# ELEMENTARY GEOMETRICAL MODELLING OF SUN SHADOWS IN DIFFERENT CULTURES

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**Abstract**

This contribution presents some reflections about teaching experiments concerning elementary geometric modelling of the sun shadow phenomenon. These experiments were performed at Grade IV, VI and VII level in different cultural environments: more than one hundred classes in towns throughout Italy, two classes in Csenyète (a Romany village in Hungary), eleven classes in Sabadell (Spain), and two classes in Asmara (Eritrea) were involved. All the Italian and Sabadell classes show rather homogeneous behaviours and the approach to a "geometric manner of thinking about sun shadows" seems to represent, for most students, a natural evolution of their knowledge about the sun shadow phenomenon, although this evolution is not at all linear and does not lead to definitive results (cf Boero et al, 1995). On the contrary, in the case of the Csényete and Asmara classes strong resistance was encountered. The aim of this contribution is to analyse the differences that we found and to try to interpret them in terms of cultural differences related to different cultural environments. I will also discuss the problem of the consequences, and legitimacy, of imposing a "geometric manner of thinking about sun shadows" on students who are unwilling to adopt and use it.

## 1. INITIAL CONCEPTIONS

Since 1976 more than 3000 protocols have been collected concerning students' initial conceptions (i.e. without any preliminary classroom activity) about the phenomenon of sun shadows. The students involved were largely III, IV and VI-graders belonging to classes in Italy where the Genoa Group Projects for primary and lower secondary school were tested; in Sabadell (Catalonia, Spain) teachers worked with VII-grade classes, performing some activities inspired by the Genoa Group Project for lower secondary school . More recently, VI-graders in Csenyete (a Romany village in Hungary) were involved, as were VII-graders at the Asmara Italian School (attended largely by middle class Eritrean children).

After some very brief information from the teachers *("In the next few months we will analyse the sun shadow phenomenon, so it is important to ascertain what you already know about it"),* an initial task was proposed:

A) *"Surely, you have noticed that on a sunny day your body casts a shadow on the ground. Is your shadow longer at 9 a.m. or at noon? Can you explain why?";* or

B) *"Write* (or, in other classes, *tell*) *everything you know about sun shadows. You can make drawings in order to explain what you think."*

In some cases, task A) followed task B).

For each task we collected: individual texts and drawings; recorded individual interviews and drawings; and recorded classroom discussions.

The results of the analyses of students' behaviours can be summarised as follows:

TASK DEPENDENCE: in every grade and in every environment, students' behaviours depended on the chosen task. Task A) more frequently than B) (on average, 25% against 14%) led to proto-geometric answers: *"The shadow is longer at 9 a.m. because the sun strikes laterally"; "The shadow is longer at 9 a.m. because the sun is low".* We call these answers "proto-geometric" because they bear an intuition of a geometric relationship between the position of the sun and the length of the shadow, something that most children are not able to represent in precise, geometrical terms. Indeed, in the case of proto-geometric answers we found a majority of drawings resembling this one:

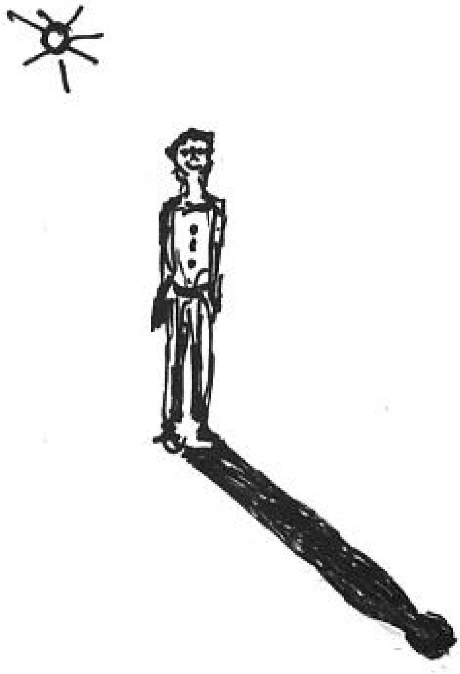
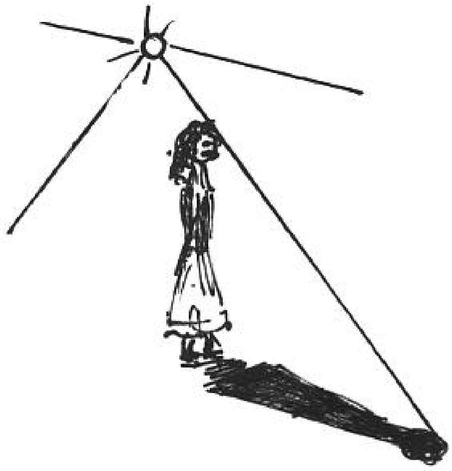


