MATHEMATICAL MODELS AND HYPOTHETICAL REASONING WHEN STUDENTS APPROACH PROPORTIONALITY

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1. INTRODUCTION

Hypothetical reasoning in mathematical problem solving has been analysed in some interesting works, which put in evidence for instance the conditions for the production of this type of reasoning or the different functions it may have during the process of problem solving.

This report has the objective of giving some suggestions in the direction of these research themes also making reference, in particular, to one of the quoted works (Ferrari P.L., 1992). More precisely, this contribution has the following aims:

- to specify some mathematical models proposed by students while they are approaching two proportional problems, before the subject 'proportionality' is treated in class by the teacher;
- to show some examples of students' hypothetical reasoning which are present both in their individual protocols and while they are discussing among themselves with the goal to prove or to refute the models proposed in class as solution strategies for the given problematic situations;
- to stress different functions of hypothetical reasoning during the process of problem solving, in agreement with some of the typologies described by Ferrari (1992) and developing them further.

The study is based on didactical experiences (carried out with 12-13 year old students) about the construction of proportional reasoning.

To allow a better understanding of the core of this contribution, it is useful to start by saying something about the usual didactical transposition of the subject 'proportionality' and the innovative method adopted by our research group, which allowed the richness of students' argumentations (described in 3.) to come out.

In reference to the first of these points, it is well known that the didactical transposition of the subject 'proportionality'(at the age of 12-13) happens in most cases without any discussion of other different solution strategies (for instance additive strategies) which may occur spontaneously but which are not correct: usually the recourse to the constancy of ratios in proportional situations is not justified at all and it is used mechanically.

It is plausible, also as a consequence of that, that the mastery of proportional reasoning is not satisfactory, at the conclusion of its study (non standard problems reveal it, as described for instance by Gagatsis et al., 1996).

On the basis of these considerations, we have planned a didactical proposal which aims at approaching proportional reasoning through appropriate problems where the recourse to the constancy of ratios has to originate as necessary, against different solution strategies which may appear natural but, indeed, are not adequate.

The discussion conducted properly by the teacher, who did not take any position with respect to the knowledge involved, was fundamental during the whole process of comparing different strategies. In other words, the discussion was conducted following the so called a-didactic modality (Brousseau, 1986). In addition, the teacher played different roles, according to different didactical phases: he (she) coordinated the discussion, solicited pupils for explanation, stressed different positions, promoted peers' verbal interaction (Arzarello et al., 1996).

Therefore the dialogue and the negotiation of meanings during mathematics class had a central role, according to constructivist principles sustained by Bauersfeld (1995), Cobb,

Yackel, Wood (1992) and Ernest (1995).
More details about the ideas on which the didactical proposal was planned and references to the literature on proportional reasoning, can be found in Pesci A. (1998a).
The experimentation, conducted with students aged 12 to 13, was the object of study for three consecutive years and widely described in three degree theses in mathematics (Valenziano, 1994, Castagnola, 1995, and Torresani, 1997).
The experiences carried out showed that the positive atmosphere established in class improved the development of students' argumentation, in particular of the hypothetical type, as it will be described at point 3.

2. MATHEMATICAL MODELS ON STUDENTS' PROTOCOLS

During the first part of the didactic itinerary, for the first worksheet, which was aimed at testing spontaneous recourse to the use of the ratio concept, we decided to use a context which was quite familiar to the students, not uncommon in school tests and frequent in literature: the mixing of colours. The following is the text which appeared on the individual worksheet:

1

Three panels of different dimensions have to be painted and equal size tins of yellow and blue are available. The panels have to be painted the same shade.

MARCO painted the first panel using a colour obtained by mixing 4 tins of blue and 6 tins of yellow.

LUISA has to paint the second panel: to obtain the same shade of the colour and with 6 tins of blue available, how many tins of yellow does she need?

PIERO has 3 tins of yellow for the third panel. How many tins of blue does he need?
Explain your reasoning in answering the questions:

For LUISA ....
....

For PIERO ....
....

