REPRESENTING THE MATHEMATICAL KNOWLEDGE INSIDE THE COMPUTER SPACE

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I. INTRODUCTION

One of the main causes of the failure in the courses of Mathematics lies in a deficient previous formation. We may add other factors exclusive of the mathematical courses, such as the difficulty for arithmetic calculations, a lack of ability in the manipulation of symbolic expressions or an important absence of spatial visualisation.

These causes are deeper than the ones that can be detected in an ordinary examination, which is usually limited to a punctual comparison between the students knowledge and the academic topics and objectives. So, it is not possible to characterise efficiently the importance and incidence of the factors that provoke the deficiencies in the learning of Mathematics.

II. EVAM

EVAM (Mathematical Acquired Knowledge Evaluation) is an Expert System designed to detect lack of knowledge and mathematical abilities deeper than the ones detected by means of an ordinary written examination.

Traditionally, it has been considered that the knowledge of mathematics is reflected in a capacity to resolve standard problems as, for example, derivatives, systems of equations or matrix calculations. So, the examinations destined to evaluate the students' knowledge only detect the success or failure in the resolution of these kinds of problems. From this point of view, the non-resolution of these problems is immediately associated with a lack of mathematical knowledge.

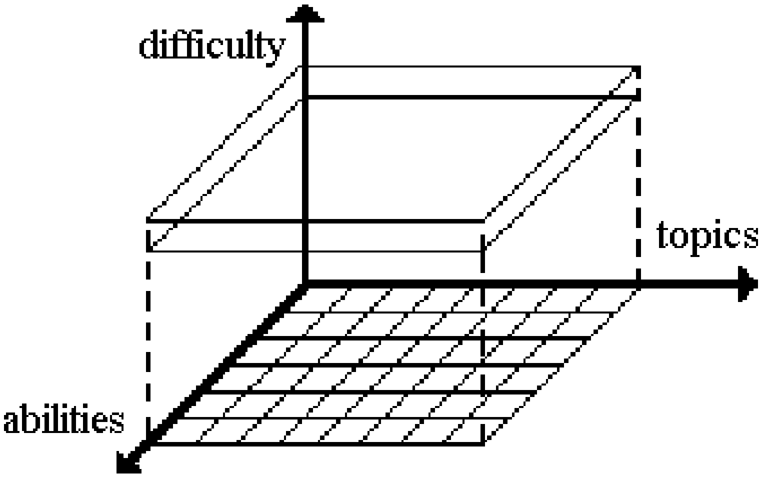
But, errors in the resolution of mathematical problems can also be caused by deficiencies in or lack of mathematical abilities, like the spatial comprehension, the capacity to develop symbolic expressions or the execution of arithmetic operations.

EVAM detects, in an automatic way, the factors that probably are responsible for the students' failure or success. We consider very useful the help of a tool to evaluate simultaneously the capacities of the students to use their knowledge and abilities in the resolution of problems.

The engine of the expert system selects interactively a battery of questions from a mathematical database through an intuitive and easy to use graphic interface. Each question is connected logically with a part of a three-dimensional space containing the mathematical model. Depending on the answer given by the user, the image representing the mathematical knowledge changes and conditions the next questions.

III. THE MATHEMATICAL KNOWLEDGE

The mathematical knowledge is represented inside the computer as a three-dimensional space, with the X-axis indicating a classification of mathematics topics, the Y-axis indicating mathematics abilities and the Z-axis the level of difficulty. Once the test is finished the final image resulting from the evaluation process can be easily interpreted in a visual intuitive way, or automatically analysed and reported by the Expert System.



The interpretation of the mathematical knowledge is represented starting from a two dimensional space, defined as the domain conjunction of the topics and mathematical abilities intersected into a reference plane.

Secondly, this plane is extended over a new dimension, which represents the different levels of difficulty in the use of the knowledge and abilities solving mathematical problems.

Then we have a three-dimensional space, and can manage the concept of mathematical knowledge we need to use. It serves as a reference for the Expert System to indicate which aspects of the mathematical knowledge (topics / abilities / difficulty) must be evaluated.

The execution of the program by the student assigns values for every one of the variables represented in the three dimensional space, corresponding with the state of the student's mathematical knowledge.

IV. TOPICS AND ABILITIES

The topics of the questions have been chosen between the common ones of the most representative technical studies in Spanish Universities. With the concept abilities, we are considering a series of concepts, capacities and skills necessary to solve a mathematical task. We have divided these into six different divisions:

- Arithmetic calculation
- Symbolic calculation
- Mathematical basic concepts
- Theoretical knowledge of theorems and formulae
- Spatial comprehension
- Mathematical abstraction

V. AN EXAMPLE OF TOPICS AND ABILITIES

Examples of different mathematics topics are very easy to find; trigonometry, series, continuity, vectors, interpolating polynomials, derivatives, integration etc. On the other hand, examples on how the mathematical abilities affect the resolution of problems are not as simple to find, as they are usually not considered by teachers.