FIGURE 1



and very few drawings like this one: FIGURE 2

In most drawings of both types shadows bear no "inner" detail of body (like eyes), of dresses, etc. (see Fig. 3 and Fig. 4).

Concerning the other answers, in the case of Task A we found an average of 39% of answers of this type: *"The shadow is longer at noon because the sun is stronger"*. This idea is scarcely represented (less than 10% on average) in answers to Task B. On the contrary, the majority of answers to Task B concern the idea of a kind of "soul" (or "double") that is revealed by the sun (22%), or the idea of the shadow as a kind of "carpet" that accompanies the person (23%), or the idea of a "reflection" of the body provoked by the sun (24%).

The drawings are quite coherent with those ideas: the sun is usually positioned on the same side of the person with respect to his/her shadow; and in the case of the "soul" (or "double") conception, as well as in the case of the "reflection" conception, shadows bear some "inner" details of the body (e.g. the eyes-see Fig. 3)

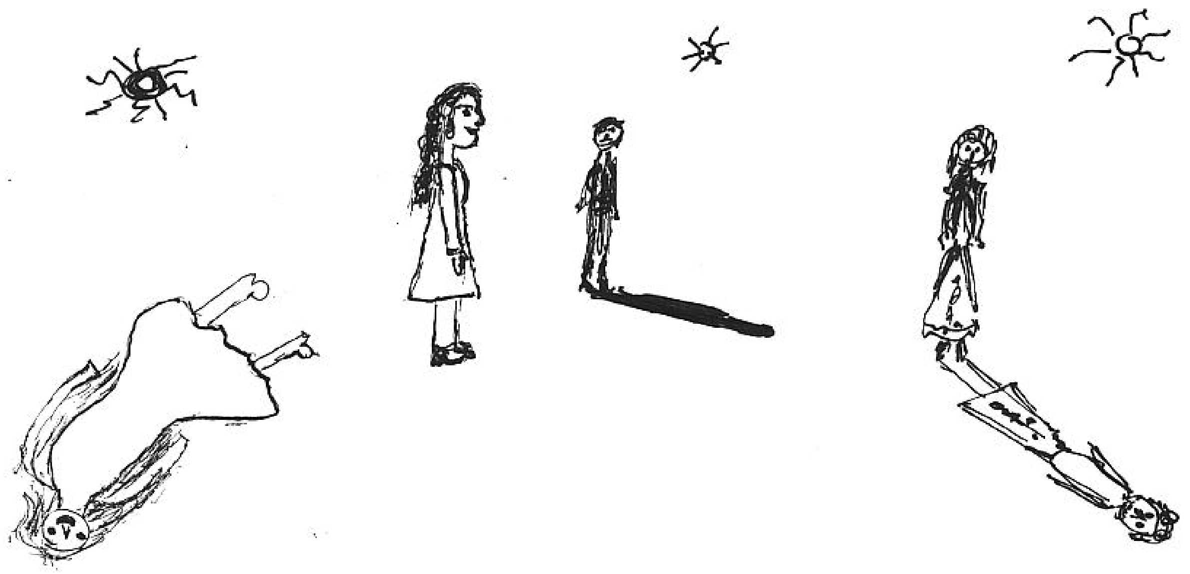


FIGURE 3: drawings related to "soul", "carpet" and "reflection" conceptions

Obviously, causality is widely expressed (more than 75%) in answers to Task A (both with proto-geometric and non-geometric conceptions). Causality is spontaneously expressed in less than 20% of answers to Task B (again, there is no difference between proto-geometric and non-geometric conceptions).

