”New math” for the 21st century

Gunnar Gjone:

Some time ago I was listenening to a presentation on the use of internet in education. Concentrating on the internet as a source of information, the presenter stated that it would be important if students could combine search criteria, by the laws of logic, to perform more efficient searches. Perhaps the proponents of the ”new math” had been too early in time for mathematics education? Over the years I had realized that many uses of computers build strongly on logic – forms of programming are obvious examples. Many uses of application programs such as word processors and spreadsheets would also be more efficient by the use of logic. Computers are ”logic machines”, so why not logic in basic mathematics education?

The ”new math” revisited

The ”new math” movement lasted in most countries from the late 1950s to the early 1970s. In many countries this reform was replaced with stressing the basic skills, i.e.arithmetic/ computation. Also in many countries these changes resulted in strong discussions and heated debates among mathematics educators. The ”new math” movement, however, was not one single reform. It had many different forms, but there was common elements. We can use concepts such as ”structure” and ”logic” to talk about these common elements, but the ”new math” movement was much more. Also it should be noted that many other reforms e.g. individualized math instruction used new math to gain momentum.

Some characteristics of the ”new math” movement

We will concentrate on two of the characteristics of the movement: the mathematical and the pedagogical. In the beginning the reform was mainly a mathematical reform, starting in the later years of upper secondary education. This is also seen by the initial activity of the OEEC/OECD conferences, e.g. in Royaumont in 1959 and the people who participated.

Very soon, however, the educators became interested in the ideas. The reform was in line with a pedagogical principle of long standing – concentrating on the basics, in the sense of ”unifyng ideas” in the subject. The unifying ideas and internal structures, used as a basis was the ideas and structures of the mathematician – sets and logic (general laws etc) to mention the most prominent features. Who else could say anything about unifying ideas than the mathematician? Not all mathematicians, however, did agree with this. Hans Freudenthal characterized this as ”the wrong perspective” for mathematics in school.

Into the beginning of the 1960s, I will argue that reform was in essence a learning theoretic reform. Weight was on ”understanding” more than on doing.
The movement lead to reforms in many countries. In some countries we find that the ”new math” reform presented some possible solutions to problems in connection with school reforms. In Norway compulsory education was extended from 7 to 9 years in the period from 1959 to 1974. What was obvious for most educators and politicians was that the mathematics taught at grades (7)8 – 9[1] before this reform was not suitable for all students at this level. The proponents of the "new math" could therefore put forward an alternative to the traditional mathematics.

The subject matter

For many mathematicians the basic mathematics are found in logic and the axioms of set theory. It was not too difficult to adapt elements of logic and set theory to elemetary teaching of mathematics. In logic one could use natural language to illustrate the basic principle, even if this at times was a little problematic, e.g. in the case of implication. Sets could be illustrated in the plane, as one earlier had introduced set theory at university level. The relationship between set theory and logic was also easy to illustrate.

[1] Before 1970 grades 8 and 9 were for a select group of students only.

Problems arose, however, when using the basic notions of set theory and logic to ”define” the more common objects of mathematics. Defining a circle as the set of points with a common distance from a fixed point, gave little intuition about the properties and construction of circles.

Another feature of the "new math" reform was the introduction of very general laws, e.g. commutative, associative and distributive laws. These were illustrated by some very simple arithmetical relations, e.g. that 2+3=3+2.

The ”new math” controversy

The reactions to the "new math" was also similar in many countries. One come to observe that students having had a ”modern” or ”new” curriculum could not compute as proficiently as students had performed before with a traditional curriculum/training.

This was one of the major criticisms of the "new math", but there were also others. It was argued that the mathematics education given at receiving institutions did not take into account the changed perspective of mathematics in the lower grades. Also criticism directed at other projects who had come under the "new math" umbrella – but had not the same foundations (e.g. individualized mathematics instructions) – lead to criticism against the reform.

I will not go into details about this controversy, which is well known to most people here. I will, however, remark that where the controversy had been hard, not much was done to save and learn from the experiences. It all came to an abrupt end. For an overview see for example Moon (1986).

The use of technology in education- a short subjective history

It was early realized that technology could play an important role in education. We also found pedagogical development that in many ways attached itself to the technology. Programmed instruction and so-called ”learning machines” were much in focus around 1960. The foremost theoretician was Burrhus F. Skinner in the USA, but this ”movement” attracted worldwide attention, and several projects were initiated in many countries. Mathematics was one of the favoured subjects for programmed instruction. The computer was not, at this stage, used intensively for programmed instruction. The reason was probably that hardware development was not suited for this kind of use.

