A project of approach to algebraic thought: experiences, results, problems[1]

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INTRODUCTION

The international literature on the problems concerning the teaching/learning of algebra underlines the crisis of traditional teaching. Many studies highlight the main cognitive obstacles that the traditional teaching of arithmetic - which mostly focuses on the results of calculus processes than on its relational and structural aspects (Kieran 1990, 1992), pose to the development of algebraic thought. Other studies emphasize the difficulties occurring in the formal encoding of problem texts (MacGregor 1991, Bednarz & Janvier 1996) and, at a more advanced level, the management and control of the algebraic language from a logical point of view (Bloedy-Vinner 1995). Some other studies emphasize the lack of awareness in the procedural-structural shift (Sfard 1991, 1994) and of flexibility and co-ordination between the various pieces of knowledge that with time overlap (Drouhard & Sackur 1997). More recent studies (Yerushalmy 1997, Kieran 1998) underline that the use of interactive graphical visualization software invites a revision of today's teaching path.

With this theoretical basis, according to the model of development of algebraic thought by Arzarello *et al.* (1992), we have started an experimental project for innovation on the approach to algebraic thought in middle school.[2] This project, which is still going on, aims at a gradual and aware approach to the algebraic objects through processes of mathematization. The project has two levels of intervention: on (and with) the teachers and the pupils. In order to understand the characteristics of our studies, it is important to know that they are framed in the 25-year-old Italian tradition of the so-called *nuclei* of didactic research, which operate in several Italian universities with the voluntary and joint co-operation of university researchers and school teachers. Many of these teachers have been following the *nuclei*'s activities for years already; some of them are today acknowledged as researchers, too, so that they now have achieved the double professional nature of teacher-researcher (TR). They are indeed careful, highly motivated teachers, who are ready to get involved with full awareness.[3].

As we said, the project concerns a compound process in which the following aspects are considered:

- the teachers' initial beliefs, their development of new attitudes and conceptions towards the approach to algebra and its feedback onto the cultural and methodological choices for the classroom-activities;
- the interaction between research director and teachers on the joint planning of the classroom interventions and the construction of questions to test suitable hypotheses of research;
- the joint study (director and teachers) of the pupils' behaviour and the qualitive analysis of their productions;
- the feedback as to classroom innovation and teacher's beliefs.

Novelty elements which don't appear in the studies we have quoted are the complexity of the process on one hand, and on the other the consideration of pieces of knowledge which are seldom studied in the literature about this level of schooling, such as: proof in arithmetical realm, relational aspects and of structural analogy in various numerical ambits, the ordered field of the rational numbers, the concept of function.

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1 Work carried out within the 40% and 60% projects of the MURST.

2 Ih the past three years the project was spread to primary school with the revisiting of the teaching of arithmetic in a pre-algebraic light (see Navarra 1999).

3 For more details on the state of art of the Italian research and in particular on the role of the teacher-researcher, see Arzarello (1999) or also Malara (1998).

In this paper, after giving a quick overview of the project, we concentrate on:
- some results of experimented activities with hints on the pupils' behaviour;
- some questions concerning the way we carry on the research and the role of the teachers in these situations;
- hints on more general problems.

OVERVIEW OF THE PROJECT

The project started in 1994, after two years of intensive reading and discussion on the problems of teaching/learning algebra carried out in study seminars with the teachers.

The need for deep study had become clear to us while we were working on a previous research project on "problems" (Malara 1993). In those times, the teachers were sceptical about studying algebraic verbal problems. They believed that it is impossible to deal with any meaningful problem of this kind in middle school, because the pupils are not accustomed to solving methods such as equations and linear systems. Many teachers even had their own conception of algebra which unfortunately was reductive and distorted[4].

Because we believed that it was necessary to give a new conceptual foundation and clearness in this field, since the algebraic language is so important overall, we decided to concentrate on algebra with a wide project for didactic innovation. We wanted to make the teachers approach algebra as a language (not only in its syntactical aspects, but also in translation and in production/communication of thought) and to promote an early use of letters in their classes. Letters should be used to codify relations on one hand, and to formulate properties in general terms on the other. All this should be carried out *through* and *for* the study of problems, either within or out of mathematics, or even of demonstrative kind.

Thanks to the critical analysis of many studies of those years[5] the teachers of our group gradually became aware of the need of a deep renewal of the usual didactic praxis and started wishing to test themselves with experimental researches on the topic.