WRITTEN AND ORAL ANSWERS: examining results from different classes at the same school, we found no substantial difference between the conceptions emerging from individual oral or written answers to Tasks A) and B). However, the oral answers are usually richer (both with IV-graders and with VII-graders).

VERBAL ANSWERS AND DRAWINGS: with the exception of the village of Csenyete (where children prefer to produce drawings and half of them have serious difficulty in writing), students preferred to answer verbally both in written answers and in oral interviews; in most cases, drawings are only produced on the teacher's request. In general drawings are rather coherent with verbal answers.

DISCUSSIONS: these reveal a characteristic phenomenon - discussions took different directions depending on the ideas expressed by some class leaders: sometimes the discussion stuck to a dominant position, sometimes they showed a wide variety of positions.

AGE - DEPENDENCE: in the Italian classes, we were able to collect many protocols from IV and VI-graders; differences according to the students' age are not very relevant (or, more precisely, less relevant that one would imagine) for both tasks. In particular, the prevailing answer to Task A *("the sun shadow is longer at noon, because the sun is stronger")* is produced by 41% of IV graders (out of 630 protocols) and by 38% of VI graders (out of 2350 protocols). A proto-geometric conception (see above) emerges in 22% of protocols produced by IV-graders, and in 26% of protocols of VI-graders.

GENDER DEPENDENCE: on average, no relevant difference. But see later!

ENVIRONMENT DEPENDENCE: similar behaviours were observed between Catalan VII-graders and Italian VI-graders for both Tasks A and B. Remarkable differences were observed between Catalan and Italian students on the one hand, and Csenyete and Asmara students on the other.

In the case of the Asmara students, both individual texts and recorded discussions show clear differences with Catalan and Italian students that can be summarised as follows:

- non-geometric conceptions are partly opposite. Considering Task A, in the case of Italian and Catalan classes the prevailing conception is *"the sun shadow is longer at noon, because the sun is stronger"*; in the case of the Asmara classes, the prevailing conception is *"the sun shadow is shorter at noon, because the sun is brighter and beats darkness";*

- non - geometric conceptions are more stable in the Asmara classes across Tasks A and B (and in other tasks, too - see Section 4.); the prevailing conception of the Eritrean children in Task A reported above is also one of the two prevailing conceptions in Task B - the other is the "soul" or "double" conception. This conception is also frequent in Task A *("the sun shadow is shorter at noon, because the sun reduces our dark image");*
- the Asmara students' non-geometric conceptions fit the sun shadow phenomenon better (see the above-reported examples);
- in some cases, the Asmara students' conceptions are explicitly integrated into a fairly developed system of thinking about nature, the Sun, etc.: *"the sun shadow is shorter at noon, because the sun is brighter and beats darkness":* this child's idea is strongly related to other ideas of the sun as the source of life, darkness as the manifestation of evil, etc. (similar phenomena are quite rare in the Italian and Catalan classes, and mostly connected to general difficulties in approaching sciences).

In the case of Csenyete students, the weakness in the mastery of written and oral language makes the comparison rather hazardous. However, drawings express a proto-geometric conception in one case out of 37 (two classes); non-geometric drawings are gender-dependent and seem to be inspired by a "soul" (or "double") idea (see Fig. 4).

