MODELLING AS A SUPPORT IN TEACHING OF MATHEMATICS

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1. INTRODUCTION

Whenever modern trends in mathematics teaching are resolved, the phenomenon of a model (modelling) arises (eg. ICMI study [7]). The word „model" may possess different meanings. For instance, in primary teaching „separated models " play a key role in the „mechanism of understanding" (e.g. Sierpinska [13], Hejny [6]), when speaking about practical aspects of mathematics teaching, modelling is associated with applications. In any case, the historical link of mathematics to „real-life" gives the word „model" a certain intuitive charge. Therefore, at initial educational levels, there is no need to speak about modelling explicitly, although an intuitive method of modelling is widely and „secretly" employed. With an increasing knowledge (not merely mathematical) , at a certain stage a new emerge of modelling with more intensive meaning may be observed . Namely, the modelling is crystallising into a method , which can be effectively used in a learning process. There are several important contributions of modelling to mathematics education (see also ICMI study [7]). Modelling is undoubtedly a mathematical process ; there is widespread agreement among mathematics educators that greater emphasis on mathematical processes is desirable at all educational levels. „How to model situations" is on the list of main goals for mathematics learning. Modelling is inevitably connected with the applications of mathematics ; teaching students how to apply mathematics is a crucial task of instruction in mathematics. Modelling may markedly affect the process of understanding , which is a vital point of learning and teaching of mathematics. Modelling can also encourage the motivation of students to study mathematics, particularly the students of engineering .

To precise the terminology is not the main topic of further considerations. On the contrary, the wealth of contents of „model", „modelling" may be in many respects productive. We also do not intend to describe real modelling problems undertaken by the professional modellers. Our aim is to present modelling as a versatile strong tool to meet the above mentioned decisive aspects of mathematics teaching, mainly on secondary or post-secondary level. A classification of modelling methods is proposed reflecting consistently viewpoints of practical use. Illustrative examples are presented and methodical recommendations are suggested.

2. MODEL, THE ESSENCE OF MODELLING

The need to speak about models emerges when investigating the behaviour of more complex real objects. For our futher purposes, any **model** can be defined as a simplified representation of a real situation. A **mathematical model** (briefly a **model** in the sequel) is a model created using mathematical tools . Then, by **mathematical modelling** (briefly **modelling**) we understand a procedure of creating models. Inevitably there is a conflict between mathematical ease and the model's accuracy. The closer the model comes to reality the more complicated the mathematics is likely to be.

It is important to realise that modelling is carried out usually in order to solve problems. Any model has a clear purpose stated at the start. Hence, modelling a given real situation, we may come to different models depending on the purpose or given problem respectively. In other words, the model construction will be affected by such viewpoints. For example, modelling a journey by car from one town to another remote town to visit friends will depend on whether we give priority to the beauty of nature or to the consumption of petrol.

Now let us turn back to our issue - modelling as an aid in mathematics instructing. We can often encounter two typical situations we will focus on next. We are facing the „problem“ to reach understanding of some concepts or we must find the solution (model) to a given problem (real situation) and use it in subsequent calculations. The former is of „classical“ fashion (more intensively appearing on lower educational levels), the latter is closer to applications (more intensively on higher educational levels).

With a view to teaching aims we will restrict ourselves at most to real situations, which can be simplified to the correspondence between two sets of numbers. With the exception of abstract modelling such correspondence will be functional, i. e. our models will be mostly functions of one real variable. Hence, elementary functions will be widely employed. There is another crucial point of modelling, namely discretness and continuity. As the bulk of teaching time at secondary level is devoted to elementary functions (or due to bad teaching?), students perceive the function concept as continuous. Despite using up-to date teaching methods even at primary level, the understanding of the function concept remains something still to be seen. The matter is more complicated . In my opinion, the step from discrete to continuous claims intellectually a lot, for many students more than they are able to handle. Instructors at secondary level did not have to realize it, but those at post-secondary level (in particular teachers of engineering mathematics) face the problem fully. Although students comprehend that a real situation has discrete character, to accept its continuous model some mental work is wanted (at least on the first occasion). For instance, when speaking about demand function , first year students of economy are acquainted with concrete examples of the dependency between the price and the amount of goods in the form of a discrete diagram, but then suddenly a demand function is given (in fact modelled) by a continuous elementary function. The only way for the students to overcome the problems , to realize the reasons (and to appreciate the role of mathematics) is doing modelling. Modelling of dependencies (laws) formulated verbally are of particular didactic value. For instance , when modelling the declaration „ more cars will cause more accidents“, the student should get the idea to draw a graph of a continuous increasing function in the first quadrant passing the origin as a (rough)model of a real situation described by the declaration(perhaps convexity or concavity are questionable).