Students had to face the problem using intuitive strategies which could prelude proportional reasoning: some of them could appeal explicitely to the constancy of ratios between the given quantities. In any case, from the answers on this worksheet the teacher derived information about the different cognitive levels in the class group. This was therefore an introductive worksheet, whose aim was that of promoting a profitable discussion.
The following results were obtained from our experimentation classes:

- the majority of the students applied the criteria of the "constant difference", that is, they maintained in every case the number of yellow tins equal to the number of blue tins plus 2. Therefore, they said that Luisa needs 8 tins of yellow while Piero needs 1 tin of blue;
- some students kept the total number of tins (10) to be used constant: Luisa needs 4 tins of yellow and Piero needs 7 tins of blue.
- only a minority of students made recourse to intuitive strategies dealing with proportional reasoning. Some observed, for instance, that Luisa's blue tins (6) are one and a half times the amount of Marco's (4), therefore it has to be the same for the yellow tins: Marco has 6 tins, therefore Luisa has to have 9. In the case of Piero the situation is simpler: he has 3

yellow tins, half the number of Marco’s, so he also has to have half the number of blue tins, that is 2, half of 4.
On students’ protocols there were therefore three different mathematical models: the first could be named ‘additive’, the second refers to the ‘constancy of the total’, the third is of ‘proportional’ type.

The second problem which promoted interesting students’ hypothetical reasoning during the collective discussion, was presented in the fourth individual worksheet, in the following version:

4

Now complete the following table in such a way that the difference between the number of matches played and matches won by each player is 30.

	Ada	Aldo	Bice	Enzo	Anna	Ivo	Gino	Emi
Matches won	2	...	30	...	...	10	...	...
Matches played	32	...	...	64	...	...	...	100

Based on the data in the completed table, can you consider all the players "equally good"? Justify your answer.

Before the description of the mathematical models present on pupils’ protocols, it is necessary to say something about the work developed in class between worksheet 1 and worksheet 4, to understand what the students’ cognitive situation was.
The collective discussion which followed the first individual work had the aim of bringing to light the different strategies used spontaneously by the students and also the reasons behind the choice of these same strategies. In any case, at this stage it wasn't revealed which strategy was the most appropriate one; the discussion was restricted to bringing forth the students' ideas by stimulating verbalisation.

The itinerary proceeded dealing with a new problematic context: tennis matches (worksheets 2, 3 and 4, presented to pupils one at a time). In worksheet 2, having given the students the number of matches played and won by each of four players, we asked them which one was the best. Also in this case we were dealing with an explorative situation. In order to solve it, it was not necessary to use ratios; in fact, the numerical data were very simple: only one of the given players won more matches than he lost, therefore he was the best. The situation was deliberately presented in such a way as to allow different resolutive strategies to be used. Nevertheless, through the students' answers, the teacher could immediately identify if any of them spontaneously resorted to a comparison of ratios, possibly also as a result of the discussion held.
Usually, the results obtained showed a significant improvement compared to the first problem.
The successive worksheet 3 showed a table to be completed with the missing data, (referring as before to the number of matches played and won by each player), in such a way that each of them could consider themselves equally as good as a particular player who had already been assigned with a numeric value for matches both played and won. In this case the use of proportional reasoning was necessary to obtain the correct answer, and therefore this workseet highlighted, more than the previous one, who applied it.
In our experiences, to solve the task the majority of students correctly used a "multiplicative law", whereas the others were wrong, using an "additive law".
At this point of the itinerary, before any discussion about the workseets 2 and 3, number 4 was

proposed, that is the worksheet reported earlier.
It aimed at making the students reflect on the use of constant difference criteria to judge the ability of a player, criteria which emerged in the completion of the previous chart. We thought of "forcing" the whole class to reflect on the possible use of difference instead of ratio, having established in previous experiments that the discussion of the difference method produces good results and aids reflection. This is in accordance with didactical research which stresses the importance of a positive use of students' errors (Borasi, 1996) .

In reference to the results obtained with the workseet 4, a good number of students (often the majority) said that the players are not equally good and some of them declared that it was necessary to refer to ratios; the others said that the players are equally good because they have lost the same number of matches.
There were therefore two different mathematical models (one 'multiplicative', the other 'additive') about which discussion was necessary.
In the following point some examples of students' hypothetical reasoning in reference to the problems proposed in workseets 1 and 4 are described, together with the different functions which have been pointed out.