So we planned an experimental project for the three years of middle school[6], with the precise intention of monitoring the same class during the three years of school[7] and of doing specific studies with other classes in the meantime.

The core of the project was to give a naive approach to the algebraic language, focusing on the control of the meaning of the writings, which are created by the need of codifying or generalizing relations among elements belonging to the given problem situations on one hand, and on the comparison of writings that derive from different but equivalent ways of codifying such relations on the other. In this approach, the properties of the arithmetical operations guided us in setting the laws of syntactic transformation, which were the goal of a difficult and slow collective activity.The core elements in our planning for the activities were: i) the early use of letters and the highlighting of the link between algebraic and natural language, with particular importance to the meanings and roles of some of the latter's linguistic elements, such as terms, signs, symbols and conventions, either in translation activities from one language to the other or on shifting from the arithmetical to the algebraic realm; ii) the progressive acquisition of the symbolic language seen as a cultural fact, in

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4 It must be underlined that most of the people who teach mathematics at this school level actually have never done specific studies on mathematics or professional training.

5 We limt ourselves to quote those by Bell (1987), Filloy and Rojano (1989), Kieran (1990, 1992), Kuchemann (1981), Sfard (1991), and the articles on the theme appeared in the proceedings of PME. It must be underlined that the critical study of research literature continued along the actual realization of the project.

6 On the three-year articulation of the experimental activities, see Malara & Gherpelli (1996) or Malara & Iaderosa (1999a).

7 It must be considered that usually in Italy the same teacher works over the three years of middle school with the same class. The class we have observed worked with the TR L. Gherpelli.

analogy to what happens in the learning of any other language, interlacing the study of syntax with translation and thought production.

Methodologically speaking, our teachers always work constructively, stimulating and orchestrating the pupils' intentions, promoting group reflection on what is gradually carried out until knowledge is eventually institutionalized. .

ENACTING THE PROJECT

We recall here briefly and in chronological order some studies that we have carried out in middle school, concerning: argumentation and proof in arithmetic, the solution of algebraic problems, some problems on the intermingling between algebra and arithmetic. We have not got room to report on other studies on the approach to the concept of function in its various (verbal, numerical, graphical, algebraic) aspects (see Iaderosa & Malara 1999)

Argumentation and proof in arithmetic

This first study concerned the analysis of the pupils' behaviour and productions in front of inquiry problems such as those reported in Table 1. We started with these questions in first grades. In these situations, the pupils must not only find out properties and regularities, but also discuss hypothetically, find out counter-examples and substantiate their opinions.

With these questions the pupils gradually understand the role of counterexamples in confuting the presumed truth of a statement and, consequently, that numerical checks - even if many - are not enough to declare the truth of a statement, because general argumentation is necessary.

In this phase we paid attention to the pupils' use of letters and on their role in the argumentation and we checked their spontaneous codifying of properties and relations, which are later discussed in class. You can find deep analyses of protocols on the difficulties and potentialities expressed by the pupils, as well as on the global results in the classes, in Gherpelli & Malara (1994). At this point, the pupils' positive attitude and the results obtained persuaded the teachers that they have found a very productive study field, which made them wish to continue the inquiry with the passage from argumentation to proof.

We started the approach to proof with the use of the algebraic code already in the second grades with: a) proofs constructed collectively (pupils and teacher); b) proofs "guided" by the teacher (who indicates the various steps and shows the way they should face the thing to achieve the goal)[8]; c) proofs that the pupils produce by themselves. We are not going to linger on this study here; for the analysis of the pupils' productions and the global results of the innovation in the classes, see Malara & Gherpelli (1996); for hints on the interaction between teachers and university researchers in the development of the research and in particular on the evaluation discussions on the protocols as to the research itself, see Malara & Iaderosa (1999a). Still, we would like to underline that the extremely positive result we obtained at the end of this school cycle consists of the teachers' awareness of what 'to prove' means and of the role that the algebraic language plays in arithmetic.

The solution of algebraic problems

Our second study, intermingled with the previous one, concerned algebraic verbal problems. As well known, this study starts traditionally after the introduction of a linear equation in one unknown and, since the pupils are so young, the problems that are normally used can be solved also intuitively and/or resorting to the right graphical representations. This way of working, as underlined also by Bernardz & Janvier (1996), does not allow them to appreciate the goodness of the algebraic method. Our choice was to suggest the study of algebraic verbal problems at two or more unknown quantities, even complex ones, before the formal introduction of equations. This would first of all

[8] the moment in which the teacher acts as a model is very important because the pupils can understand what they must really do and produce.

justify, in front of the pupils, the opportunity of studying them as a mathematical object, according to the historical path as well as to our syllabuses, and more in general it would educate to the "principle of economy", typical of mathematics, which privileges the study of the representative schemes of a plurality of situations.

