents are usually not addressing realistic situations. It is argued that this mismatch cannot be overcome by the design of better application tasks, but that this mismatch is the symptom of a general overestimation of the changing role of mathematics in everyday life. Mathematisation is thus problematized as located in a field of idealistic myths. After a brief discussion of who benefits from this overestimation, ways to overcome the situation are outlined.

Résumé. Cette contribution présente une enquête de littérature pour défendre la revendication provocante que, concernant l'applicabilité quotidienne des contenus enseignés, les mathématiques d'école secondaire sont largement inutiles. Bien que la qualification mathématique dans l'école secondaire inférieure soit officiellement considérée un préalable pour la pleine participation d'apprentis dans les occupations professionnelles et privées dernières, l'utilité individuelle de mathématiques d'école secondaire est discutable. De présents exemples pour l'application de ces contenus mathématiques d'habitude n'adressent pas des situations réalistes. Il est soutenu que cette discordance ne peut pas être surmontée par le design de meilleures tâches d'application, mais que cette discordance est le symptôme d'une surestimation générale du rôle changeant de mathématiques dans la vie quotidienne. Mathematisation est ainsi problematized comme localisé dans un champ de mythes idéalistes. Après une discussion brève de ce qui profite de cette surestimation, les façons de surmonter la situation sont exposées.

1. Content-centred education

Given that mathematics education, especially in secondary schools, is the source of hardship for a wide proportion of school students and serves as a gatekeeper for further opportunities in life, mathematics education requires a solid legitimation. Although mathematics education is increasingly claimed to develop meta-mathematical competences such as modelling, argumentation and problem-solving (e.g., KMK, 2003), or demanded to engage critically with social applications of mathematics (e.g., Skovsmose, 1994), official curricula are still organised around contents. Rather than elaborating on meta-mathematical competences or critical agency and allowing teachers to choose contents which fit to these aims, curricula usually still follow the traditional approach to provide a list of contents that have to be taught, learned and tested (e.g. KMK, 2003). For example, the contents of the German curriculum for the 8^{th} to 10^{th} year of schooling comprise the Thales theorem, quadratic, exponential and trigonometric functions and elementary probability theory. The vast body of research in mathematics education focussing on approaches to teach specific contents more efficiently illustrates the dominance of the belief that the contents are of central relevance to the learning of mathematics; and the claim that these contents would be relevant for the mastery of vocational and private situations is widely used to defend the dominance of content learning both by scholars (criticised by Lundin, 2011, and Pais, 2013) and by students (discussed in Kollosche, 2017).

2. Usefulness in decline

The proclaimed everyday usefulness of the contents of secondary school mathematics is confronted by two objections. Firstly, research in the field of situated cognition provides evidence that, on the one hand, mathematical competences required in recurring out-of-school situations are usually acquired 'in practice', and that, on the other hand, mathematical competences acquired in school are rarely applied in out-of-school situations (Lave, 1988). Consequently, it is doubtful whether secondary school mathematics is necessary or even helpful to cope with out-of-school situations. Secondly, the mathematical demands in workplaces and in critical engagements with socially relevant mathematics tends not to match the contents being taught in lower secondary school. In the German case, Heymann (1996/2010), himself a passionate defender of mathematics education, admitted that most of the contemporary *contents* of secondary school mathematics are useless for the wide majority of learners, and that in later occupations learners will usually use only a small part of the mathematics that they have learnt in school. Borovik (2016) points out that developments in automatization and digitalisation have severely limited the usefulness of secondary school mathematics. One the one hand, technological developments allow to have reoccurring mathematical tasks performed by computers or to altogether substitute them by automatized processes. On the other hand, the mathematics needed to produce and maintain this technology exceeds secondary school mathematics by far. OECD's Programme for the International Assessment of Adult Competencies (PIAAC) analysed the statements of adults concerning their use of mathematics in work (OECD, 2013). Although it stays unclear how language, culture and occupational fields may influence the worker's identification of a certain procedure as 'mathematical', the results indicate, firstly, that workers in technologically leading countries, such as Norway, Germany, the Netherlands and Japan, use less 'numeracy skills' than workers in other countries (p. 144), secondly, that young workers use less 'numeracy skills' than aged workers (p. 153), and thirdly, that the extent to which numeracy skills are used depends much weaker on educational attainment than the use of reading and writing, ICT or problem-solving skills (p. 156). All these findings support the assumption that, in the course of technological progress, knowledge of mathematics becomes less and less important in the workplace. The same argument could be brought forward concerning the use of mathematics in private life where technology is also increasingly used to perform mathematical tasks.

3. The myth of usefulness

Despite its fading usefulness, secondary school mathematics is still presented as relevant for everyday life. The concept of mathematical literacy as utilised in OECD's Programme for International Student Assessment (PISA) is only one representation of this phenomenon. Several studies criticised that, in spite of their alleged realistic nature, the supposed applications of mathematics do not contribute to coping with realistic problems but obscure mathematical contents in a realistic disguise which then has to be unpacked by students (Dowling, 1998; Meyerhöfer, 2005). On a naïve basis, it could be argued that better examples for the practical use of secondary school mathematics have to be developed. Following a deconstructive approach based on Foucault, one might however ask in how far this mismatch between the assumed and described relevance of secondary school mathematics is productive in the socio-politics of mathematics education (cf. Kollosche, 2016). Dowling (1998) argues that applications in school mathematics create the myths that mathematics was omnipotent and everywhere relevant, thus creating the ideological conditions to use mathematics as a social technique of power. Lundin (2012) claims that secondary mathematics is a part of the broader 'standard critique' in mathematics education, which constantly criticises the presumably deficient *status quo* of mathematics education and formulates goals for improvement, thus both legitimising school mathematics as the institution it is idealistically supposed to become, and mathematics education research as the institution which will ensure this evolution. Kollosche (2017) proposes that both teachers and students have an interest in assuming the everyday relevance of mathematics against all contradictory experience, as this assumption provides meaning to their obligatory engagement with mathematics. All in all, the myth that secondary school mathematics was useful for the mastery of everyday work and private life constitutes a win-win situation for everybody who is affected by the institution of school mathematics: Students and teachers experience their daily work as more meaningful, researchers can legitimise the need for their income- and prestige-generating activities in mathematics education, and decision-makers can use mathematics as a presumable flawless argumentative and organisational tool.

4. Hope for secondary mathematics education

Alternative philosophies of mathematics education are possible and have been proposed. Heymann (1996/2010) suggests that after an elementary education in mathematics, which might be completed with covering today's contents of the first seven years of schooling, students should be given the choice to learn more advanced mathematical contents or to use their existing mathematical abilities to develop further meta-mathematical competences and discuss social applications of mathematics critically. Courses in advanced mathematics could be opened for students who want to continue their study of advanced mathematics later and would secure the education of mathematical expertise for higher education and specialised jobs. Fischer (2003) sees the goal of mathematics education for all the other students in enabling them to critically interact with mathematically further educated experts and mathematically organised procedures in society. Vohns (2017) provides a recent example of how Skovsmose's (1994) and Fischer's (2003) programs might materialise in the mathematics classroom when he proposes classwork on the critical examination of the mathematical modelling of measures for poverty which are applied in policy and social science. Taking these approaches seriously would widely agitate the traditional list of contents in secondary mathematics education and might lead to an inclusion of approachable forms of a wider range of socially and technically applied mathematical theories, including non-Euclidian geometries, graph theory, cryptography and inferential statistics. However, before such alternative approaches towards mathematics education can be explored in school, a critical mass of concerned protagonists will have to break through the win-win situation around the myth of the relevance of today's contents of secondary school mathematics.

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The dialectics of mathematization and demathematization

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Abstract. Many scholars have argued that mathematization as a social process can be only investigated if its antagonist, the process of demathematization, is also taken into account (e.g. Keitel, 1989; Gellert & Jablonka, 2006; Skosmose, 2014). Although these authors posit the relation between mathematization and demathematization at the centre of the discussion about the interrelations between mathematics and the social in modern societies, a systematic conceptualization of this relation is still due. In shifting the conceptual focus from the content of the processes of mathematization and demathematization to their form, I will attempt to qualify the relation between mathematization and demathematization as a dialectics. I will conclude by suggesting to confront this formal conception with empirical processes of mathematization in order to unfold its analytical potential.

2. Introduction

Since the 1980ies, all investigations of the multifaceted interrelations between mathematics and society were grounded in two related theses: 1) The number of mathematizations which are 'colonising' the social and material world is steadily growing; and 2) "[W]e should observe these developments critically, as they could do damage at all of us" (Davis & Hersh, 1986, p. 17). In the late 1980ies, Chevallard (1989) gave the first of these two founding theses an unexpected twist:

> "No modern society can live without mathematics. [...] In contradistinction to societies as organized bodies, all but a few of their members can and do live a gentle, contented life *without any mathematics whatsoever.*" (Chevallard, 1989, p. 49)

That is to say, the expanding *mathematization* of the social does not require more mathematical skills of the individuals which participate in mathematized practices, but quite the contrary is the case: The mathematization of the social implies a *demathematization of its members*. While "society as a machinery is more and more mathematised, our daily life is more and more demathematised" (ibid., p. 52). Therefore, many scholars argued that the social process of mathematization can be only adequately understood if the "reverse" (Gellert & Jablonka, 2007, p. 1) process is also considered: the accompanying demathematization of the social (e.g. Keitel, 1989; Keitel, Kotzmann & Skosmose, 1993, Gellert & Jablonka, 2007; Straehler-Pohl, 2017). Although it is commonly agreed that mathematization and demathematization *relate* to each other, there are subtle differences about the *nature* of this relation among the scholars that contributed to this discussion (see *Table 1*).[1] In the next section, I will argue for a characterization of the relation between mathematization and demathematization as a *dialectics* (II.). Subsequently, I will conclude by suggesting to confront this formal conception of a dialectical relationship between mathematization and demathematization with empirical processes of mathematization in order to unfold its analytical potential (III.).