3. DIFFERENT FUNCTIONS OF HYPOTHETICAL REASONING

In this paragraph three typologies of functions covered by students' hypothetical reasoning are described: they have been stressed, as said before, by P.L. Ferrari (1992) who analized a long term experience where the verbal language, as in our experience, had an important role.
These three typologies of hypothetical reasoning were crucial in the development of students' skill in describing and discussing their models of solution. We believe that in particular the third of them ('change of data'), which was the most frequent during collective discussions, allowed the students also to understand what 'to check a model to solve a problem' means.
For each of these typologies the most meaningful examples are presented, along with some comments from the point of view of the involved mathematical models.
Students' argumentations are excerpted both from protocols and from collective discussions, as specified in each case.

A) EXPLORATIVE

Sometimes hypothetical reasoning is explorative and it helps the students to arrive, step by step, at the solution. It happens that "the pupils put themselves in a particular case and try to draw some conclusion; this may be useful ... to make explicit some relationship involved in the problem situation" (Ferrari, 1992, p.131)

Examples

" *If the panel of Marco is 2 m long, for 1 m he uses 2 tins of blue and 3 of yellow. Luisa uses 6 tins of blue, it means that her panel is 3 m long. I know that for 1 m, 3 tins of yellow are needed, therefore for 3 m, 9 tins are needed.*
Luisa has to mix 9 tins of yellow".
(PROTOCOL - Alessandra)
It is clear that the recourse to the hypothetical measure of Marco's panel helps Alessandra in the individuation of the fundamental couple (2,3) which represents the basic relationship between the given numbers of blue and yellow tins, and allows an easy solution.
"*..For instance, if you have a panel twice the size, you will also use twice the amount of paint: that is, 4x2 tins of blue and 6x2 tins of yellow.*"
(DISCUSSION - Paolo)
Paolo presents to the schoolmates his intuitive proportional reasoning: it seems plausible that with a hypothetical panel which is twice the given size, you must use twice the quantity of paint. The duplication is indeed the simplest case of proportionality and the first step towards more general cases.

B) PLEONASTIC

It may happen that “in some situations, mainly in complex problems or in problems involving proportion or comparisons, there are children who regard the data as hypothetical even though they are given without conditions.

... One may conjecture that pupils use pleonastic ‘if’ to insert the data in a sort of complex elaboration structure which allow them to focus on data and, at the same time, on some inferential steps they want to point out” (Ferrari, 1992, p.132).

This is also a common situation for experts: pleonastic ‘if’ is often only the first link of a chain of successive argumentations.

Examples

“For Luisa 9 tins of yellow are needed, because if she has 6 of blue, which are one and a half times those of Marco, she must have 9 of yellow, because they are one and a half times those of Marco.

For Piero 2 tins of yellow are needed, because if he has 3 of yellow, which are half those of Marco, he must have 2 of blue, because they are half those of Marco”

(PROTOCOL - Giorgio)

Under the conditional ‘if’, Giorgio puts data which instead are given, but this form of argumentation is only the first step of his elaboration, produced with the scope of underlining the relationship between 6 and 4 (or between 3 and 6 in the case of Piero) and allowing him to get the solution.

It is also possible to find wrong solutions which use pleonastic hypothetical reasoning, always with the aim of grasping a mathematical regularity in the given situation: in the following quotation an additive relationship between 4 and 6 (6=4+2) is indeed underlined and the same relationship is imposed to find the results: it must be said that looking only at numbers, this additive relation is correct, but it is indeed not adequate as a model for the given situation (all of which has to be the object of the collective discussion).

“ In my opinion Luisa must use 2 more tins of yellow, because if Marco has used 6 tins of yellow, that is 2 tins more to obtain the green, Luisa also, to obtain the same shade of colour, must add 2.

In my opinion Piero must use only 1 tin of blue to obtain a shade of green equal to that of Luisa and Marco, therefore if he has 3 tins of yellow, 3 - 2 = 1, 2 tins of blue are subtracted and 1 remains.”