Table 1

Examples of inquiry activities in arithmetical realm
Discuss the following information on natural numbers saying why they are true or false to you. Is it true that: 1. The sum of two even numbers is divisible by 4? 2. If the product of two numbers is even (odd), each of the two factors is even (odd)? 3. A number divisible by 3, i.e. of the 3n kind, is always odd? 4. The square of any even number is divisible by 4? 5. All numbers divisible by 3 are also divisible by 9? 6. If 3 divides two natural numbers, it divides also their sum or their difference? 7. The product of two even numbers is divisible by 7? 8. There are values of n so that 5n+3 is: divisible by 5, divisible by 3, divisible by 2? 9. The sum of two numbers that divided by 5 have remainder 1 is again a number that divided by 5 gives remainder 1? 10. The sum of four natural successive numbers is always even?

In such hypotheses we wanted to study the pupils' behaviour as to: i) the formal translation of the relations expressed by the verbal text; ii) the transformation and the elaboration of the relations in order to get to the resolutory equation;iii)the naive study of equations for the determination of the unknown quantities and the solution of the problem.

The problems we gave to the pupils, some of which you can see in Table 2, usually have a schematic text that requires - except in some cases - a rather easy formal translation. The number of unknown quantities ranges from two to four whereas the relations are mainly additive, multiplicative, or both.

The didactic steps that we developed with the pupils focuses on the control of the plurality of representations possible for the same relation, which is achieved by asking the pupils intermediate questions that induce them to ponder on the equalities induced by the given relation. For example, if we consider the relation :*"segment* $\overline{AB}$ *is bigger than segment* $\overline{CD}$ *by 4 (cm)"* we ask the pupils the following questions: 1. *Which of the two segments is the major one (and which is the minor one) ? ;* 2. *Once you chose the major one (the minor one), write down how you obtain the other one;* 3. *Once you chose the major one, write down the difference between the two;* and we induce the simultaneous control of the following writings: $\overline{CD} = \overline{AB} - 4$; $\overline{AB} = \overline{CD} + 4$; $\overline{AB} - \overline{CD} = 4$.

This way the pupils get used to representing in many ways a relation between two quantities and then choosing the more convenient one according to the problem examined. The solving method suggested is by substitution; it is presented as "the game of the exchange", that should be applied one or more times until they obtain an equation with only one unknown quantity, which must be solved. This equation, in the problems we consider, can be reduced, thanks to the properties of arithmetic operations, to the structure ax+b=c, with a, b and c as natural or rational numbers.

The difficulties, beside the number of unknown quantities and the kind of relation, concern: i) the translation of relations into terms of equalities and their transformations; ii) the choice of the unknown quantity through which the other can be expressed; iii) the management of the resolutory equation. A detailed report of this study can be found in Malara (1999).

In general, we can say that the impact of this activity in the classes is quite strong, both for the involvement and the results achieved. Almost all the pupils realize that they are in front of a general method for solving this kind of problem, based on translating into formulas the information contained in the texts and then elaborating the formulas they obtained. The fact that they face the solution of algebraic problems without any instructions into the syntax of the algebraic language makes the pupils give wrong syntactic transformations mainly due to the lack of control over the meanings of the writings obtained; this kind of problems gives a reason to study syntactic questions in themselves and independently from the context, concentrating the attention on the arithmetical properties. This aspect of the problem is quite interesting both for the pupils, who eventually understand why they study literal calculus and how important it is, and for the teachers who realize how vital it is to talk to the pupils in order to negotiate the meanings of simple algebraic expressions, usually taken for granted.