Table 1. Overview of the terms indicating the relation between mathematization and demathematization

Authors	Relation between mathematization and demathematization

[1] Due to the limited space of this paper, I unfortunately am unable to provide a close reconstruction of the different conceptual nuances as they are indicated by the different terms in Table 1. Instead, I will focus on those aspects that I view as central to my endeavour of conceptualizing the relation as a dialectical one.

Chevallard (1989)	„contradistinction" (p. 49), „paradoxical" (p. 52), „dialectic" (p. 52)
Keitel (1989)	„contradiction" (p. 9), „paradoxically" (p. 11), „dilemma" (p. 11)
Keitel, Kotzmann & Skovsmose (1991)	"contradictory" (p. 250), "paradox" (p. 251), "parallel" (p. 251)
Gellert & Jablonka (2007)	"reverse" (p. 1) "dialectically" (p. 1), "paradox" (p. 9)
Skovsmose (2014)	"accompanied" (p. 442)
Straehler-Pohl (2017)	"reinforce" (p. 40), "apparently antagonistic" (p. 41), "dialectical" (p. 41)

2. The Dialectics of Mathematization and Demathematization

Gellert & Jablonka (2007) define mathematization purely formal as "a process in which something is being rendered more mathematical than it has been before" (p. 1). Conceptualized this way, the process of mathematization is universal and specific at the same time: It is *universal* because the quality of the particular 'something' (=*X*) which is to be mathematized is not limited at all – in principle, anything can be handled "by means of mathematical insight and techniques" (Skovsmose, 2014, p. 442); however, it is also *highly specific* because, as soon as we decide to mathematize a certain thing, process, practice, etc., we limit ourselves to *one* particular form of seeing the world and thus also to one particular form of seeing these things, processes, and practices. Whenever we are debating about concrete premises and decisions during a process of mathematization of a particular *X*, we should not belie us about the fact that we have always already chosen the *mathematization as such.* That is, we have excluded all other possible forms of seeing the world. At a more general level, this means that "choice is always meta-choice; it involves a choice to choose or not" (Žižek, 2003, p. 66).[2] So, whenever we are faced with the choice to choose between different premises that will lead to different mathematical models, we also choose the *choice to mathematize as such.* To put it another way, we automatically position ourselves on the side of mathematization and we are thus implicitly actualizing the primordial distinction which is the one between mathematization and non-mathematization.

In opposition to mathematization, demathematization indicates that the mathematics which is brought into action within the process is usually "operating beneath the surface of the practice" (Skovsmose, 2014).[3] As soon as mathematical models which aim at regulating a social practice are materialized in different forms of technology, mathematics seems to disappear from the (visible) surface of social practices (Keitel, Kotzmann & Skosmose, 1993). This 'disappearance' of mathematics from the visible surface obviously does not mean that mathematics is factually absent in the concomitant technologies, but, instead, it merely refers to the observation that mathematics brought into action quasi-automatically switches its "mode of presence" (Chevallard, 1989, p. 49) from an explicit towards its implicit form:

> „Implicit mathematics is formerly explicit mathematics that has become 'embodied', 'crystallized' or 'frozen' in objects of all kinds – mathematical and non-mathematical, material and non-material –, for the production of which it has been used and 'consumed'" (Chevallard, 1989, p. 50).

The explicit mathematics which is utilized to construct a certain technology becomes 'crystallized' in the process of mathematization. In this way it gets out of sight of the individuals using this technology at a later point in time. Therefore, every mathematization entails a demathematization, while the phenomenon of demathematization can actually be conceptualized as the *inner negation of mathematization*. That is to say, "an entity [here: mathematization] is negated, passes over into its opposite [here: demathematization], as a result of the development of its own potential" (Žižek, 2002, p. 180). Although dematheматization is a process which is opposed to the process of mathematization, we should not treat the two processes as indifferent to one another: "Quite generally, what is distinct in an opposition confronts not only *an* other but *its* other (Hegel, 1991, p. 187). Demathematization is not *an other* of mathematization, but it is precisely *its*

2 Here, we must note that the only free choice or political act is then 'not to' choose the choice as such which can be put into the formula "I prefer not to […]" (e.g. I prefer not to … choose between these and that premises in the modelling process, but boycott the choice as such).

3 For empirical examples see Skovsemose (2014) or Keitel, Kotzmann & Skosmose (1993).

other and thus mathematization and demathematization can be interpreted as "two sides of the same coin" (Straehler-Pohl, 2017, p. 41).

However, with respect to our primordial distinction (mathematization/non-mathematization), demathematization is not a cross of the limit to the side of non-mathematization (as the absence of mathematics in opposition to its presence), but, instead, we fully remain on the side of mathematization. In other words, demathematization is located fully *inside* the coordinates of mathematization and should be merely interpreted as a transition to another state – the implicit mode of presence. At this point, we can conceptualize the dialectics between mathematization and demathematization: The two distinctions that constitute their dialectical relationship are 1) the distinction between mathematization and demathematization and 2) the distinction between mathematization and non-mathematization, whereby these two distinctions are at the same time bound together *via* self-referential closure. To grasp the social process of mathematization in its *form*, we must conceive this process as the unity of the difference of a particular mathematization and its demathematization *as the two distinguished sides of the form* (content). While the form of mathematization as such is itself only one side of the primordial distinction (mathematization/non-mathematization) and thereby immanently referring to its other: the side of non-mathematization. Hence, we actually have to negate the negation itself, that is, we have to move to the negation of the negation (the side of non-mathematization), to really imagine a new *form* of social practice that is beyond *its mathematized form*. Therefore, I argue that it is very important to distinguish between *the other* of mathematization at the level of content, which is demathematization as being simply the implicit mode of presence (in opposition to its explicit mode), and *the Other* of mathematization at the level of form, which is non-mathematization as being the absence of mathematization as such (in opposition to its presence).

If we do not assert this Other (with a capital 'O'), we will be caught in an endless self-referential loop: One mathematization that is regulating a social practice can only be replaced by another mathematization, while the primordial distinction (mathematization/non-mathematization) functions as the blind spot stabelizing the whole endeavour. Every problem which is caused by the implementation of a certain mathematization can be only solved *inside* the realm of mathematization, that is, the solution will always be a new, 'better' or more sophisticated mathematization and the mere oscillation between mathematization and demathematization as two modes of presence constantly reproduces the presence of mathematization as such, the unity of this difference, or simply: *the form of mathematization.*

3. Concluding Remarks

This line of argumentation allows us to draw two conclusions: *Firstly*, every particular mathematization entails its own demathematization, which renders invisible that it depends in its own constitution on an extra-mathematical, subjective act, which is in its most general form: the distinction between mathematization and non-mathematization.[4] S*econdly*, the dialectical dynamic of mathematization transforms social practices beyond the level of our consciousness because the technological materializations of mathematical models function as black boxes in a radical sense: We are not only blind for their functioning, but we are apparently often, and this is way more frightening, blind for our own blindness, that is, we do not experience them at all. Since the number of mathematizations that surround us is nowadays "growing exponentially" (Ernest, 2001, p. 287), the primordial distinction between mathematization and non-mathematization seems to fall more and more into oblivion. However, we cannot limit ourselves to the discussion of the problem of how we can make "*implicit mathematics explicit"* (Keitel, 1989, p. 12) because then we would focus on the 'hidden' content of the mathematizations only. On the contrary, we also have to deal with another, maybe even more important, question: Why does mathematics brought into action assumes quasi-automatically this peculiar two-sided form? And moreover: How did this form historically came into being?

In this paper, I rudimentarily outlined a form analysis of the process of mathematization by conceptualizing the vivid dynamics between mathematization and demathematization as a dialectics. I conclude by suggesting that this purely formal conception of a dialectical relationship between mathematization and demathematization could form a point of departure to investigate empirical processes of mathematization in contemporary late-modern societies, since it is unquestionably an important task for mathematics education

[4] In mathematics, the first one to prove that it is not possible for any formal system to bootstrap its own conditions of possibility was Gödel (1931) in his famous paper *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*.

research to analyze these empirical processes *in content and form*.

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Mathematisation as a ruled practice: Questioning the production of knowledge of school practices under a normative Wittgensteinian perspective.