Let us take integration as an example of a mathematics topic. We will show six different problems on the same topic, integration, featuring each of the six abilities enumerated before.

Problem 1. *Calculate the value of the definite integral* $\int_{1/2}^{1} 4x \; dx$

The integral resolution itself is supposed to be without any difficulty: $2x^2\Big|_{x=1} - 2x^2\Big|_{x=1/2} = 3/2$. If a different solution appears, it is almost sure that the problem lies in a wrong *arithmetic calculation* rather than in the integral itself.

Problem 2. *Which change of variable simplifies the resolution of* $\int \sqrt{r^2 - x^2} \; dx$ *?*

The problem does not need any especial analytic knowledge, just a medium ability to realise *symbolic calculation*.

The change $\begin{cases} \mathrm{x} = \mathrm{r}\sin(\mathrm{t}) \\ \mathrm{dx} = \mathrm{r}\cos(\mathrm{t})\,\mathrm{dt} \end{cases}$ simplifies the resolution of the integral converting it into $\int \sqrt{\mathrm{r}^2 - \mathrm{r}^2\sin^2(\mathrm{t})}\;\mathrm{r}\cos(\mathrm{t})\,\mathrm{dt} \;=\; \mathrm{r}\int \sqrt{1-\sin^2(\mathrm{t})}\;\mathrm{r}\cos(\mathrm{t})\,\mathrm{dt} \;=\; \mathrm{r}^2\int \cos^2(\mathrm{t})\,\mathrm{t}\,\mathrm{dt}$, which can be easily solved.

Problem 3. *Solve* $\int \dfrac{6x^2 + 2x + 3}{2x^3 + x^2 + 3x} \; dx$

This integral is immediate if the student possesses the *mathematics basic concepts* of logarithm properties. The numerator is the derivative of the denominator, so the solution is $L\left(2x^3 + x^2 + 3x\right) + C$.

Problem 4. *Which value* $e \in [1,2]$ *satisfies the Medium Value Theorem for* $f(x) = 2x$ *?*

The calculation is extremely simple, but the *theoretical knowledge of theorems and formulae* is absolutely necessary to expound the problem: $\int_1^2 2x \; dx = (2-1)\,f(e)$, that immediately leads to the solution $e = 3/2$.

Problem 5. *Propose the integral that calculates the volume of the part of the cylinder* $x^2 + y^2 = 2x$ *limited by* $z = 0$ *and* $z = x^2 + y^2$.

Spatial comprehension is required to "see" the problem in space, then the integral is simple to define: $V = \int_0^2 \int_0^{\sqrt{2x-x^2}} \left(x^2 + y^2\right) dy \; dx$

Problem 6. *Knowing that the revolution surface generated by a function rotating around the OX axis is* $S = 2p\int_a^b f(x)\sqrt{1 + [f'(x)]^2} \; dx$. *What is the integral expression that determines the area of a cone of radius 1 and height 2 ?*

No theoretical knowledge or symbolic and arithmetic calculations are required. It is only necessary to use *mathematical abstraction* to determine which function $f(x)$ and interval $[a, b]$ generate the cone. Obviously, the cone is created by the line $f(x) = 2x$ in the interval $[0,2]$, so the solution is $S = 2p\int_0^2 2x\sqrt{5} \; dx$

VI. THE KNOWLEDGE BASE

The knowledge base represents the mathematical knowledge necessary to develop the application. From this base, the software will obtain the questions used in the evaluation of the students. To avoid a static execution, a set of inference rules are defined, and determine which is the fittest question in every moment, attending a series of necessities that are modified as the system receives information.

The system knowledge base contains the information and questions necessary for the student evaluation. Each question contains information relating to:

- Topics the question is referred to (one question may imply the knowledge of several mathematics topics).
- Abilities necessary for its resolution.
- Difficulty level.
- Possible answers, considering not only the correct one but also the wrong ones, evaluating the magnitude of the errors.
- Resolution time, depending on the difficulty level.

VII. THE GRAPHIC INTERFACE

It has been verified that many students make errors induced by the use of the mathematical notation. To avoid this problem we have decided to use a friendly graphic interface.

For example, for the following expressions:

(a) **`2+3*x`** (b) **`2+3x`**

we have detected that frequently students interpret wrongly the precedence of arithmetic operators. With a sample value x=5, in expression (a), many times the answer appears as 25 (instead of the correct one, 17). Using the representation (b), this problem doesn't appear.