From the viewpoint of practical teaching reflecting diversity of milieu a certain classification (or sorting) of modelling types seems to be useful.

3. QUANTITATIVE AND QUALITATIVE MODELLING

Quantitative and qualitative modelling are concepts (and methods) currently used in applications. While the former is traditional and quite transparent, the latter , though not new, has achieved revival and progress in recent time.

Quantitative modelling is based on the applications of approximation theory, statistical analysis, numerical analysis, among others. It works usually with concrete data (numbers) obtained by experimental measurements and the result is a concrete mathematical structure (equation,function,etc.). The resulting part of quantitative modelling is mostly performed by mathematically educated professionals or with the help of mathematical software. This seems to be rather out of the scope of the purpose of modelling as given above, but the principle of quantitative modelling will be of use in combination with other types of modelling (particularly visual).

Qualitative modelling (eg Davis [2], Dohnal [4]) has been developed to model real life situations which are described by vague, inconsistent and sparse data. The variables are not represented by numbers, but at most by their signs. Briefly, in qualitative modelling only four „qualitative" values are considered,

positive	negative	zero	not relevant(unknown)
+	-	0	*

and for them an algebra with two operations (+ and .) reflecting usual algebra of numbers is defined. In an intuitive way a number of rules may be derived, as for instance

$$(+) \; + \; (+) \; = \; (+)$$
$$(-) \; . \; (+) \; = \; (-)$$
$$(+) \; + \; (-) \; = \; (*).$$

For example, the last equation is to be understood as the addition of negative „amount" and positive „amount"- the result must be (*) (unknown), because the sizes of „amounts" are not known (in fact the result could be + or - or zero). Mathematical expression containing numbers may be transformed to qualitative one by so called **degradation**; for instance, the equation

$$0{,}02x_1 - 5x_2 + 100x_3 = 0$$

is degraded to

$$X_1 - X_2 + X_3 = 0 \; .$$

In this way a real situation can be described as a **qualitative model**. For example, the qualitative model

$$X = Y + Z$$

may represent the sum of total revenues of two subsidiary enterprises.

Following the patterns given above dynamic qualitative models may be constructed. Let x = x (t) be a function of time t describing the trajectory of variable X. Consider an interval I, where x (t) and its first and second derivatives do not change the sign. Then a **qualitative behaviour** of X is defined as a triplet

$$[\, X \; , \; DX \; , \; DDX \;],$$

where DX is the first and DDX the second qualitative derivative of X, ie DX is the degradation of dx/dt and DDX is the degradation of d^2x / dt^2 . Some interpretations :

[X, DX, DDX]	
+, + , +	X is positive, increases, the rate of increase grows
+, + , 0	X is positive, increases, the rate of increase is zero (linearity)
+, + , -	X is positive, increases, the rate of increase slows
+, - , +	X is positive, decreases, the rate of decrease slows .

Trajectories may be simply depicted graphically, for example the case +,-,+ may be represented by an decreasing convex curve in the first or the second quadrant.

After creating a qualitative model , we look for so called **qualitative solutions** with respect to qualitative behaviour of variables, as the following example shows :

Example : *Let*

$$X = Y + Z$$

be a qualitative model. Then, for instance

X DX DDX Y DY DDY Z DZ DDZ

\- + - + + - + - +

is not a qualitative solution of the model, because

X Y Z

(-) ¹ (+) + (+) .

Qualitative modelling may seem to be highly sophisticated for modelling as a teaching aid, but actually there is not too much „mathematics " in it. The basic idea of qualitative modelling, namely to grasp „trends", is extremely valuable. It is implicitly included in the principles of visual and analytic modelling. Qualitative modelling is an up-and-coming discipline, which has become an important tool of contemporary engineering. In recent times models were created of Financial flows, Taxes, Unemployment, Interest payments, Ecology analysis ,etc. .

4. VISUAL AND ANALYTIC MODELLING

Visual modelling is based on shape characteristics. The result of visual modelling , a **visual model**, is a geometric structure (in most cases a curve) satisfying given properties; properties are usually formulated in mathematical language, ie in terms of statements. The resulting structure is depicted graphically , its analytic form (if known) is not essential. Visual modelling apprehends the shape, hence it is of qualitative nature.