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Abstract. This is a discussion about the mathematisation processes of cultural practices, regarded as didactic strategies necessary for the production of significant mathematical knowledge in school contexts. We consider initially there is a connection between any symbolic production and the purposes of this production. Moreover, the language is understood as a game that in the use of the words under certain rules establish definitions and meanings that must be given to things. Therefore, we discuss on the mathematisation under what is called "a normative perspective", which emphases the performative character of the mathematical meanings in a symbolic game whose rules do not reside in the individuals, but in the set of social and cultural practices from which the subjects participate.

Résumé. Il s'agit d'une discussion sur les processus de mathématisation des pratiques culturelles, considérées comme des stratégies didactiques importantes à la production de connaissances mathématiques significatives dans les contextes scolaires. D'abord nous considérons qu'il existe un lien entre toute production symbolique et les objectifs de cette production. En plus, le langage est compris comme un jeu qui, dans l'utilisation des mots sous certaines règles, établit des définitions et des significations qui doivent être données aux choses. Par conséquent, nous discutons de la mathématisation sous ce qu'on appelle "une perspective normative" en mettant en évidence le caractère performatif des significations mathématiques dans un jeu symbolique dont les règles ne résident pas chez les individus, mais dans l'ensemble des pratiques sociales et culturelles dont les sujets participent.

1. Language and practice as ruled activities

Our theoretical storyline premise is that there is a connection between any symbolic production and the purposes of this production. In addition, the language is no longer understood as a mediator and communicative agent of certain cognitive operations; but as a game that when moving the use of the words under certain rules establish definitions and meanings that must be given to things. Thus, by thinking about the mathematisation under a normative perspective, we seek to put the performative character of the mathematical meanings produced by the school practices in a symbolic game whose rules do not reside in the individuals, but in the set of social and cultural scenarios from which the subjects participate.

According to Luria, "our intellectual operations involve such verbal and logical systems; they comprise the basic network of codes along which the connections in discursive human thought are channeled" (Luria, 1976, p. 101). If theoretical thought develops, the system of codes will consequently become more and more complex by including not only words (more precisely, meanings, which have a complex conceptual structure) and sentences (whose logical and grammatical structure allows them to function as the basic apparatus of judgment) but also more complex verbal and logical "devices" that makes it possible to perform different operations of thinking without reliance on direct experience.

Distinctively, for Wittgenstein, speaking a language is part of an activity guided by rules, a form of life (Wittgenstein, 2014, § 23); following a rule is part of our practices under certain conditions. And as "forms of life" our philosopher understands all our habits, ways, lifestyles, actions, institutions in which our activities are based and intertwined with language.

For this purpose, Wittgenstein coins the term "language games" not only to establish the ruled character of linguistic activities, but also to understand how people interact according to the forms of life and practices they carry out. Thereby, cooking, farming, or business, as well as explaining, imagining, describing, questioning, reporting, are all practices, language games, and they can take place within and across different domains or subfields.

The analysis of "language games" by Wittgenstein, in particular about what the philosopher exposes through his work *Philosophical Investigations*, does not seek an essence of language, or a "prescriptive" philosophy that identifies the problem of the essence, or simply to deny this path to point the correct one (Vilela, 2010). Wittgenstein (2014, § 90) proposes a grammatical reflection, that is, a reflection that aims to remove misunderstandings concerning the use of words, caused, among other things, by certain analogies among the forms of expression in various areas of our language, or even among different *language games*.

What's the meaning of a word? Wittgenstein would tell us this question is wrong since it suggests just one and definitive answer. It depends on which language games are in use and their set of activities enmeshed (Wittgenstein, 2009, § 96). For our philosopher, the structure of a Language is the structure of a reality. Hence, choosing the more adequate meaning is the result of following the rule regarding the system of reference, which works as a horizon of intelligibility (Régnier et al, 2016)

This point of view also implies that human activities are complex ruled, dynamic, and interchangeable games, and the world of culture is no longer a system of structures, but the variable result of interchanges among different activities.

For Wittgenstein, practice is a priority conceived according to our actions, forms of life, and language accordance (Bloor, 2001). Here, it is noteworthy the understanding given by Theodore Schatzki to Wittgenstein's words. For him, practices are, first of all, organized nexuses of activity; open-ended sets of doings and sayings organized by understandings, rules, and *teleoaffectivities* (Schatzki, 1996, my emphasis).

Moreover, the actions that compose a practice are either bodily doings and sayings or actions that these doings and sayings constitute. By 'bodily doings and sayings' Schatzki means actions that people directly perform bodily and not by doing something else. To say that actions are 'constituted' by doings and sayings is to say that the performance of doings and sayings amounts, in the circumstances involved, to the carrying out of actions (Schatzki, 2001). According to Miguel (2014) we always practice the language with the whole body and not just with culturally ruled vibration sounds emitted by our vocal cords. In this sense, to stage or perform a practice is the same as staging or performing a ruled language game; that is, both endeavors involve disciplining the body to make it follow the rules of that game.

Let us see, for example, the ways in which female workers involved in the world of school kitchen (lunch ladies), run bodily actions to prepare a specific dish or food for children (Ogliari & Bello, 2016). Dealing with the control of quantities, measurements and proportions (a pinch of salt, a tablespoon of oil, five kilos of rice for 130 children), they perform their job by calculating the quantities of ingredients to obtain certain flavors, textures, as well as servings that satisfy certain number of people. One way or another, they follow certain algorithms. This does not take away the possibility of our lunch ladies to come up with other processes. However, they have to know other recipes, to combine other ways to prepare, sometimes rehearsing, trying; making it up and following, unequivocally, another set of a ruled and normative language game; that is, a new algorithm that will lead them to the preparation of servings with certain flavors and textures. The symbolic production that guides their performance in the kitchen is guided not only by the rules and understandings contained by the procedures, but heavily by the teleoaffective component in question.

In that sense, we can also consider mathematics as practices, as language games or at least as a set of rules that govern our ways of doing and sayings in composing practices. By considering mathematics as a family of activities with a family of purposes (Wittgenstein, 2009, p. 273), Wittgenstein offers us an understanding of "Mathematisation" or mathematics in action, [...], that is, a heterogeneous and dynamic set of ruled symbolic stagings.

Many contemporary readings (Gottschalk, 2004, 2007; Shanker, 1987) of Wittgenstein's reflections about mathematics have pointed out that the originality of these reflections has been, primarily, their contribution to the emergence of a normative conception of mathematics, which is not compatible with logicist, intuitionistic, formalistic, or anthropological conceptions such as Ethnomathematics (D'Ambrosio, 1985). In addition, we can see *a priori* the numbers or algorithms as being invariably mathematical objects but they are, first and foremost, signs whose meanings are assigned in relation to performances and actions guided by rules and purposes.

2. Mathematisation as a didactic game.

To pay attention to the different language games, to the different uses a word can have, and recognize the different meanings assigned to that word claims, in our view, a reflection on the language. For Vilela (2010), Wittgenstein's philosophy is like a therapy that consists of traversing the various uses of a concept in the linguistic practice, undoing exclusive footage, inserting them in the plurality of uses. And this is what we propose to do with the notion of mathematisation.

From this perspective, when we put in check the uniqueness of meanings, we put in evidence the implications of these analyses for the teaching of mathematics; especially when it comes to subvert some linguistic "misunderstandings" that reinforce certain meanings over others in the field of school educational activity. The mathematisation as a didactic-pedagogical principle enforces a set of rules regarding the practice of teaching, among them that of "teaching" to solve problems, to understand and propose mathematical wordings, and almost always assign meanings strictly formal when describing, even mathematically, everyday activities. Thus, it is pertinent to remember that it is in the use of the contents, methodologies, the conduct of teachers and students, and especially in their purposes, that the meanings will arise as forms of production of the "mathematical" knowledge. In other words, questions that enunciate different contexts of use, related to different scenarios in mathematical texts and contexts, advocate the mathematics as unique, so that all the existing relations in the different contexts in which mathematics is needed only makes sense by its formalization. Thus, the mathematisation, even if related to different contexts of human activity, is taken as a "practice in itself", that is, a practice with universal characteristics that is worthy by itself. Acting this way is an acculturating strategy that typifies speeches and produces their own language games.

Lave (1996) gives us proof that to investigate the relationship that people have with mathematics in a given situation converges to the understanding of the existence of certain combinations and transformations of the relations of the so-called mathematical entities (whether of numbering, form, or measure) in the current activity, which ultimately lead to an apparent "disappearance of mathematics" in overcoming an impasse on a daily basis and not a relationship with pre-established images by formal or even scholar mathematics. Gottschalk (2008) states that all educational trends within constructivism assume that the mathematical objects preexist, "whether in the empirical, mental, or in the social intersubjectivity."

The condition of non-existence of universality of meanings, as well as these meanings just make sense regarding practices, helps us to take the problem of non-transfer of learning. If we suppose that our lunch ladies successfully performed a practice of measurement in the kitchen, we could ask: Do they have the ability to measure in any situation? Or would it be more prudent to say that they have learned to measure from the use of spoons, hands, "to the naked eye"? Out of the kitchen, could they measure anything with the measuring instruments similar to those used at the school? Or on the contrary, could they directly apply what they have learned at the school to improve their recipes? Each practice is governed by a different set of rules.