In the same way, a simple integral, such as:

$$\int \frac{9x-2}{(3x-1)(x^2-2x)}dx$$

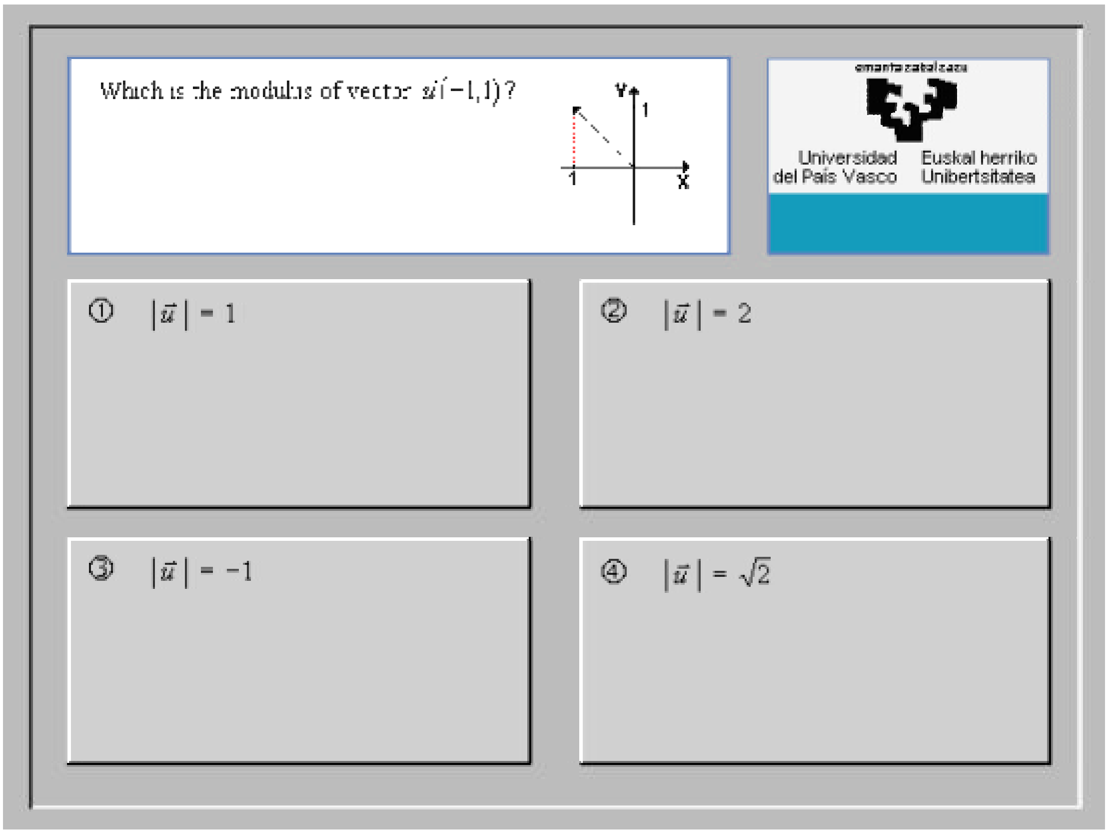
can be difficult to understand if it is presented under the following format:

∫ **`[(9*x-2)/((3*x-1)(x^2-2*x))]dx`**

and it is then not possible to determine if the origin of the error lies in a mathematical misconception or in the difficulty derived from the notation.

The graphic presentation of each question consists of a screen divided in the following way:

- An upper window containing the question. This window can contain graphics and formulae to clarify the question.
- Four options to select the answer, contained in windows that can be selected by means of the keyboard arrow keys or the mouse pointer.
- A small chronometer indicating the time left to answer.



VIII. OPERATION AND EXECUTION

EVAM is written in C++ language with the following program modules:

a) **Necessities establishment**. The system initial application creates a sheet placing in the plane abilities – topics which regions are considered more interesting to start the test. The previous studies of the student should be taken into account to select the adequate level of difficulty and the test finalisation condition.

b) **Initial test**. The system selects the set of questions that better fits the necessities sheet and the selected difficulty level. A first approximation of the student's knowledge is obtained.

c) **Store and evaluate answer**. All the answers collected by the system are stored, and used to modify interactively the necessities sheet and the difficulty level. This task is declared as a set of inference rules.

d) **Select next question**. With the updated necessities sheet, EVAM searches the optimum next question from the database. The search uses Boolean operations to check the maximum coincidence between the necessities sheet and knowledge that represent the question. When one topic is sufficiently covered, the necessities change giving priority to other uncovered topics.

e) **Represent results**. With the information obtained by the process, the system calculates several coefficients extracting information about the student's knowledge and abilities. Finally, an individual or group report is elaborated.

IX. BIBLIOGRAPHY

- Barroso C. 1994. *ENS-AI: un sistema experto para la enseñanza*. Teoría de la Educación, Vol 6, pp. 237-251.
- Sleeman D. 1984. *An attempt to understand students' understanding of basic Algebra*. Cognitive Science, 8 pp. 387-412.

X. ACKNOWLEDGEMENT

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