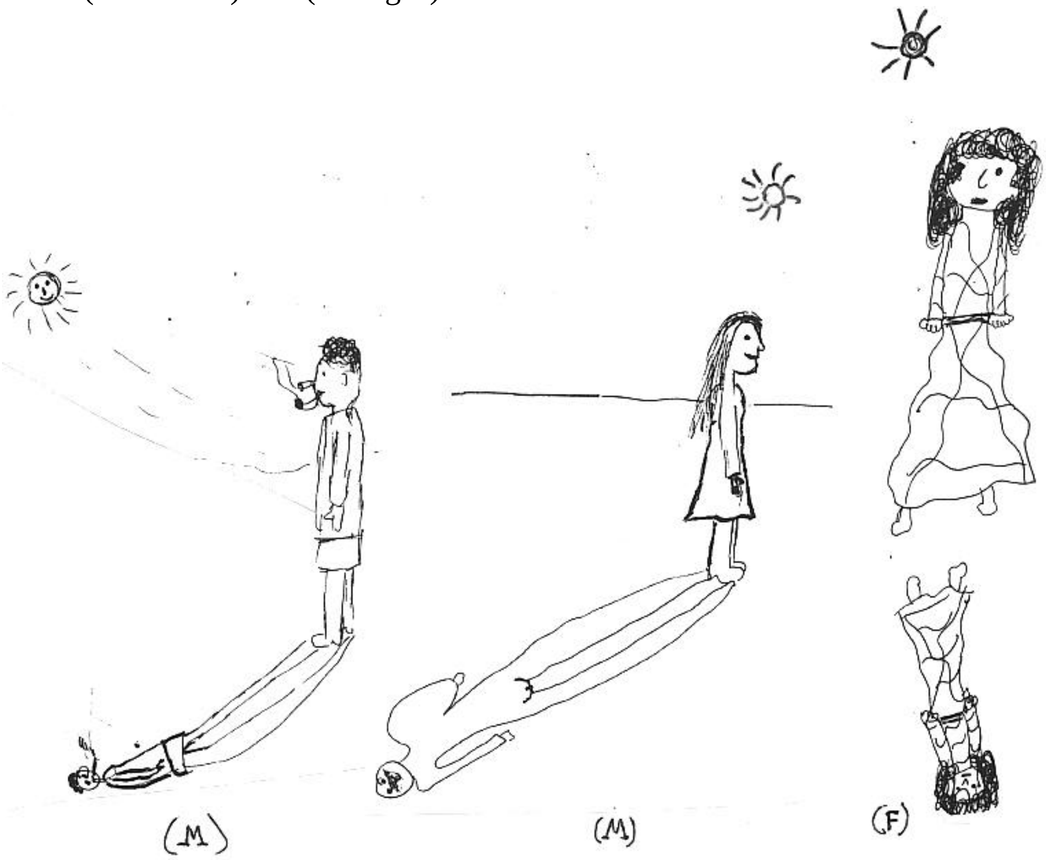



FIGURE 4

DISCUSSION: with the exception of the Asmara (and, perhaps, Csenyete) children, non-geometric as well as proto-geometric conceptions appear as perception-inspired adaptations (Piaget, 1926; 1936) to a specific task. Although perception-inspired, some non-geometric conceptions scarcely fit the sun shadow phenomenon, in the sense that it is easy to find specific observations and experiments which invalidate students' hypotheses. This is the case with the answer *"the sun shadow is longer at noon, because the sun is stronger"*.

It would be interesting to understand better why this conception is so frequent in the Italian and Catalan classes, and rare in the Eritrean classes. It might depend on the fact that, compared with Eritrea, Italy and Catalonia experience fewer sunny days, and shadows are especially "evident" at noon. On the contrary, sun shadows in Eritrea are very short at noon (the noonday sun there is exactly over the head twice a year). Some relevance may also lie in the fact that some Eritrean children explicitly relate the sun shadow phenomenon to a system of knowledge about the sun and other entities, which in turn seems to be connected to environmental culture. This fact might also explain the greater stability of the Eritrean students' conceptions across the tasks, while the task-dependence of the Catalan and Italian students' conceptions might depend on their "isolation" from the children's system of knowledge and environmental culture.