Table 2

Algebraic verbal problems at two or more unknowns done in seventh grades
1. Consider the triangle *ABC,* lying on the side $\overline{AB}$, about which you have the following information: the perimeter measures 60 cm; side $\overline{BC}$ is bigger then side $\overline{AC}$ by 5 cm; side $\overline{AB}$ is bigger than side $\overline{AC}$ by 10 cm. Calculate the length of its sides. Can the triangle be right angled? If yes, what would be the right angle and which the hypotenuse?
2. Determine the two numbers such that the larger one exceeds by 50 three times the smaller one, and the sum of the two is 110.
3. There are 224 animals, cats and dogs, in a courtyard. The number of the cats is 6 times the number of the dogs. Calculate how many cats and how many dogs there are in the courtyard.
4. Massimo goes to a pizzeria. He gets a pizza, a dessert and a coke and spends 8300 £. The dessert cost 1500 £ less than the pizza. The pizza cost 400 £ more than twice the coke. Calculate how much were the pizza, the dessert and the coke each cost.
5. The perimeter of a rectangular trapezium is 96 cm. The length of the larger base is 20 cm more than the smaller one. The difference between the length of the larger base and of oblique side is 13 cm. The difference between the measure of the oblique side and of the height is 10 cm. Calculate the lengths of the two bases, the height and of length of the oblique side.

Syntactic and structural aspects

This study was developed, in parallel with the previous ones, in the class tutored by R. Iaderosa, a teacher-researcher with a refined mathematical background, who more than the other teachers was sensitive to the transversal questions underlying modern algebra. It examines a series of syntactic problems that arise either on shifting from the various numerical ambits or within the same ambit because of the predominance of the additive model over the multiplicative one. One *leitmotiv* was to constantly compare the operations of addition and multiplication, in order to force the pupils' attention not only towards the analogies but also towards the differences between the two. One of our research hypotheses was, in fact, that this very comparison could avoid improper tranfers or mixtures of properties from one structure to the other, which always give vent to the several and persistent errors testified in literature (see for instance Fishbein & Barach, 1993 or Fishbein 1994).

On this basis, we enacted an experimentation with second- and third-grade classes which focused on a series of tests (some of which are reported in Table 3) explicitly conceived to force a comparison between the two structures. These tests ask the pupils to transcribe and elaborate arithmetic or algebraic expressions by operating an analogical transfer from the additive structure to the multiplicative one (usually in natural numbers) and vice versa, in a context with many operations and in situations of distraction, which requires a good control between the additive-multiplicative notation and the multiplicative-exponential one.

The aim was to investigate on the pupils' ability to see the structure of some expressions and modify it by analogy. In Malara & Iaderosa (1988) we report a sample of their productions, which show that almost half of the pupils keep a good control of both notations. As we had foreseen, they have difficulties in dealing with powers and some of the weaker pupils simply try to explain powers as products. In particular, the pupils identify quite well the (direct and inverse) distinction between addition and multiplication, but in the meantime they cannot control the operation of power, especially in the passage from the multiplicative-exponential notation to additive-multiplicative one.

A posteriori, these tests allow the teachers to detect which level of conceptualization the pupils have achieved about arithmetical operations and their properties and about the characterization of the two structures, which can be hardly imagined when such tests are conceived and administered.

Table 3

Comparison between additive ambit and multiplicative ambit

Seventh grade.

1) Calculate, by using as much as possible the properties of powers:

a) $3^3 \times 3^2 + 3^3 + 3^2$; b) $5 \times 2 + 5^2$; c) $(2 \times 3^2)^2 + (2 + 3^2) \times 2$

2) Transform the following writings by substituting to any addition sign a multiplication sign, and to any multiplication sign the power:

a) $(2 \times 3) + 5 + 7 \times 2 \longrightarrow$; b) $(2 \times 5) \times 7 + 2 \longrightarrow$; c) $(5 + 2) \times 4 \longrightarrow$.

3) Transform the following writings by substituting to any multiplication sign an addition sign, and to any power a multiplication sign:

a) $2^3 \times 5 \times 7^2 \longrightarrow$; b) $5^3 \times 2^4 \times 3 \longrightarrow$; c) $(5 \times 8) \times 2 \longrightarrow$; d) $(5^2 \times 2^2) \times 3 \longrightarrow$.

4) Look at the following writing and decide for each of them whether it is true or false:

a) $2 \times 5 + 3 \times 4 = 5 \times 2 + 4 \times 3$; b) $2^5 \times 3^4 = 5^2 \times 4^3$

Look again at a) and b). Is it possible to go from one to the other like in the previous exercises?

Eight grade.