It is noteworthy that according to Wittgensteinian's understanding of practice, there is no distinction between theoretical and empirical approaches. The normative condition of language imposes a normative condition of knowledge. Hence, practicality and knowledge are both constitutive of the unique process.

The mathematisation as a didactic principle that highlight the democratization of teaching mathematics should not load only tracks of educational trends that have established themselves in the educational discourse. It should consider that the symbolic production is ruled by the practices constituted, whose circulation is inherent in our different forms of life.

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Ethnomathematical study on folk dances: "mathematisation" of the garbs

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Abstract. This proposal for oral presentation is included in a doctoral project in Mathematics Education. Part of this project aims to analyze and understand the mathematical structure inherent in various elements of folk dances characteristic of Northern Portugal and Galicia (Spain), specifically choreography, accessories, and music. We expect to develop an ethnomathematical study on elements of folklore, within a process of mathematization built on cultural practices. Regarding the accessories, two folk groups' garbs were photographed, in order to study the geometric patterns present on them.

Résumé. Cette proposition pour la présentation orale est incluse dans un projet pour un doctorat dans l'Éducation de Mathématiques. La partie de ce projet a l'intention d'analyser et comprendre la structure mathématique inhérente dans de divers éléments de caractéristique de danses folklorique du Portugal du Nord et de la Galice (Espagne), spécifiquement la chorégraphie, les accessoires et la musique. Nous nous attendons développer une étude d'ethnomathematical sur les éléments de folklore, dans un processus de mathematization a tiré parti des pratiques culturelles. Concernant les accessoires, les costumes de deux groupes folkloriques étaient photographiés, pour étudier le présent de dessins géométrique sur eux.

1. Ethnomathematical study on folk dances: "mathematisation" of the garbs

Mathematical activity is a human activity and so it constitutes a cultural activity (Gerdes, 2007a). Therefore, mathematics must be understood as a knowledge that all cultures produce, not necessarily equally to each other (Bishop, 1988). D'Ambrósio (2001, 2002, 2008) conceived the word "ethnomathematics" to designate mathematics practiced by distinct cultural groups, identified by common goals and traditions. Bishop (1986, 1988) determined the existence of six basic universal activities (counting, locating, measuring, designing, playing, and explaining) through which mathematics as a cultural product has developed not only in our culture but in all cultures. According to Bishop (1986), since these are universal activities, mathematics exists in some form, to some extent and with more-or-less significance for individuals within all cultures. In this sense, we agree with Bassanezi (2002) when he argues that each cultural group has its own way of mathematising the reality, and we should not ignore it in the educational field.

In this line of thought, Gerdes (1988, 2007b, 2013) uses an ethnomathematical approach to mathematise an old tradition of culture *Cokwe*, from the Northeast of Angola, specifically their drawings (composed only by points forming a grid and lines involving the points), known in the local language by *sona* (singular, *lusona*). These drawings are usually executed in the sand and serve to illustrate stories, legends and riddles (Gerdes, 2013). The mathematical potential of *sona* was object of continuous research by Paulus Gerdes. In this regard, Gerdes (2007b) analyzes a particular category of *sona*, monolinear, whose elements are *mirror-generated curves*, and describes some of its basic properties. In a multiphase process, these curves originate matrices. In a different sense, Barton (2008) ensured that the location of an object in two dimensions is, according to the dominant mathematical approach, determined by using the *Cartesian Coordinate System* or the *Polar Coordinate System*. However, in languages *Tahitian* and *Maori*, the location of an object is carried out with reference not to one but to two origins - the speaker and the interlocutor - and, consequently, the amplitude of two angles – one for each origin (Barton, 2008). Another aspect we consider important to invoke here is the study of symmetry from a cultural point of view, which has been widely explored and systematized by Washburn and Crowe (1988), appearing as an element present in artifacts all over the world.

In the previous investigations, ethnomathematics appears as a methodology to "mathematise" cultural practices, by recognizing and presenting the mathematics presented there. In this line, the doctoral project that motivates this proposal aims to analyze and understand the mathematical structure inherent in various elements of folk dances characteristic of Northern Portugal and Galicia. During the project, bibliographic collection about choreographic folklore of Northern Portugal and Galicia will be carried out and two folk dances will be studied. In particular, we intend to study three elements that constitute folk dances, specifically the choreography, the accessories, and the music. We will study the curves inherent to the movement of the dancers, the mutual locations of the elements of the pairs and the symmetry, not only in the consecutive positions of the pairs but also in their own garbs. Therefore, the research strategy to be used is a study with ethnographic characteristics, because it will be a descriptive study of the culture of a community or of some of its fundamental aspects (Baztán, 1995), which are, in this case, folk dances. Ethnography is an attempt to describe the culture or certain aspects of it (Bogdan & Biklen, 1994). The data collection will be carried out in a natural environment, through several methods, and complemented by information obtained through the direct contact of the researcher with this environment (Bogdan & Biklen, 1994).

2. Study on folk groups' garbs

Since this study started with the accessories, two folk groups' garbs were photographed and geometric patterns present on them were studied.

One of the folk groups was the *"Grupo Folclórico de Vila Verde"*, from the district of Braga, Portugal. The garbs that these dancers wear reflect the socio-economic reality of the region. The *"Traje de Encosta, Festa ou Domingueiro"* (figure 1) was used in the wedding day or in festivity days, being later left for the shroud. The *"Traje de Noivos"* (figure 2) is the continuation of the previous garb, now in the ceremonial version. The *"Traje de Ribeira, Feira ou Lavradeira"* (figure 3) was used in the fairs. The *"Traje de Trabalho Rural ou de Uso Comum"* (Figure 4) was used for agricultural work.

Figure 1. *Sunday Garb*

Figure 2. *Wedding garb.*

Figure 3. *Fair garb.*

As we can observe in the previous figures, the totality of the garbs of the dancers of *"Grupo Folclórico de Vila Verde"* displays reflection symmetry of vertical axis (Crowe, 2004). However, there are exceptions that deliberately break with this symmetry. For example, the linen jacket belonging to the *"Traje masculino de Encosta, Festa ou Domingueiro"* (figure 5) has a red embroidered motif in the central part that cancels the isometry. Also, the sweetheart handkerchief placed on the left side of the *"Traje feminino de Encosta, Festa ou Domingueiro"* (figure 6) makes it impossible for the vertical axis reflection to leave the whole invariant.

Figure 4. *Day by day garb.*

Figure 5. *Example of vertical reflection symmetry breaking.*

Figure 6. *Example of vertical reflection symmetry breaking.*

Figure 7. *Symmetrical arrangement of gold pieces.*

As we can observe in the previous figures, the totality of the garbs of the dancers of *"Grupo Folclórico de Vila Verde"* displays reflection symmetry of vertical axis (Crowe, 2004). However, there are exceptions that deliberately break with this symmetry. For example, the linen jacket belonging to the *"Traje masculino de Encosta, Festa ou Domingueiro"* (figure 5) has a red embroidered motif in the central part that cancels the isometry. Also, the sweetheart handkerchief placed on the left side of the *"Traje feminino de Encosta, Festa ou Domingueiro"* (figure 6) makes it impossible for the vertical axis reflection to leave the whole invariant.

The reflection of vertical axis present in garbs is also visible in the disposition of the gold pieces in filigree abundant in the *"Traje feminino de Noivos"* (figure 7). See how the pieces are carefully arranged on the black coat on a search for symmetry. However, as they are all different filigrees, this goal is not achieved.

The other folk group studied was the *"Agrupación Folclórica Cantigas e Agarimos"*, from Santiago de Compostela, Spain.

Some pieces of the garbs that these dancers wear also displays reflection symmetry of vertical axis (fig. 8).

Figure 8. *Example of vertical reflection symmetry.*

Figure 9. *Decorative shape repeated in centre and in each of the cuffs.*

However, there are clothes in these garbs that exhibit distinct patterns. For example, in a man's dress jacket, the same decorative shape is repeated in the central part of the jacket and in each of the cuffs (figure 9). Note that the three black bands (a wide band, a band with squares and a narrow band) appear precisely on these three parts of the jacket, keeping the same order from the top to the bottom on the wrists and from the left side to the right side in the centre.

Another example of pattern visible in these garbs is the continuity of decorative motifs that appears along the outline of women's dress coats (figure 10 and figure 11).

Figure 10. *Continuity of the decorative motif along the outline of the coat.*

Figure 11. *Continuity of the decorative motif along the outline of the coat.*

In figure 10, there is a repetition of a symmetrical motif, which causes the coat, viewed in a global way, to exhibit symmetry of vertical axis reflection. The motif that is repeated (figure 12), forming kind of a frieze, presents four reflection symmetries and four rotation symmetries. However, in figure 11, repetition of an asymmetric motif occurs, the repetition of which does not allow the coat to exhibit vertical axis reflection symmetry. The motif that appears repeated (figure 13) has no symmetries.