## 2. TOWARDS GEOMETRICAL MODELLING

In most cases, the first tasks were followed by a relatively long activity (some 10/12 hours over 3-4 weeks) conducted in the experimental classes: students were invited to observe the sun shadow phenomenon in the schoolyard and draw and verbally report their observations. Drawings and verbal reports were compared and discussed in the classroom, then individual and/or collective syntheses were produced and compared with what really happened outside. Two or three cycles of this kind were usually needed in order to reach exhaustive and satisfactory descriptions of the phenomenon.

We have observed that initially, even after the observation of sun shadows in the schoolyard, no precise geometric relationship was established between the position of the sun, the top of the object casting the shadow and the extremity of the cast shadow: even most of seventh graders who produced a "good" answer to Task A *("The shadow is shorter at noon because the sun is high")* do not go beyond a graphical representation where the only link between the height of the sun and the length of shadows is through a qualitative geometric relationship (see Fig. 1). In addition, the fact that the geometrical model (cfr. Fig.2) is not an immediate consequence of observation, but seems a rather sophisticated cultural construction, finds a counterpart in the history of ancient mathematics (cf. Serres, 1993). For these reasons, a relatively long sequence of activities (10-15 hours) was planned in order to approach the geometrical model. Gradually students discovered the shadow space; they learned to indicate its borders with their hands and draw it in this way:

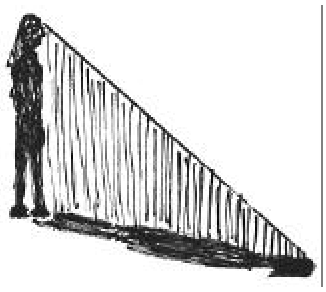


FIGURE 5

They also learned to point at the sun with their arms and represent its movement in the sky. As a consequence of both tracing the border of their own shadows and pointing at the sun with their arms, the introduction of the geometrical model of sun shadows was a reasonable and rather natural outcome of this activity (if some students anticipated it, the teacher exploited their drawings; in the other cases, the teacher themselves proposed the geometrical model - see Fig. 6). For more details, see Scali (1997).

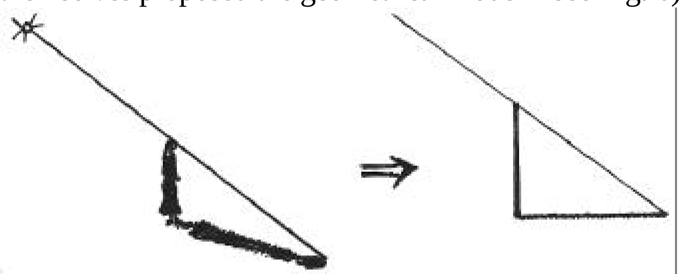


FIGURE 6

## 3. THE GEOMETRICAL MODEL AS A CARRIER OF MANNERS OF THINKING OF THE SUN SHADOW PHENOMENON

However introduced, the geometrical model of shadows changed the students' manner of thinking of the sun shadow phenomenon. This change was revealed through students' verbal productions. A careful analysis was performed on the Italian students' protocols produced during the observation-and-report activities (see Section 2). Before the availability of that sign, most (80%) of the students' oral and written reports about their observations contained no trace of causality or conditionality relationships between the position of sun and the length of shadows. The most frequent comments were along the following lines:

- *"In the morning, the sun is low and sun shadows are long";*
- *"At noon, the sun is high and sun shadows are short";*
- *"During the morning, the sun goes up and sun shadows become shorter and shorter".*

The introduction of the geometrical model of sun shadows strongly affected the emergence of causal or conditional relationships in students' verbal descriptions: the percentage of spontaneous descriptions containing explicit traces of causal relationships (*"in the morning, the shadow is short because the sun is low"*) or conditional ones *("if the sun is low, the shadow is long")* increased from about 20% to more than 58%!

How are we to interpret the available data? The most reasonable interpretation seems to be that of a change of paradigm in the manner of looking at the phenomenon (see Section 5).