5) Calculate, by using as much as possible the properties of powers:

a) $2^3 \times 2^2 + 2^3 + 2^2$	$a^3 \times a^2 + a^3 + a^2$
b) $(2 \times 3)^2 + (2^3)^2$	$(ab)^2 + (a^3)^2$
c) $(2 \times 3^2) + (2^2 \times 5)^2$	$(ab^2) + (a^2 b)^2$

However, it must be underlined that this study, which was quite appreciated in research contexts[9], gave vent to contrasting reactions among the teachers, some of which consider such approach to be too sophisticated and advanced for this school level.

The interconnection between (algebraic) fractions and rational numbers

This study moves from some discussions we had in our group about the possibilities our project could give to the analysis of rational numbers in middle school, with the intention of creating continuity with upper secondary school.

There are some epistemological divergencies between the modern, structural vision that concentrates in middle school on the introduction of the various numerical ambits, rational numbers in particular, and the old, consolidated teaching tradition that deals with fractions from a merely operative point of view, without even getting to the concept of rational number as class of equivalent fractions. In this tradition, the approach to the operations aims at determining their result for particular couples of fractions and it almost never arrives at explaining its laws of correspondence in general. The comparison of fractions, moreover, is usually carried out passing from the decimal representation of the quotient between numerator and denominator (which of course is often approximated), and the pupils are never asked to see how a fraction varies as its numerator and/or denominator varies[10] This procedure, unfortunately, doesn't allow us to move to the comparison of fractions in general terms, nor does it allow the conceptualization of how a generic fraction varies when its terms vary.

Our study is based on the hypothesis that an early approach to the use of letters allows us to face elementary questions about fractions from a general point of view, so that the pupils can develop highly flexible, effective and transparent conceptual models as to: rational numbers, order and the operations among rational numbers in general terms.

One specific aim is to induce the analysis of the meanings that different representations of the numerator and/or denominator of simple numerical or algebraic fractions convey, so as to avoid - or at least limit - stereotyped behaviour or classical errors on transforming them. From a conceptual point of view, we want to bring the pupils to the awareness of: i) how to recognize equivalent

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9 We are talking about the Working Group on algebra at PME 22 (1998, Stellenbosch, South Africa) and the European Conference Cerme 1 (Osnabruck, 1998)

10 There is an interesting study on the topic by Lopez-Real (1998).

fractions[11]; ii) how to compare fractions without resorting to the decimal representation; iii) the reasons that are at the basis of the general definitions of addition and multiplication (through questions such as their being independent from the representing fractions, the embedding of natural numbers into the non-negative rational numbers); iv) the calculus simplifications produced by reducing a fraction to the minimal terms and, specifically for addition, by resorting to the minimum common divisor of the denominators; v) the existence of opposite and reciprocal for each rational number different from zero (and through these concepts up to the operations of difference and division); vi) of Archimedean but incomplete nature of the order structure and of the laws of monotony.

The experimental work started this year and the research turned out to be very delicate and subtle. The topics we have already faced concern the four points we mentioned; some of the activities connected are reported in Table 4, some others look at the problem of defining in general terms the operations and their legitimacy as to the corresponding ones in natural numbers. For more details on this research, see Malara (1999a); as to the results, some are still to be analyzed[12], yet we may say that the teachers acknowledged that this new way of looking at rational numbers is very productive. In particular they underline that:

- using letters allows metacognitive teaching on the properties and algorithms of the operations;
- beside allowing the pupils to distinguish the number from its representation, working on the multiple representations of a number in the ambit of natural numbers is very important because, here too, it makes them easily accept equivalent fractions as different representations of the same rational number.
- paying attention to the structural aspects allows an easy approach to two aspects which, didactically speaking, are rather delicate: a)widening the concept of number; b)widening the numerical ambit.

Despite the open problems, the very positive result heightens the teachers' awareness on two important points: a) the usual didactic of rational numbers is reductive both from a cultural point of view and as to the pupils' skills; b) paying attention to the structural aspects allows the pupils to face by themselves important problems such as the construction of the arithmetical operations in a new and wider numerical ambit.