Figure 12. *Decorative motif.*

Figure 13. *Decorative motif.*

3. Brief conclusions

In the scope of the doctoral project, the bibliographical collection on choreographic folklore, as well as the beginning of the study of the folk dances, in particular accessories, allowed to study the symmetry present in the garbs of two folk groups. Briefly, this study will be extended to other groups. Nevertheless, the research initiated has already made possible to study the interrelationships between mathematical ideas and other elements and cultural constituents (Gerdes, 2007a). Symmetry is until now the salient mathematical idea, which is used both in the formation sense and in the intentional and episodic break. The vertical axis symmetry is the most frequent in the costumes of *"Grupo Folclórico de Vila Verde"*. However, it is also visible in *"Agrupación Folclórica Cantigas e Agarimos"*. There are also distinct patterns in these garbs: the repetition of decorative shapes in the central part of jackets and in each of the cuffs, and the continuity of decorative motifs that appears along the outline of coats.

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Mathematization of Selected Real-Life Aspects by Applying Dynamical Systems

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Abstract. Drawing in on what mathematics is, many reasons which justify its teaching become evident. Especially in the context of education, it is important to consider mathematics not only from the scientific perspective, but also include facets that explore mathematics from further perspectives. Throughout history, mathematics has occupied a predominant place in the school curricula. Mathematics has achieved this status for cultural and social reasons rather than because of its self-value. What I want to do in this paper is to develop an updated version of Felix Klein's book: *"Elementary mathematics from a superior point of view*" – a milestone in the didactics of mathematics, published in 1908 by the University of Göttingen. This paper aims to provide novel ideas by describing selected social, economic, and ecological procedures that might solve mathematics-related problems of a person. Therefore, I present several case studies situated in real-life situations which apply dynamical systems (Romero, 2013) as mathematical models.

1. Introduction

Throughout the following sections, from the point of view of mathematical modelling, I try to answer the following questions:

a. What are the mathematical developments of the 21st century that mathematics teachers at all levels, in particular secondary school teachers, should know about and how can they be made accessible?
b. How can we use other mathematical models that are not normally used in daily classroom teaching?

Currently, there are certain issues in the area of mathematical knowledge which relate to the profession of mathematics education. These are problems related to the general question of how to unite and structure curricula in a way that the teaching of the different subjects becomes correlated. This is important due to the fact that the majority of subjects are disconnected from the real world as well as from the sciences themselves. Because of this, students might not conceive the true usefulness of mathematics necessary for their general education. In many countries, e.g. Spain (BOE 5 and BOJA 171, 2007), with the exception of few educational communities, "a tradition has been generated in the way of organizing curricula in mathematics, reducing teaching to a work based on algorithms that does not allow to students to understand the role of mathematics in society" (Aravena & Caamaño, 2007, p. 8). This tradition is deeply rooted in the school system and has shown to be detrimental to achieve more elaborated aims in the learning processes of our students. Furthermore, it enlarges the disparities of achievement among students.

There are extra-mathematical situations (Romero et al., 2015) in the teaching of mathematics in the secondary school curricula which are primarily constructed in order to motivate students to develop mathematical techniques. This is a long-term process. These situations are not always motivated by problems (intra- or extra-mathematical) that can be solved. Conversely, modelling work that entails the investigation of possible models can lead to the increase and consolidation of an exceptional practice.

Since a few years, researchers in mathematics education have focused their attention on the design of activities based on the mathematical modelling of real-life situations. They do so with the aim to guarantee a gain of competences within the learning process of our students as well as our educators.

2. "News" Arguments

"Man cannot discover new oceans unless he has the courage to lose sight of the shore." André Gide

It is necessary to modify the curricula innovatively by the integration of new aspects, new models, and new creative themes (Izquierdo et al., 2004). There are many mathematical domains (Romero, 2011) which remain almost unexplored in primary school education (or in secondary schools), but are organized in an original and creative way that would enrich the classroom activities. These are, among others, graph theory and optimization, dynamic systems, fractals, topology, information processing, code theory and cryptography, and modelling.

One of our main aims is to challenge and provoke our students in order to make them use mathematical models (Duperret, 2009); for example the use of models which are designed for the description and analysis of aspects close to their daily life, so that they consolidate with their mathematical knowledge. Further, this aims to establish and implant solid cognitive principles for the future conception of such. Also, it is important to situate the descriptions, analyses, and dilations of the practical contexts in their everyday living environments as "experimental mathematics".

Throughout the development of the sciences, the various concepts as well as the use of mathematical theories were always considered. The mathematical results obtained are useful for the study in several areas. Presumably, without mathematics, societies would not have reached the levels of development they now incorporate.

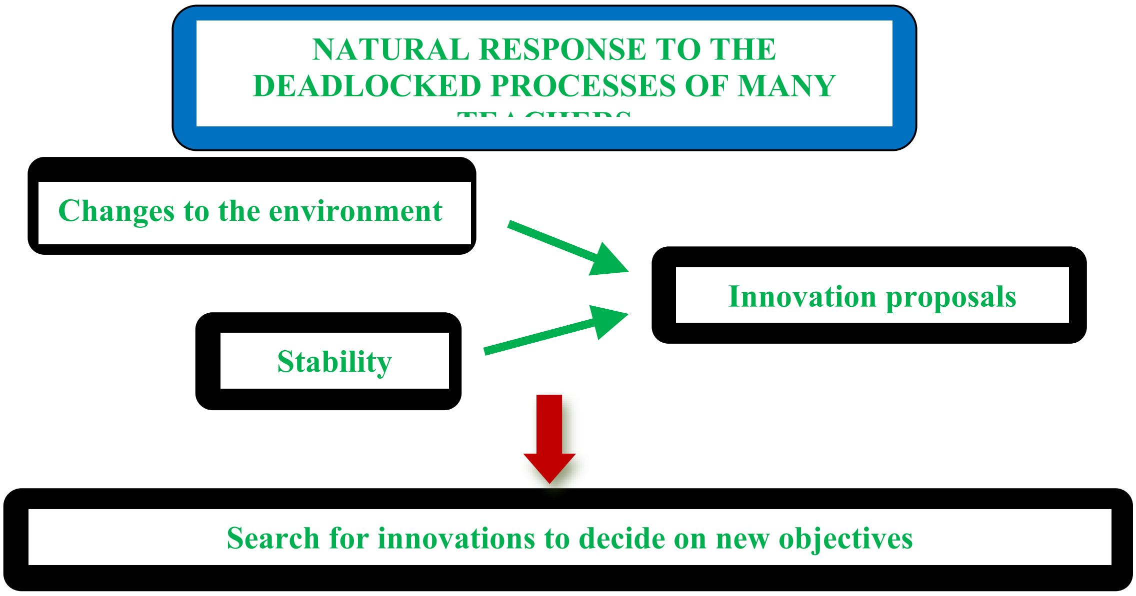


Figure1. New arguments against immobility

3. Dynamical Systems: Ideas

a) In the 17^{th} century, Newton, in his study on gravitation, discovered that Differential Calculus (DC) is a mathematical tool that can be applied to study various other physical phenomena. He successfully applied DC in order to make precise calculations on the orbits of the planets around the sun. The theory of Differential and Integral Calculus has been used to explain a large number of real-life events.

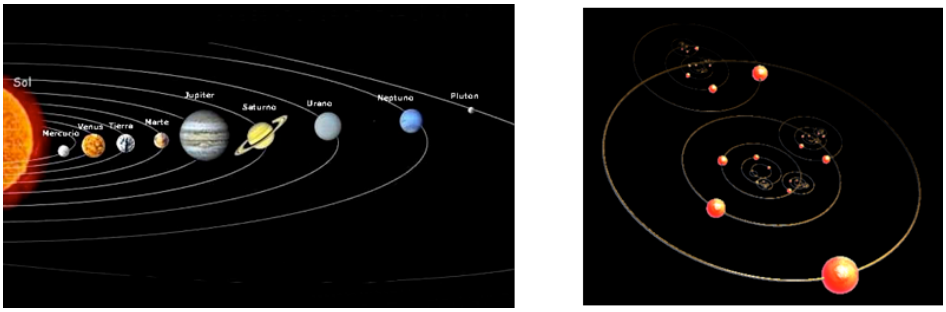


Figure 2. Planets

b) In the 18th to19th century, Euler, D'Alembert, Lagrange, Jacobi, Legendre, Hamilton, Fourier, and many others extensively developed theories on the movement of the planets Heat, Fluid, etc. This clearly illustrates the extend to which Differential Equations (ODE's) have contributed to the development of Physics.

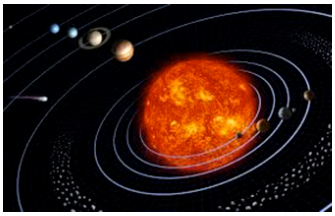


Figure 3. Orbits

c) In the 19th to 20th century, ODE´s that were made within practible studies have shown to have clear predictable results; however, accurately determine those was not always possible.

3.1. What is a Dynamical System (DS)?

It was Henri Poincaré who started the qualitative study of the differential equations in Mathematics, but it was only about 40 years ago that the dynamical systems were established as an independent field of research studies. This was, for example, thanks to the outstanding work not only of mathematicians, but also of engineers among which we can mention Smale, V. Arnold, Lyapunov, etc. Drawing in on the concept of dynamical systems, we could say that they are a rather new area of mathematics which belong to th research area of deterministic systems (Romero, 2013). This entails that they include the consideration of situations that depend on any given parameter, e.g. time, which vary according to established laws. Those considerations then allow us to reconstruct the past and to predict the future. Theoretically worded, we could say that a dynamical system (**Alligood** et al., 2009) is a way of describing the path through time through all points of a given space.