## 4. A CRUCIAL TASK: WHAT HAPPENS WITH SUN SHADOWS IF WE MOVE TO ANOTHER PLACE?

Once the sign of shadows had been introduced in the classroom and students were familiar with it, the following task was proposed:

(motivated by reasons such as the following: *"Our schoolyard is no longer available"... "In this season, trees cover the sun"...* etc.)

"*We need to move to another place* (another schoolyard, the terrace on the school roof , etc.) *in order to record shadows during the day. If we measured the length of the shadow of the same object at noon on the same day, would we get the same length*?".

In the Italian and Catalan classes, most students (more than 75% in each class; average value: 82% out of 237 students from twelve classes in Italy; similar results in Catalonia) use the geometrical model of shadows as the crucial tool for producing their hypotheses (see Scali, 1994). The hypotheses produced differ: half the students draw one sun with two rays producing the shadows of the two objects, and in this way get different lengths.

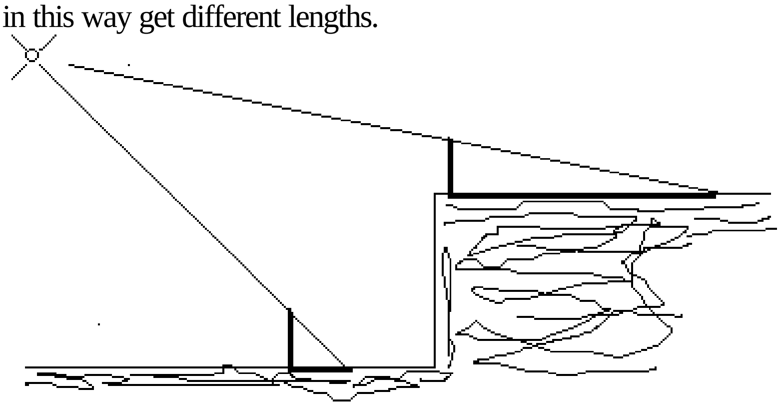


FIGURE 7

Other students remember that sunrays are parallel, draw parallel rays and get equal lengths; other students try to co-ordinate knowledge about parallel rays with rays originating from the sun, and draw one big sun with two parallel rays, or two suns... or stick to the contradiction and produce no answer. All the produced hypotheses have a common feature: they are built by exploiting the geometrical model of shadows. The other students produce hypotheses which are difficult to interpret, or are far from the content of the task: in some cases, only a few drawings of possible shadows of the two objects are produced, in other cases only some texts, usually revealing misunderstandings of the task (for instance: *"We must go to another place, and we bring our stick with us; then we put the stick in a vertical position and follow the border of the shadow on the ground with a piece of chalk, then we measure it"*).

A common feature of students' behaviours in the two Asmara classes is that the majority of students reversed the above-reported relationship between the geometrical model and the process of production of hypotheses: for 62% of the students, production of hypotheses depended on their conceptions about sun shadows and general principles; then half of these again featured a graphical representation of the hypothesised situation. A typical text is as follows: *"Our body remains the same, and the sun is the same, so also the shadow must remain the same, because it is our image and the light of the sun is equally strong. I can draw it [...] I see that sunrays are parallel, as we have noticed in our experiments".* When graphical representations were produced, in some cases they led to confirmation of the knowledge about parallelism of sunrays; in other cases students perceived a contradiction between their hypothesis and the intention of drawing two sunrays coming from the same sun. But the fundamental point to be stressed is that, for all these students, graphical representations were a consequence of a hypothesis produced through a process based on non-geometrical conceptions and general principles.

DISCUSSION: the available data represent a challenge for interpretation of the main difference emerging from the Italian and Catalan classes on the one hand, and the Asmara classes on the other: the reversed relationship between production of hypotheses and intervention of the geometrical model.

One interpretation might be based on the unavoidable differences between the management of the didactical activities; indeed we know that it is impossible to achieve an identical treatment of the same didactical variables (Artigue, 1988). In particular, the level of mastery of the geometrical model might be different. But the fact that students' behaviours were similar in Italy, in Catalonia, and at Asmara as concerns the use of that model before (Italy and Catalonia) or after (Asmara) the production of the hypotheses shows a comparable level of mastery of the geometrical model of shadows.