Table 4

Examples of the activities given
Seventh and Eight grade
1. Build fractions equivalent to the give ones, according to the indications expressed by the scheme $(\leftarrow \; \rightarrow)$
$\leftarrow \frac{15}{25} \rightarrow$; $\leftarrow \frac{10 \cdot b}{14} \rightarrow$; $\leftarrow \frac{3 \cdot 2. \cdot a}{14} \rightarrow$; $\leftarrow \frac{18 \cdot k}{2 \cdot a \cdot 15} \rightarrow$
2. For each couple of fractions in brackets, decide which is smaller, and explain why:
$\left\{\frac{15}{17};\frac{13}{19}\right\}$; $\left\{\frac{19}{36};\frac{11}{24}\right\}$; $\left\{\frac{235}{352};\frac{115}{176}\right\}$
Eight grade
3. We ask you to build equivalent fractions for each of the following fractions; find out whether there are any difficult cases and why they are difficult:
$\frac{5}{7}$; $\frac{2 \cdot 11}{5 \cdot 9}$; $\frac{p}{k}$; $\frac{3 \cdot a}{12}$; $\frac{5+4}{11}$; $\frac{a+b}{2}$

[11] At most, the pupils manage to conceptualize how to pass from one fraction to an equivalent one, but they can't say in general when two fractions are equivalent. This happens also in higher school levels.

[12] Then more problematic results, which must be analyzed, concern the comparison of rational numbers.

4. Following the strategy you prefer, compare each couple of fractions and explain which is the smaller one $\left\{\frac{3}{4};\frac{5}{6}\right\}$; $\left\{\frac{11}{24};\frac{17}{36}\right\}$; $\left\{\frac{2a}{17};\frac{3a}{22}\right\}$

Seventh and Eight grade

5. Replace the letters with the numerical values that make the following equalities true:

$$\frac{k+1}{4}=2\,;\ \frac{8}{k+1}=4\,;\ \frac{4}{5}=\frac{12}{k+2}\,;\ \frac{15}{18}=\frac{k}{k+1}\,;\ \frac{k}{3}\cdot\frac{4}{m}=\frac{5}{6}\,;\ \frac{2}{3}\div\frac{c}{d}=\frac{6}{21}$$

THE TEACHERS' ROLE AND CONTRIBUTIONS IN THE RESEARCH

The teachers' contribution to the developing of the project is remarkable and manifold. Once the hypotheses are made clear and the guidelines for classroom intervention are agreed on the specific goals, they often create autonomously the tests they want to give to the pupils. Before actually doing such tests, we always discuss them within the group in order to assess their potentiality and difficulties in advance; on this occasion, the director of the research can point out the aspects lacking in the tests and suggest how they could be refined, or he/she can simply approve and appreciate their quality. For example, the tests created by L. Gherpelli for the enhancement of argumentation in arithmetic (reported in Malara & Gherpelli 1996) and some about rational numbers, reported in table 4, have been appreciated for their quality and effectiveness. Moreover, the questions in Table 3, concerning the generation of expression, starting from the given ones, by analogical transfer from the additive-multiplicative notation to multiplicative-exponential one and vice-versa, were created by R. Iaderosa and, as we said, obtained different reactions among the teachers. Some appreciated them mainly as research tools, other teachers thought they were pointless, since they were by no means remarkable to the pupils' eyes and they were optional within a general didactic frame for the development of algebraic thought[13].

In general, however, most of the tests for the experimentation are selected from among a group of proposals coming from the various members of the team after a careful analysis of the possibilities and difficulties that they could give the pupils. This happened, for example, with the tests concerning the approach to proof: we discussed the difficulties in the syntactic elaboration of the expressions created as translation of the hypotheses and in the interpretation of their transformation to achieve the thesis, or with algebraic problems.

We paid particular attention to the formulation of the text in the tests, giving the utmost importance to the immediateness of communication rather than to linguistic perfection. Sometimes these texts are formulated (or re-formulated) by using the children's slang or particular class-codes. You'll find examples for this in questions 1 and 5 of Table 3, where the teacher uses the word "calculate": this verb is improper for the kind of performance she wanted to promote (which however had already been explained orally to the pupils) but, in her opinion, it is somehow neutral whereas more appropriate terms could cause some difficulty[14]. Similarly, in Table 4 the teacher uses an unusual graphical code, which she specifically created in order to let the pupils work simultaneously on the simplification of fractions and on the generation of equivalent fractions, because she wanted to put together two activities that are usually experienced as separate.

The team discusses and selects also the problems suggested by the director of the research, and when they are accepted their texts are often revised so that, as we said, the pupils find them easier. Sometimes their proposals are refused, because they seem too advanced; this happened, for example, with the construction of the concept of rational number. On that occasion the director

[13] The teacher's opinions are so different because each of them has a different cultural background and therefore they have different conception for each topic examined. This shows how real the cultural debate within the group is and how each participant is fully autonomous in his/her role.