To explain what is meant by the term dynamical system (DS), we need to consider two steps:

1. We must choose a base space; for example, that could be the earth, the sun, the planetary system, a car, a company, a sphere. These base spaces are called topological spaces; we will denote them by *D*.
2. We must define a function in this base space:

$f: D \rightarrow D$ / x any one is associated with $f(x)$

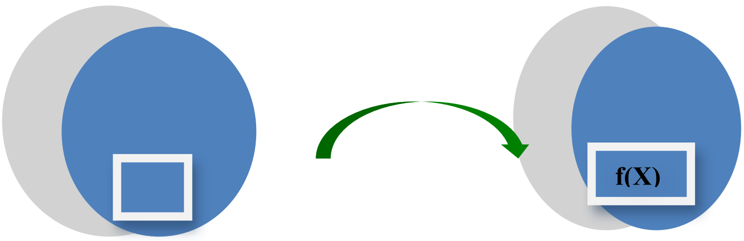
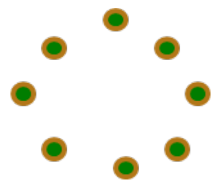


Figure 4. Function in the base space

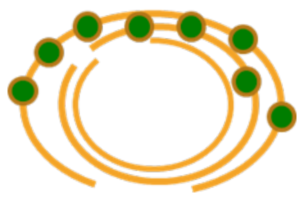
In case that it is possible to repeat the process *f(x)* → *f(f(x)),* an orbit of *x*, which are the points *x, f(x), f(f(x)), f(f((x)))* is generated.
Each point of D has an orbit; consequently, the study of these orbits is the Dynamical System in D due to f. This may give rise to different DS's which depends on the behaviour of their orbits; for example, *periodic orbit, quasiperiodic orbit*, or *other types.* They are used in order to study how the points are attracted to other sets:

- Single point orbit to Fixed point:

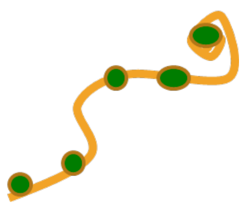
- Periodic Orbit to Finite number of points that, by the repetition of the process, return to the initial position with a finite number of times.



- Quasiperiodic orbit: the process where the points move near a periodic orbit is repeated.



- Another type: the points are attracted to another set



3.2 Case Studies

The activities that I am going to present next stem from a Spanish national project denominated ESTALMAT (**ES**timulate to the **TAL**ent **MAT**hematic).
They refer to mathematical models of processes that evolve with time, i.e. discrete DS's. They can be contrasted to differential equations, which are models to describe situations where changes occur at specific times rather than continuously.

In the next section, I present selected mathematical models which display processes that evolve over time, that is, discrete DS's that often involve an iteration process.

3.2.1. Case 1*, related to economy: THE BANKS – Working with the concept of BANK CREDIT*

Suppose my son, Enrique, in the middle of the current economic crisis, receives the offer to lend money from a bank. The interest rate charged by the bank is 0.5% per month. Enrique's actual repayment capacity is a maximum of 200 € per month. How much money do we want the bank to provide, reasonably?

A. Ideas for the solution

A. Ideas for the solution

a.1) A naïve answer could be: EVERYTHING YOU CAN GET!
a.2) Let's analyze the question to understand what we are asked for:
Let's call D_0 the amount of money we want to borrow from the bank, and ***Dn*** our bank credit after **n** months. The calculation that we come up with is the following:
* After one month, we would owe the bank

$$D_1 = D_0 + 0.005\, D_0 - 200 = (1.005)\, D_0 - 200$$

* After the second month, our debt would be

$$D_2 = (1{,}005)\, D_1 - 200 = (1.005)\, (D_0 + 0.005\, D_0 - 200) - 200) = (1.005)\ 2\ D0 - 200\ (1.005) - 200 = (1.005)\ 2\ D_0 - 200\ (1.005 + 1)$$

a.3) Following this process, in how much debt would we be after 6 months? And after a year?
a.4) What is the general formula for n-months as a function of D_0?

$$D_n = 1{,}005^n D_0 - 200\left(1{,}005^{n-1} + 1{,}005^{n-2} + \dots\dots\dots + 1\right) = 1{,}005^n D_0 - 200.\frac{1.1{,}005^n - 1}{1{,}005 - 1}$$

$$D_n = 1{,}005^n D_0 - \frac{200(1{,}005^n - 1)}{1{,}005 - 1}$$

$$D_n = 1{,}005^n D_0 - \frac{200(1{,}005^n - 1)}{0{,}005}$$

Once the above situations are resolved, a reasonable task for our students would be to confront them with the following scenario: An arbitrary amount of money is presented in several situations in order to sufficiently illustrate the different behaviors of the dynamical system in question.
One concrete example that allows us to introduce the concept of DYNAMICAL SYSTEMS in a natural way is the following:

• *If we loan more than 40,000 €, Enrique will have to pay back a monthly increasing debt.*

• *If we loan less than 40,000 €, Enrique will have to pay back a monthly decreasing debt.*

• *What if you loan 40,000 €?*

These three types of behaviour illustrate the concept of a *fixed point or an equilibrium point* of a dynamical system that does not change over time. Hence, we can set up the following equation:

$$D_n = 1{,}005 D_n - 200 \Rightarrow D_n = \frac{200}{0{,}005} = 40.000$$

B) Conclusion

To be familiar with the laws that govern the dynamical system allows us to predict the future. After the students fathomed this out, they were able to answer the question of how much money they would recommend Enrique to loan from the bank.
The answer further entailed expressions with a sequence of values:
Dn with **n** ? **Z** such that the value of ***Dn*** is determined by the previous values D_{n-1}, D_{n-2}, D_{n-3}, ..., . We call these equations equations in differences and thereby give rise to discrete dynamical systems. In this context, the term discrete refers to the parameter time, which could be every month, every year, every hour, etc. The equations in differences, i.e. discrete dynamical systems, more simplistic or more complex systems arise through the iteration of functions.
For example, they appear in:
A) The concepts of recurrence
B) General terms

C) Geometric progression
With the formula of addition, we explain how we obtain the expression of the sum of the terms of any geometric progression of ratio r as well as the initial term a_0 :
$a_0, a_1, a_2, a_3, \ldots\ldots\ldots., a_n, \ldots\ldots.$
finite or not with
$a_i = a_{i-1}.r$ for i ≥1
where $a_0 + a_1 + a_2 + a_3 + \ldots + a_n$ is the sum of the n terms of the geometric progression

$$\sum_{i=0}^{i=n} a_i = \frac{a_0 r^n - a_0}{r-1}$$

3.2.2. Case 2, related to the EVOLUTION OF THE POPULATION OF A SPECIES: Malthus growth model.

An Italian mathematician of the XIII century, Fibonacci, was the first to describe the growth of populations with a mathematical formula, today known as the Fibonacci sequence. Because of its general validity, it is still highly valued by the research application fields of the natural sciences.
In the above sections, we already discussed selected contributions of the seventeenth century where mathematicians proposed first attempts of mathematical models that aimed to be applicable to other sciences. Descartes, for example, applied them to physiology; however, these showed to contain a large number of errors and are therefore inaccurate.
In the eighteenth century, Malthus, another mathematician, set up an equation that showed the interdependence of the world population and global food resources. Later, this model became known as the 'Malthusian Catastrophe' because the calculations clearly revealed that the food resources are unsustainable in order to maintain the world population. Consequently, Malthus predicted famines and wars. Malthus presented his ideas in the form of differential equations including the necessary parameters in order to determine the exact points in time where the quantity of food would not be enough to subsist the entire population. Due to its accuracy, many people recognized this model as valid; however, the mathematically forecasted famine did not occur for several reasons. First, the British society of the eighteenth century underwent a significant demographic transition. Second, in the upper and middle classes of society, birth control was established; therefore, the birth rate transitioned from an exponential growth to a logistic growth. Third, due to agricultural development, food resources grew faster than expected.
In alliance to the Malthusian model, in 1838, Verhulst set up a model to describe a population's process of change. In that model, the growth rate is exponential at the beginning (as in the case of the model of Malthus) but alters its growth rate at a certain point in time; i.e. when the members of a society start to compete against each other over highly limited goods. As a result, the growth rate of the population in question decreases (García Rodríguez, E. et als, 2015).

3.2.2.1. Suppose we want to study the evolutionary process of a determined species concerning its growth rate from the moment when the number of individuals equals x_0. We further decided to measure time in years and denote it by x_k. The growth factor of said population between two consecutive years is 0,3. It is our aim to identify the number of individuals in year k.
1. Establish the relationship between x_k and xk-1.
2. Obtain a formula to obtain x_k as a function of k.
3. What types of behavior does the model have?