In the 1960s we saw many projects for using technology, and optimism was expressed by several leading educators. However, the development was seen connected to large computers. When smaller computers entered the scene one could start thinking of computers for a school or group of schools. In the 1970s two developments took place that was especially important for mathematics education. In the beginning of the decade we witnessed the advent of the pocket calculator, and at the end of the decade – the personal computer.

The pocket calculator was an obvious tool for mathematics, performing routine tasks of arithmetic quickly and with great precision. Since the 1970s we have seen the (pocket) calculator develop into an advanced tool, concerning the tasks it can perform in mathematics. Near the end of the decade the computer became ”personal”. this lead to a renewed development of ”programmed instruction” although the word was not used, having developed a negative connotation by many educators. We also saw an incredible development in computers for education, both in hardware and software. The boundaries between calculators and computers have largely been erased. Leaving us now with a powerful tool to do mathematics. The tools are now indispensible in applied mathematics, and has also proved to be very important in pure mathematics research. I will argue that it has not found it’s place in schools, even though there are many promising developments.

We are still – in the late 1990s trying to solve the question: How should computers and calculators be integrated into mathematics education? Computers and calculators outperform the human being in computations, and it has been a continous debate for the last 20 years what consequences this should have for mathematics education, especially in arithmetic. We can list several problems: with the technology, resources, teacher education, influence from outside school , does it ”deliver”?

Even if the development has been incredible, I will argue that the basic functions of computers in education has remained the same over the last 40-50 years. The categories introduced by Robert Taylor around 1980: The computer as a *tutor, tool* or *tutee* (Taylor, 1980) still gives the basic functions of computer use in education.

Problems with the technology

It seems that in some countries the technology has entered mathematics education without much preparations. Calculators have been easy to use, and have been widely introduced in many countries. However, there has been no widely accepted theory, and even symbol manipulating calculators have been introduced in some countries without a foundation. I will here briefly report on an evaluation on the use of symbol manipulating calculators on Norwegian upper secondary mathematics exams. It was clear to me on examining the exam papers that a comparatively large group of students failed to see an obvious mathematical solution, but instead relied on the calculator to perform operations.

> Consider the following problem:
> Use $\sin(2x+x) = \sin 3x$ to show that $\sin(3x) = 3\sin x - 4\sin^3 x$

The ”standard” way of solving this problem, would be to start with $\sin(3x)$ and then to expand this by first using $\sin(2x+x) = \sin 3x$ and then the formula for sin to a sum. What many students did, however, was a very different approach. It is possible to ask a symbolic manipulating calculator if [$\sin(3x) = 3\sin x - 4\sin^3 x$]?, and then the calculator might respond with “TRUE”. This many students did. Another method used was to have the calculator draw the two graphs of the functions $\sin(3x)$ and $3\sin x - 4\sin^3 x$. By observing that the graphs were coinciding, they concluded that the functions were equal. I think these two “methods” will make most mathematicians somewhat uneasy. We can ask ourselves how the calculator arrives at “TRUE”. It can only be calculated at a finite number of points. With the graphs it is obvious that the resolution of the calculator window plays an important role.

Another example which I consider to be problematic with the technology is the calculation of percentages (%). On a cheap calculator the following sequence might be obtained:

50 [+] 10 [%] (gives) 55	50 [×][%] (gives) 25 (and then) [×][%] (gives) 6.25
50 [–] 10 [%] (gives) 45	50 [÷][%] (gives) 100
50 [×] 10 [%] (gives) 5	
50 [÷] 10 [%] (gives) 500	

Then the important questions are: Why do we get these answers? What is the logic behind what we see? For some of us this situation gives rise to a question of introducing structure and logic in mathematics education. To discuss these questions we need also to discuss the goals of mathematics education. What kind of knowledge do we want the students to have in mathematics? Before we go into this let me first outline two perspectives on mathematics education that has been argued.

Mathematics education between usefulness and ”Bildung” [2]

[2] I use here the German word “Bildung” because I do not find an English word that will give the same meaning. “Bildung” signifies the more general aspects of education. We find a similar meaning if we say “an *educated* person”.

In many connections the usefulness of mathematics is stressed – mathematics is in the curriculum because it is useful – in society at large, and for the individual in daily life. It is useful to be able to calculate efficiently. Mathematics is also useful to solve many problems that occur in daily life. As societies become more ”technical” the demand for mathematical skills in daily life increases. Mathematics so as to be useful stresses the tool function of mathematics.