[14] The group discussed and criticized this word, but the teacher preferred to keep it owing to the reasons we've said.

underlined the convenience of characterizing the classes of equivalent fractions by doing with the pupils the proof of the following statement starting from particular cases "if *the fraction c/d* is equivalent to the *reduced fraction a/b,* then c/d can be obtained from a/b by multiplying its terms by a same natural number different from zero"[15]. The teachers reject this proposal because they find it difficult even in the particular cases, although the concepts are simple and the pupils have experience in using letters and in using the principle of substitution. Nevertheless, our experience says that with time the teachers reconsidered the proposals formerly rejected and eventually experimented with them.

Moreover, we must consider the important role the teachers play in developing the classroom discussions (starting discussions, for construction, for balance and/or institutionalization), in understanding each pupil's personality and assessing their contributions in the collective activities, also with reference to their character. On leading such work, the teachers manage to play the double role of participant and observer (Eisenhart 1988), even if in some cases, as reported in Garuti & Iaderosa (1999), they can meet various difficulties. Still, they are precious when we evaluate the pupils' protocols, because they know who made them and in which atmosphere, which goes beyond the director's analysis. There is an interesting example on the different points of view from which a protocol can be assessed, which states the importance of joint assessment, in Malara & Iaderosa (1999a).

Another thing that should not be neglected is their contribution at the beginning, when we consider and identify specific research problems and we formulate the hypotheses to be verified etc., but most of all their work at the end, when it comes to writing down important parts of research reports.

Beside the positive aspects, however, we would like to consider also the limits they put on the developing of the project. As underlined in Malara & Iaderosa (1999a), once the guidelines for the realization of the project are decided, the actual carrying out of the research presents crucial difficulties among which, beside those concerning organization, there are: the cultural background of the teachers (some of them had a degree in maths, some in other scientific disciplines), their opinions on didactic matters (teaching tradition, trends in programming the lessons, different importance given to the various activities, etc.), personal preferences and, what is more important, the different ways of seeing their own role (more as a teacher, more as a researcher or something in between). Moreover, their autonomy often makes them neglect the common planning of the guidelines for discussion, the recording of the discussions (they prefer to witness them through their diaries, which are usually based on their memory rather than on original recordings); they refuse to have a silent observer (disturbing and inhibiting); they forget to make a joint analysis of the protocols (since there are so many, they usually pre-select them and offer only prototypes for the analysis in the group). All these questions eventually influence both the choice and sharing of the activities which are the object of experimentation and the ways of carrying out the research itself (for more details see Malara & Iaderosa 1999a or Malara 1999b)

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[15] The demonstrative steps that should be developed with the pupils are:

Step 1: start from a particular fraction, for example 3/7, suppose that 3/7=c/d and prove, in collective activity, that necessarily 3d=7c. What you must use here is that: i) if 3 is divisor of the product 7c, since it isn't divisor of 7, it must be divisor of c; ii) it is possible to express c as 3 by some other number and, giving a name to this number, say q, we have c=3q; iii) by substituting in the equality 3d=7c we obtain 3d=7x3q and by cancellation d=7q.

Step 2. Generalize the result by substituting 3/7 with any reduced fraction a/b along the reasoning we have just done.

FINAL CONSIDERATIONS

Though this is a very general overview, it probably gives an idea of what we have done so far and of how complex it is to carry out research for innovation by working this way, typical in many Italian research groups, that we can call it the "Italian model" of research for innovation.

First of all, this way of developing research requires a long time for confrontation of the parts involved and cannot be done without cultural and didactic mediation with the teachers (we must not forget that, beside being managed personally by the teachers, the objects of our researches belong to a specific program in school that must go hand in hand with a more general framing concerning the teaching of all the areas of the discipline and take into account the didactical times). Moreover, we often find ourselves working "without protection", since the object of our study is the teachers' and pupils' behaviour in front of innovation, which sometimes requires that we go back to the studies we have already done in order to highlight and refine some aspects that had remained in the shade or unsolved.

From a more general point of view, owing to the way they are carried out and to all the variables involved, these researches cannot be reproduced in the classical sense. They are prototypes useful to give new visions and new directions to most of the teachers and they are interesting for those who need a different, more appropriate approach to certain topics; but their usefulness is related to the ability of the teachers in evaluating and appreciating the questions in play and, as underlined by Zan (1999), it concerns a very delicate aspect: the question of the teacher training and her/his culture.

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