A. Ideas for the solution

1. Establish the relationship between x_k and x_{k-1}

$$x_1 = x_0 + 0.3x_0 = 1.3x_0$$
$$x_2 = x_1 + 0.3x_1 = 1.3x_1 = 1.3(1.3x_0) = 1.3^2 x_0$$
$$x_k = x_{k-1} + 0.3x_{k-1} = 1.3x_{k-1}$$

2. Obtain a formula to obtain x_k as a function of k

$$x_k = x_{k-1} + 0.3x_{k-1} = 1.3x_{k-1} = 1.3(1.3^{k-1}x_0) = 1.3^k x_0 \Rightarrow x_k = 1.3^k x_0$$

B) Conclusion

Which kinds of behavior can you extract from the equation?

On the basis of the mathematical expression we can say:
1) The mathematical model is reasonable in the early stages. That is, the model serves as far as it serves.
2) The evolution of the species itself allows us to state that the model is not valid.

In short, we can presume that our students, while accomplishing the task, consider three aspects of the GROWTH MODEL OF MALTHUS:

1. Is the model viable? When would it be?
2. In which aspects would the model differ in case that growth is proportional to the existing population with constant ***d***?
3. Would the model serve to study a population that extinguished, i.e. where the decrease of a population was proportional to the existing population?

3.2.2.2. Practical example

A bacterial culture is supposed to follow the growth of Malthus. Initially, there are 1000 bacteria; after one hour, there are 1250 bacteria.
a) How many will be there after 4 hours?
b) Using a table, find out how long it will take until there are 5000 bacteria.

A. Ideas for the solution

Applying the Malthus model, we come up with the equation $x_k = d^k x_0$, where $x_0 = 1000$.
After one hour, we have $x_1 = 1250$. Therefore, we can deduce:

$$1250 = d^1 1000 \Rightarrow d = \frac{1250}{1000} \Rightarrow d = 1.25$$

The model is governed for the equation $x_k = 1.25^k 1000$

a) After four hours, the number of bacteria would increase as followed:

$$x_4 = (1.25)^4 1000 = (2.4414).1000 = 2441,4$$

b) A table that shows how much time would have to pass until the bacterial culture grows to 5000 bacteria is being provided below:

k=1	***d^1=1,25***
k=2	*d^2=1,5625*
k=3	*d^3=1,9531*
k=4	*d^4=2,4414*
k=5	*d^5=3,0517*
k=6	*d^6=3,8146...*
k=7	*d^7=4,7683*
k=8	*d^8= 5,9604*

$$5000 = (1.25)^k 1000 \Rightarrow (1.25)^k = \frac{5000}{1000} = 5 \Rightarrow (1.25)^k = 5$$

$$k \log_e(1.25) = \log_e 5 \Rightarrow k = \frac{\log_e 5}{\log_e 1.25} = \frac{1.6094}{0.2231} = 7.2134$$

3.2.3. Case 3, *related to a MODEL OF ECOLOGY*

Suppose the following example: In Spain, we have 1250 individuals of a protected species of birds. Experts believe that the existing bird population decreases by 7% each year; either by natural causes, or by poachers. There is also a captive breeding programme which increases the bird population by 5 individuals each year.
Questions that we could present to our students are:

a. Write the relation of recurrence that relates the existing population in year k, x_k, with which there was year $\boldsymbol{k-1}$, $\boldsymbol{x_{k-1}}$.

b. Determine a formula that allows to obtain $\boldsymbol{x_k}$ as a function of $\boldsymbol{k}$.

c. Suppose the conditions do not change, is this species in danger of extinction? (It is established that a species is in danger of extinction when the number of its individuals is less than 100.)

A. Ideas for the solution

We would ask our students to make the necessary calculations.
Prior to discussing the results, we would ask the students to sketch a graph that visualizes the model adequately with respect to the following questions:

1. Note that the issues we have resolved in the previous exercise can be addressed as follows: We take a cartesian representation system for each point $\boldsymbol{(x,y)}$ where $\boldsymbol{x}$ is the number of individuals at time $\boldsymbol{k-1}$, and $\boldsymbol{y}$ is the number of individuals at time $\boldsymbol{k}$.
2. Check whether the question raised above responds to the dynamic system that is expressed by the equation

$x_k = 0.93\, x_{k-1} + 5$

3. Draw the dynamic growth curve.
4. Starting from the amount of birds in this year ($\boldsymbol{x}$) and with the help of graphic representation, give the amount of birds that will be there next year as well as in two years.
5. How large would the population of birds need to be this year ($\boldsymbol{x}$), so that there would be an increase of the population in the following year?

How large would the population of birds need to be this year ($\boldsymbol{x}$), so that there would be a decrease of the population in the following year?

6. B. Graphic visualization of the dynamic behavior of the system of Case III.

One way to represent the case is to start from a plane where the axis OX is $\boldsymbol{x_{k-1}}$ and the axis OY is $\boldsymbol{x_k}$:

$x_k = 0.93x_{k-1} + 5$

We take the graph $\boldsymbol{y = x}$ as an assistant.

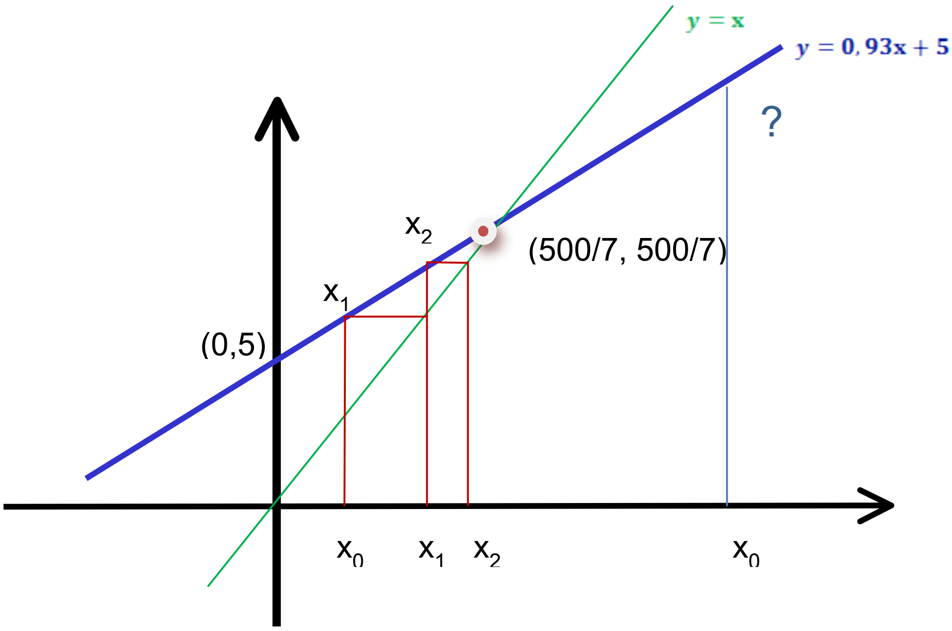


Figure 5. Graphic visualization

In general, how does the dynamical system behave over time (regardless of the starting point)?
The system has an attractive equilibrium point ***Pe (500/ 7,500 / 7)*** at the intersection of the two graphs ***y = 0.93x + 5*** and ***y = x.***

4. Modeling and formalization: Dynamical Systems of one-dimensional linear applications

Let ***L: R → R*** be a linear application, that is, ***L (x) = ax*** with ***a* ∈ *R***

4.1. Case 1: *a <1.*

p=1. The first option to investigate the ***dynamical system generated*** by this application would be to take a starting point, for example, 1, and calculate its orbit (projections on the axis OX of the points obtained).

$1, a^2, a^3, a^4, a^5, \ldots$

The orbit converges to point 0.

p≠1. For any other starting point the result stays the same because the orbit of the generic point ***p*** is

$p, a\,p, a^2\,p, a^3\,p, a^4\,p, a^5\,p, \ldots\ldots$

The Dynamical Analysis that follows this finding is:

a) If |a| < 1, all orbits converge to point 0

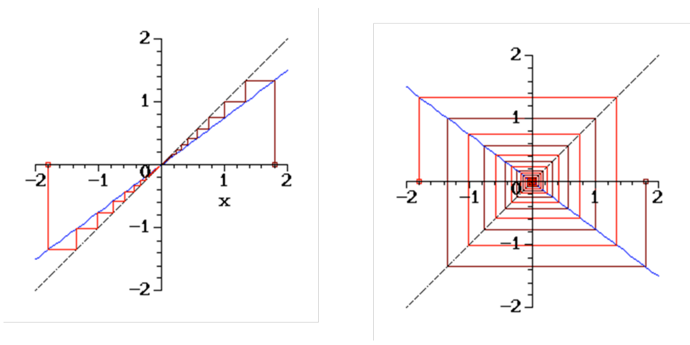


Figure 6. Dynamical Analysis-1

b) If |a| > 1, any orbit other than point 0 diverges in modulo to infinity

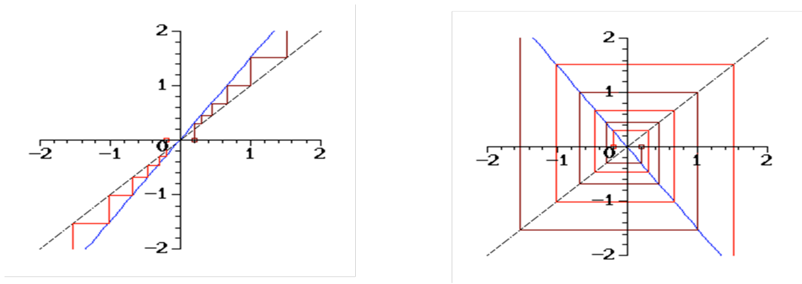


Figure 7. Dynamical Analysis-2

c) If a = 1, all points are invariant

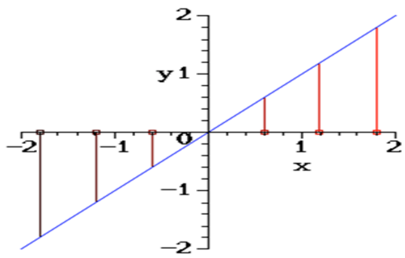


Figure 8. Dynamical Analysis-3

d) If a = -1, all points except 0 are of the period 2

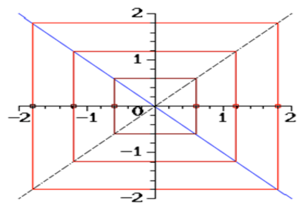


Figure 9. Dynamical Analysis-4

5. Ideas concerning other Dynamic Systems

Studying linear dynamics, analogous studies can be done with multiple other dynamic systems.