But on the other hand, we can consider mathematics – as most other school subjects – as being part of a more general *education*. Concerning this perspective let us us list some aspects of mathematics:

- generalizations
- connections and integration
- definitions
- mathematical thinking
- insight
- structures
- logic, reasoning, proof

I hesitate to put these two perspectives as opposites, but in the history of education we see sometimes what might be called ”a pendulum movement” from one extreme to another. The ”new math” movement I will argue emphasized the perspective of ”Bildung”, where as one of the reactions – ”back to basics” – argued for mathematics to be useful, by the introduction of more calculations and manipulation. Looking back in history it can be argued that many sides of education can be seen as a movement from one extreme to another. I will not argue this ”movement” here, but use it as a background for looking at the present situation. My view is that introduction of technology will change the perspective of mathematics in schools. Let us, however, first look into the problems of formulating goals for mathematics education with technology.

Goals for mathematics education in the technological age

Is it possible to agree on goals for mathematics education? In the PISA[3] project a number of competencies are outlined, which could be taken as a starting point for our discussion:

Reproduction, Procedures, Definitions, Computations. This class includes knowledge of facts, representing, recognising equivalents, and recalling mathematical objects and properties, performing routine procedures, applying standard algorithms and developing technical skills. Manipulating expressions containing symbols and formulae in standard form, and doing calculations, are also competencies in this class.

Connections and Integration for Problem Solving. In this class connections between the different strands and domains in mathematics are of importance and information must be integrated in order to solve simple problems. To solve tasks students will have to make a choice of strategies and a choice in the use of mathematical tools. In this class students are also expected to handle different methods of representation, according to situation and purpose. The connection component also requires students to be able to distinguish and relate different statements like definitions, claims, examples, conditioned assertions, and proofs. From the mathematical language point of view, decoding and interpreting symbolic and formal language and understanding its relations to natural language is another key skill in this class.

Mathematical Thinking, Generalisation and Insight. For items in this class, students are asked to mathematise situations: to recognise and extract the mathematics embedded in the situation and use mathematics to solve the problem; to analyse; interpret; develop own models and strategies; and make mathematical arguments, including proofs, and generalisations.

(Downloaded from PISAs homepage: http://www.pisa.oecd.org)

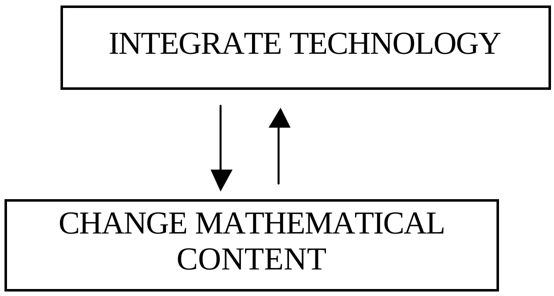
If we consider these goals from the perspective of technology, we see that technology enters each competency in a fundamental way. The second, and especially the third competency concentrates on *abstractions* and *generalisations* hence going further than the computational

[3] PISA = Programme for International Student Assessment (OECD)

aspect. Let us consider how information technology could be used to strengthen these competencies.
I will argue that it will be necessary to use some of the elements of the ”new math” – structure and logic. In this discussion, two important questions arise:

- How should technology (calculators/computers) be introduced into the mathematics curriculum?
- How should the mathematics curriculum be changed to take the use of technology into mathematics?

These questions are similar, but represent two perspectives, from the technological side and from the mathematical side. They are interdependent, as we have illustrated in the diagram below.



Our concern here is the changing of the content in mathematics. Let us consider an important element of the competencies outlined in the PISA document, that of *mathematising*.

In many counries we are now in a period of stressing applications of mathematics. This is not a goal to be neglected, but I will argue that in this technological age we must rethink the concept of application, hence I favour the general term *mathematising*. How to apply mathematics is changing. It is common today to adopt a more complete picture of the application process, taking into account all elements of the process, as well as the use of technology. See for example NCTM (1989).

Mathematising
If we look at how mathematising is described in the PISA document: *to analyse; interpret; develop own models and strategies; and make mathematical arguments, including proofs, and generalisations* I will argue that knowledge of elementary logic (and set theory) will greatly help in this process. We must ask the question if being able to perform the basic operations should take so much time as it has traditionally done in school mathematics.

It could also be argued that as probability is coming into school mathematics in many countries, elementary set theory would have a natural place in the curriculum

A new ”new math”
As mentioned above one of the major reasons that the "new math" was abandoned (at least in some countries) was the poor performance of the "new math" students in arithmetic. In later years this is an often heard complaint about mathematics education. Unlike earlier criticism there is now no one instance to attack, so it has been a criticism largely without an address. However, it has been directed against all that is not ”traditional” – i.e. against most new projects.[4] It has also been directed towards the use of computers and calculators in school mathematics. Many university mathematicians react negatively to the use of calculators, and in some university courses in some countries (graphic) calculators are not allowed on examinations.