<u>Example</u>*: Let „ as the price of goods rises, demand falls" be a statement (the law of decreasing demand in economic theory). As its visual model may be drawn as any decreasing function in the first quadrant (the demand is on horizontal, the price on the vertical axis). It can be convex or concave ; in case we get additional information about the rate of decrease, for example that decrease is slower rather than faster, we will draw a decreasing convex function as an improved visual model.*

<u>Example:</u> *As the output of a statistical investigation can be a collection of data (couples) expressing the relationship between two variables x and y. Graphically it may be interpreted as a collection of points [a,b] in the plane (discrete graph). There is a standart requirement to draw a continuous curve passing through the given collection of points, „ modelling "the shape of deployment of points. The resulting visual model will be the sketch of a curve or possibly better as an improved visual model together with the characterization within the class of elementary functions (linear, quadratic, exponential,etc.); for this a good visual imagination of the shapes of elementary curves is desirable. Such information has decisive importance for subsequently finding such a function in an analytic form using quantitave modelling (approximation).*

Analytic modelling is based on analytic characteristics. The result of analytic modelling, an **analytic model,** is given in the form of elementary function. Then we speak about linear models, quadratic models,etc. or modelling by linear function, modelling by exponential function, modelling by hyperbolas, etc . Analytic modelling requires good knowledge of the real situation under consideration, otherwise problematic results may be obtained.

<u>Example :</u> *We will solve the task to find a linear model of demand function. A demand function reflects the law of decreasing demand . It is linear, i. e. it has the form* $Q = f(P) = aQ + b$*. As mentioned above, it must be a decreasing function. Hence*

$a < 0$ and (obviously) $b > 0$.

5. ABSTRACT MODELLING

Abstract modelling is a universal type of modelling. It concerns real situations (phenomena) whose common feature is the existence of „ internal structure" that is able to transform certain input quantities into output quantities. To grasp the behaviour of such phenomena mathematical system theory proved to be vital (e.g. Mesarovic,Takahara [10], Meznik[11]).

Abstract modelling is based on the utilization of mathematical systems theory tools. The result of abstract modelling, an **abstract model** , is given in terms of mathematical systems theory. There are not many key concepts of mathematical systems theory needed to be able to create simple abstract models. A **system** *S* is any subset of X x Y (cartesian product of sets), where X (**inputs**), Y (**outputs**) are nonempty sets. Let C be an arbitrary nonempty set and suppose there exists a partial function r: C x X into Y such that (x,y)∈ S if and only if there exists c ∈ C with y = r (c , x). Then C is said to be the **set of states** , r the **response** .

Example : *We will create an abstract model of a lift. First, we must state the"problem" which is to be solved for the lift. The natural "problem" is simply the usual operation and function of a lift, namely to meet requirements of persons to reach a selected floor. Our task is to describe the lift as a system. Consider n floors 1,...,n. Choose the set of floors as inputs, ie X = {1, ,n}; an input is realized by pushing the button corresponding to a selected floor. As outputs we choose the reaction of the lift, formally the number of floors the lift goes up (+) or down (-), i e. Y = {-n+1, -n+2,...,1,0,1,...,n-1}. Supposing there are no technical limitations, we can describe the lift as system S ⊆ X x Y, S = {(1,0),(1,-1),..., (1, -n+1), (2,1),(2,0),...,(-n+2),...,(n, n-1),(n,n-2),...,(n,0) } (notice, that for example (n,-1) does not belong to S). For the construction of the set of states realize what is determining for assignment different outputs to the same input. It can be seen that a distinguishing role is played by the set containing the information about the floor the lift was entered in, i e. C = { 1,...,n}. Then response r can be expressed in the form*

$$y = r(c, x) = x - c,$$

where c is entered floor and x is demanded floor. Of course, there may be other ways how to describe the lift as a system.

6. FINAL REMARKS

Mathematical potentialities at any educational level have two components- acquired knowledge and developed abilities. A good didactic of mathematics instructing should utilize methods that develop desired abilities simultaneously with the effective fixation of a required extent of knowledge. The development of abilities should have priority, because when abilities are desirably developed, then the effective fixation of knowledge may be gained., but not vice versa. Modelling belongs undoubtedly to the methods developing abilities, particularly reasoning, perception, space visualization and imagination, intuition, estimation of results, adaptibility, creativity, orientation in nonstandard situation and others. The use of modelling and its different types should respect many aspects. Teaching of mathematics will benefit from modelling on condition that its implementation will be natural, purposeful and perhaps nearly spontaneous and unconcious ; up to at least secondary level there is no need to speak about modelling as it is at all. When employing modelling for real-life problems solving or for illustrations taken from applications at secondary level, more caution is advisable. Material borrowed from friendly disciplines (physics, chemistry,etc.) does not address and attract students too much, because they claim extra mental effort (besides others). Theoretically simple, quite comprehensible and transparent situations are much more effective. Therefore the examples mentioned above for illustration are mostly of an „economy for all" nature. Of course, the usage of abstract modelling concerns higher educational levels. Engineering at doctorate level provides universal methods of how to grasp nonstandard situations. In a broader sense, modelling is encouraging the whole mathematics teaching process.

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