Another possible interpretation is that students' spontaneous conceptions of the phenomenon were different in the two situations and differently "adapted" (in Piagetian terms - see Piaget, 1936) to the phenomenon. Generally, the conceptions Italian and Catalan students brought with them did not stand up to elementary experimental testing. On the contrary, the Asmara students' conceptions fitted the phenomenon rather well. The discussions about these conceptions evolved differently in the classrooms: in the case of the Asmara classes, students could speak of *"our preferred manner of thinking, although you* (the teacher) *want us to reason geometrically";* the validity of the consequences of their conceptions as concerns the description of common facts was not put into question (according to the Asmara students' non-geometric conceptions, at noon sun shadows are shorter than in the early morning!). In the case of the Italian and Catalan classes, the prevailing non-geometrical conception *("sun shadows are longer at noon because the sun is stronger")* was rapidly perceived by the students as a "mistake". But even this interpretation does not explain everything. Why, when they are so conscious of the fact that the mathematics and science teacher naturally prefers a completely geometrical solution, do the Asmara students not submit to the didactical contract implicitly established with her and which is made so explicit in some students' comments?

A more general cultural problem might have emerged here, concerning the deep relationships between Asmara students' behaviours and their environmental cultures.

## 5. CAUSALITY IN GEOMETRIC AND NON-GEOMETRIC RELATIONSHIPS BETWEEN THE SUN AND SUN SHADOWS

Let us consider a typical non-geometric conception as expressed by an Asmara student: *"At noon, sun shadows are shorter because the sun is brighter and beats the darkness".* If we analyse this comment and others produced by the same student, we find an agent (the sun) whose brightness beats darkness and so reduces the length of shadows (conceived as a manifestation of darkness). The same student in other circumstances speaks about the fight between light and darkness, life and dead, etc. This student's comment expresses a system of knowledge (not an isolated conception); the causal agent is related to darkness, life, death, and so on. Causality is related to an intention, to a will.

The situation we have just considered is surely an extreme one - in other cases, there is no clear evidence for the existence of a strong system of interrelated conceptions in which causality takes the meaning of an agent's will. But some verbal hints suggest that other Asmara students brought an idea of "somebody" who provokes shadows (or, reciprocally, an idea of "animated" shadows). The following comments from one student are extremely interesting (they were collected during an interview with a small group of Asmara students): *"Approaching sunset, the shadow makes a big effort to reach the light of the sun"* (explain better) *"I mean that the shadow must become longer and longer in order to reach the light"* (try again, I do not understand) *"The light is not so strong, shadows are going to prevail, so they must make a big effort to develop"*.

The existence (in more or less explicit and conscious terms) of a system of agents capable of will, endeavour, etc. might explain why, in the case of the task involving shifted objects, such a large number of Asmara students produced a kind of reasoning where principles and conceptions prevailed: the sun/shadows system is controlled by an integrated system of agents; changing one variable (the place for observing shadows) is perceived as a minor change. This is confirmed by a considerable number of answers in which students simply state that *"The length remains the same, because the change of place is not important"*.

When the geometrical model of shadows is introduced, causality spreads in the description of the sun shadow phenomenon (see Section 3.). In this case, the necessity of short shadows whenever the sun is high emerges as a necessity carried by the sign of shadows: this sign encapsulates the cause/effect relationship. Modern sciences were developed on this fundamental principle: geometric signs (as well as differential equations and other mathematical objects) become the carriers of necessity and encapsulate the cause-effect relationships. The world of intentional agents (with their wills, their conflicts, etc.) is removed from scientific explanation; it may possibly remain in the background, as some Christian scientists explain well, but laws of nature are mathematical laws!

## 6. GENERAL DISCUSSION AND CONCLUSIONS

The opposition established at the end of Section 5 seems to lead to a clear conclusion from the educational point of view: some conceptions, some manners of reasoning are in complete contradiction with the needs of scientific education. Geometric conceptions must be acquired by students as a condition for entering mathematical modelling, one of the key points of the scientific view of natural phenomena.