[4] There has been criticisms directed against new curriculum reforms and movements in some countries, but they don’t seem to have the same strength as in the beginning of the 1970s.

In any event, calculators and computers are here to stay, and will be used by students whatever the teacher/school says. The school can not control the use outside of schools, and one way or the other, mathematics education has to take that fact into account. In many countries they have been adopted from an early grade.

Let us look into the use of calculators a little more closely. If we extend the examples given above, we will on some (simple) calculators get the following:

1[×]2[=] (gives 2) (then) [=][=][=] … (gives the sequence 4, 8, 16, …)

Again the question, what is the logic behind this "operation"? The sequence of pushing the buttons gives the impression of the [=]-key as an operator. For this reason we will find in our students' papers expressions like the following:

2 + 5 = 7 + 3 = 10 + 4 = 14

It should, however, be noted that ways of pushing the calculator keys also has many interesting properties to be discovered and investigated. For instance, again on some calculators:

1[+]1[=][+][=][+][=][+][=] ….

will give the Fibonacci sequence: 1, 1, 2, 3, 5, 8, …

These examples lead us to ask the fundamental question what is "equality" in mathematics. Equality is very important in many areas of mathematics. We have equality between numbers, variables, functions, sets, … etc. How is equality now defined? In more advanced mathematics equality is defined as an *equivalence relation:*

R is an equivalence relation, means (for all elements *a, b, c* in the domain):

*a*R*a*

if *a*R*b* then *b*R*a*

if *a*R*b* and *b*R*c* then *a*R*c*

This was also the definition found in textbooks for schools in the "new math" period. Perhaps students will be better equipped to use the technology with this definition, than with looking at the equal key as an operator? In general the notion of *relation* has disappeared from school mathematics.

Calculators and computers and their functioning are based on logic, so a basic understanding of logic will help us better master these tools. A spreadsheet will be more efficient if the construction involves use of propositional logic. On the other hand, computers with mathematical software allow us to experiment with logic. Most mathematical programs, such as Derive and Maple has logical functions build in, and so have spreadsheet programs. As the calculators get more advanced, I will argue that efficient use is almost impossible without understanding the underlying logic.

Another important element, that was introduced in the "new math" period, namely implication, has been abandoned. Implication is also a very central element of most of mathematics. An understanding of implication is instrumental in working with reasoning and proof in mathematics. Proofs and proving, which most mathematicians will argue is the most central part of mathematics, today has a very low status among students in schools. In his address to the International Congress of Mathematicians in Berlin in 1998, Mogens Niss listed this as what he called one of the "major findings" of mathematics education research:

> STUDENTS' ALIENATION FROM PROOF AND PROVING. There is a wide gap between students' conceptions of mathematical proof and proving and those held by the mathematics community. Typically students experience great problems in understanding what a proof is supposed to be, and what its purposes and functions are, as they have substantial problem s in proving statements themselves. Research further shows that many students, who are able to correctly reproduce a (valid) proof, do not see the proof to have, in itself, any bearing on the truth of the proposition being proved. (Niss, 1998, p. 774)

Mogens Niss then goes on to argue that proofs and proving should be (re)introduced in mathematics education. I will raise the question perhaps mathematical software could play a role in this?
The use of such software could show the need for proof. With a geometrical program like *The Geometer's Sketchpad* interesting relationships can be seen on the computer screen. To answer the question: *Is it always so?* it is necessary to give a proof. Many have also noted that computer programs (macros) are similar in many respects to proofs, and the correctness of a program (proof) can often be "checked" by the computer immediately. On the other hand there is also a danger of getting a mathematics without proof, because one can "see" something on the computer screen and be convinced – and then what is the need for a formal proof?
I have argued above that the elements found in the "new math" movement could play a role in mathematics education with technology. But a new "new math" should be very different in many respects from the "new math" of the 1960s.

Perspectives

Computers and calculators are mathematical machines. Their functioning is to some extent based on logic. However, the development also contains many problems where perhaps other aspects than the logical has had consequences for the construction. Above I showed how percentages are calculated on some calculators. Another example is the order of calculations.

The calculators we have today for schools are – for some functions – modeled after how mathematical expressions are written, not how they are carried out. Consider how you would calculate the following:

$$\sqrt{(3^2+4^2)}$$

You would take 3 and square it, then take 4 and square it, then add the two results, and then take the square root. The RPN (Reverse Polish Notation) introduced in some calculators is similar to this method of calculation.

This shows that the use of technology has its problems and pitfalls. Therefore a mathematics education should also be such that it allows us to look into the "inner workings" of the machines. Mathematics is the best way to discover the mechanisms of computers.

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