5.1 The quadratic family
The quadratic family is constructed by the applications

$f(x) = x^2 + p,\ p \in R$

5.2. The logistic family
The logistic family generates from the applications: $\boldsymbol{f_c: [0,1] \to [0,1]}$ of the form

$f(x) = cx\,(1\text{-}x),\ c \in R$

5.2.1. Examples: Logistics Family Graphics

Given the family

$\boldsymbol{f_c: [0,1] \to [0,1]} \,/\, f_c\,(x) = cx(1\text{-}x),\ c \in R$

we can see the behavior of f in differing iterations.

Example 1: c=0.75

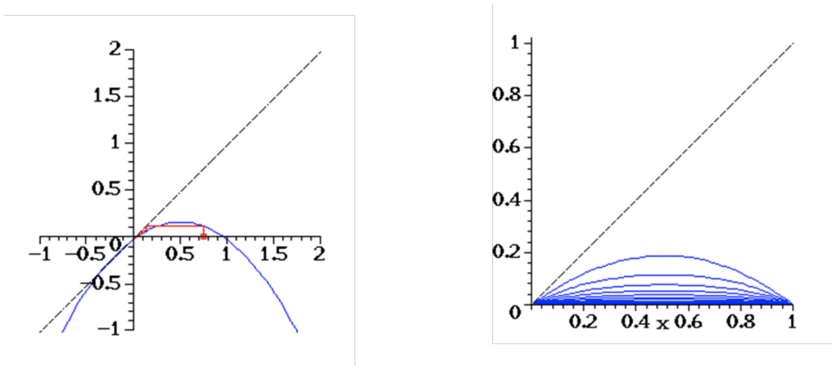


Figure 10. The interval [0,1] is invariant

Example 2: c=1

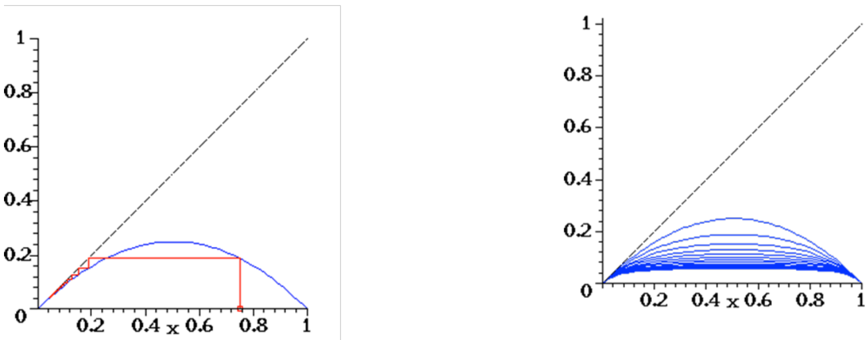


Figure 11. The point **0** has become indifferent although it still attracts the orbits of all points of (0,1). Point 1 is still fixed

Example 3: c=1.5

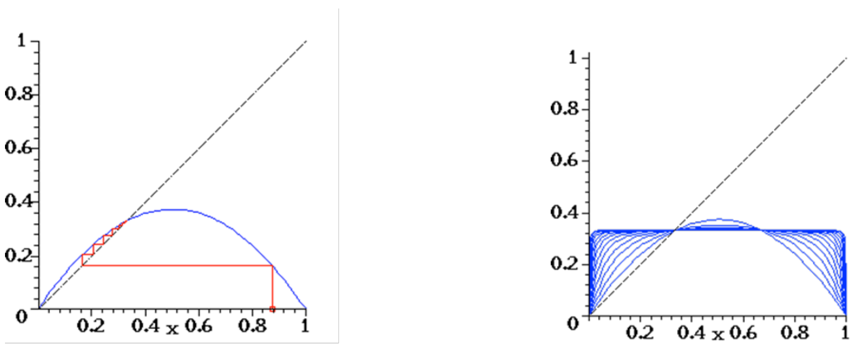


Figure 12. The orbit of any point of (0,1) tends to 1/3.

Example 4: c=2

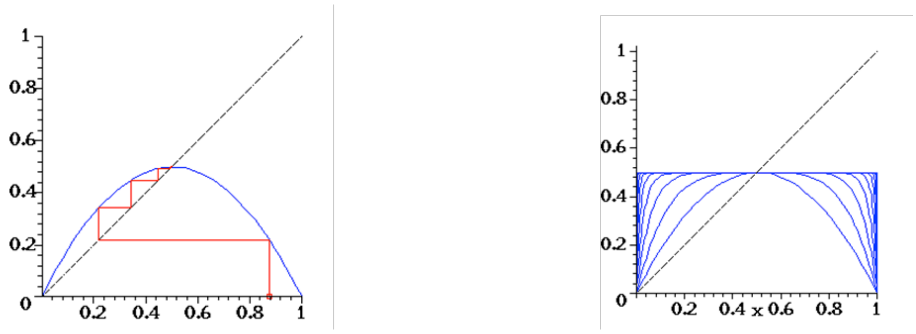


Figure 13. The orbit of any point of (0,1) tends to 0.5 which is now super attractive.

Example 5: c=2.5

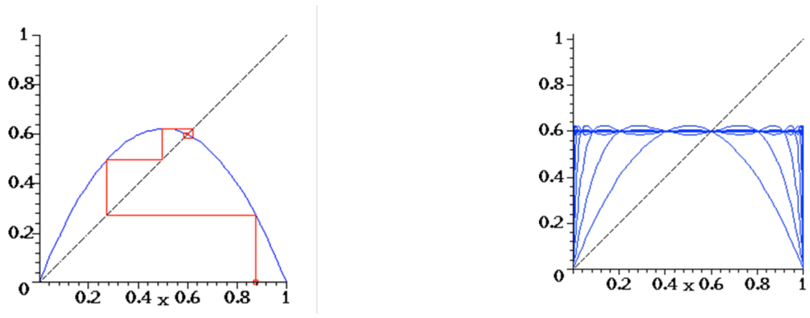


Figure 14. The orbit of any point of (0,1) tends to 0.6 which is an attractive point.

NOTE. The fundamental idea of this experience with secondary school students is to introduce dynamic systems. This is done by encouraging research on new concepts through mathematical models that are used to solve real-life problems.
In addition, in order to fulfil the task sufficiently, our students need to deepen their concepts of:
A) Fixed point
B) Eventually fixed point
C) Attractive point
D) Repulsive spots
E) Indefinite points

6. Conclusion

In this paper, I drew attention to certain mathematical models (as well as other abstract systems) that are defined as dynamic systems**.** They are applied to explain various phenomena of real-life situations. Furthermore, it showed that they are, possibly contrary to common beliefs, cognitively available to our secondary school students.

With the help of the presented case studies, I aim to provide answers to questions that are raised by the teachers of Mathematics. In general, teaching and learning science should include activities that enable the individual to construct (Blanch et al., 2004; Izquierdo et al., 1999) their way of feeling, thinking, speaking, and acting about the world around us by choosing those scientific models as one of the possible referential points. This is especially true regarding the teaching of mathematics. By applying these models, the path of the student should be guided by the path of research (Bonil et al., 2010). Furthermore, these models should strengthen the students' ability to contrast their self-generated information to the pre-existing information. Working on DSs showed to be one way to exercise the students' imagination or to understand, for example, certain approaches in ecology. A critical question that could be raised by students is *Do mathematical models provide answers to certain influences on the fundamental conditions of certain species?*

As a final conclusion, I would like to end with the following argument, quoted from the Real Decreto 1513/2006 (Ministerio de Educación y Cincia):
"... *The real possibility of using mathematical activity in contexts as varied as possible.*

Therefore, their development in compulsory education will be achieved to the extent that mathematical knowledge is applied spontaneously to a wide variety of situations, coming from other fields of knowledge and everyday life ... ".
"*... It is necessary to apply those skills and attitudes that allow mathematically reasoning, understanding mathematical argumentation and expressing and communicating in mathematical language, using the appropriate support tools, and integrating mathematical knowledge with other types of knowledge to give a better answer to the situations of life of different levels of complexity ...* "

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