I would like to challenge this easy and apparently necessary conclusion. Indeed, I think that we must ask ourselves a question: can the development of science (i.e. the production of new ideas) as well as the transmission (or, if we prefer, the reconstruction) of scientific knowledge be completely free from compromises with non-scientific reasoning, manners of viewing, etc.?

If we think that reasoning by analogy, reasoning by symmetry, aesthetics, etc. has an important role in innovative scientific work, as well as in teaching and learning, then we must consider that production (and reconstruction or transmission) of science must be protected from an over-evaluation of the shape of the scientific product (as it is presented in a scientific paper or in a graduate textbook). Following the analysis performed by Douek (1999) on the distinction between proving as a process and proof as a product, we must clearly distinguish between the product and the process leading to it. During the process, "scientific" methodologies (experimenting, measuring, applying mathematical models, etc.) mix with other kinds of considerations that are sometimes very far from official science: principles of "simplicity", aesthetic intuitions, metaphors, etc. become important ingredients. The Asmara students' behaviours are not so far from an authentic innovative work in science! (cf Feyerabend, 1975, particularly Chap. 4, 6 and 12). At the end of the process, the shape of current scientific communication prevails and non-scientific components of the process are removed ( they may be kept in informal presentations or in private communications as curious components of "heuristics" backing the innovation).

If we can agree that reasoning by principles and non-scientific conceptions must be protected as a potential resource in innovative intellectual work, the complexity of the teacher's tasks dramatically increases. Consequentially, new studies are needed to develop didactical engineerings which put before the class both scientific knowledge, mathematical models, etc and informal (possibly “non-scientific") forms of reasoning for explicit, equilibrated discussion.

Another, delicate question is related to the study reported in this paper. In the case of Asmara and Csenyète students (but also in the case of some slow - learners in Italian and Catalan classes) analyses show a *persistent* conflict between the geometrical representation of the phenomenon and the external representations produced by the students. These representations tend to *remain* the expression of an affective relationship with the represented object, culturally rooted in a wider system of conceptions. The *persistent* presence of the symbolic meanings of sunshadows, strongly involving the students' emotional life (cf the body shadows drawn by the Csenyète students in Fig. 4; and some comments by the Asmara students reported or alluded to in Sections 4 and 5) raises a delicate *legitimacy* problem.

Particularly, we may ask ourselves: Is it *legitimate* to "force" the assimilation of "scientific" culture (based on mathematical tools for describing, interpreting and forecasting phenomena and situations) when it is so far removed from the cultural experience of students' social environment and their personal conceptions ? If we consider this choice *legitimate* (as an enrichment of the way of considering reality necessary for surviving in our complex society), we must consider *how to make Asmara and Csenyète students aware of that choice and save their cultural roots* (particularly, their symbolic relationship with the phenomenon).

If we are to go further in this direction, we must take into account the fact that geometrical modelling of sunshadows does not only concern the *description* of the phenomenon, but also the *manners of thinking* about it (see Sections 3 and 5). This remark can help clarify the delicate anthropologic and cognitive issues involved in the Asmara and Csenyète experiences; it also suggests some directions for better analysis of the difficulties met by the Asmara and Csenyète students in the assimilation of the sunshadows geometric models (analysis of causality relationships in their everyday experience).

Finally, we would like to point out the fact that, as with much ethnomathematics research, our study may have interesting implications for the analysis and interpretation of phenomena that occur in "normal" Italian and Catalan classes with slow learners, particularly those students coming from "poor" regions (ones which are often defined also as "culturally backward"). Beyond the specificity of the Asmara and Csenyète students' culture, we might identify some universal aspects concerning the behaviour of subjects not sufficiently conditioned by the so-called Western culture's rationality models (cf Barton, 1996; Gerdes, 1996) when they tackle the sunshadows phenomenon and its geometric "rationalization".

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