(PROTOCOL - Valeria)

C) CHANGE OF DATA

Sometimes, when it is necessary to check the validity of a solution strategy, “conditional forms are used to deal with the same problem situation with different initial data. ... This use of conditional forms is closely related to the emergence of the alghoritm as an autonomous object, separated from the data on which it is performed” (Ferrari, 1992, p.131)

In these cases, the function of hypothetical reasoning is that of discussing one of the mathematical models developed in class. What follows refers to the two main models proposed by the students, the ‘additive’ and ‘proportional’ models.

Examples

“I want to say something interesting.

Imagine you have to paint a house in the same shade of green. If you use 1000 buckets of blue and 1002 of yellow, do you think you’ll get the same shade of green ?

(DISCUSSION -Alessio)

Alessio is contrasting the validity of the ‘additive’ strategy and his contribution is of a theoretical nature: he wants to convince his schoolmates that when dealing with a large quantity of paint and the difference between blue and yellow remains two, the quantity of each colour is almost the same, so it is impossible for the final colour to be the same as that obtained from 4 tins of blue and 6 of yellow.

"<u>If you have 2 tins of blue and 2 tins of yellow</u> you get one shade of green. If you add another 2 tins of yellow you get a lighter green, almost yellow.
But <u>if you have a lot of green</u> and you add 2 tins of yellow, you don't get such a yellowish green, but the original green colour remains almost the same."
(DISCUSSION - Paolo)
What Paolo means is that 2 tins of yellow has much less of an effect on a lot of green than on the mixture of 2 tins of blue and 2 tins of yellow.
His intervention is in accordance with Alessio's argumentation: they both stress that the effect of 2 tins of yellow on a lttle quantity and on a big quantity of the same mixture cannot be the same.
"I would say to Giovanni (who is convinced that the strategy of the 'constant difference' is right)*: you say that, in order to have the same shade, the yellows have to be always 2 more than the blues.*
But <u>if there are 2 yellow tins</u>, there have to be 2 less blues, that is, 0. It is not possible to obtain the same shade: yellow comes out. Therefore the same difference does not work."
(DISCUSSION - Paolo)
This time Paolo makes recourse to a different and interesting argumentation to contrast the 'additive' strategy and his explanation is very clear.
It is interesting to note that the types of argumentations proposed by the students during the collective discussion are the same that an expert could use to test the validity of a supposed model: that is to change data, from the lowest up, and to think (it is not necessary to try in practice!) of the validity of the model itself in reference to the considered context.

The following quotations are excerpted from the same collective discussion: they show further use of hypothetical reasoning with the function of changing data, this time in reference to worksheet 4.
The meaning of the expression 'to be equally good' is the object of discussion:
"<u>Greta</u>*: In my opinion 'to be equally good' means to lose the same number of matches.*
<u>Paolo</u>*: I don't agree with Greta: it is different <u>if you play two matches and you lose one</u> of them and <u>if you play 20 matches and you lose one</u>.*
Also in this case, Paolo is contrasting the additive strategy. He proposes again, in this new context, his same type of argumentation: one match lost weighs differently on a total of 2 or 20 matches played, therefore the criterion to look only at the matches lost cannot be valid.
During the following discussion Greta says clearly that the example used by Paolo has convinced her that her criterion is no good.

<u>Alessio</u>*: I say: how is it possible for two players to be equally good when the first plays 70 matches and wins 20 of them and the other plays 100 matches and wins 50 ? And <u>if one plays 50 matches and wins 0</u> ?*
The recourse to the extreme case with 0 emerges again, as in the previous problem.
<u>Pamela</u>*: In my opinion Marco* (who says that the players are equally good) *is wrong because if a player has played 32 matches and has won 2 matches, the difference is 30, the matches lost are more than the matches won.*
But <u>if a player has played 80 matches and has lost 30 matches</u> it means that the matches won are 50 and in this case the matches won are more than the ones lost.
The argumentation proposed by Pamela is different but efficacious: she wants to stress that one player, who wins more matches than he loses, cannot be considered of the same ability of

a player who wins less matches than he loses, even if they both have the same number of matches lost (that is the same difference between played and won matches).

4. FINAL OBSERVATIONS

In agreement with the literature, the problems which are new and in which students' experience can orient and give suggestions, seem the most suitable to promote hypothetical thinking. When an algorithm is available to solve a problem, hypothetical reasoning does not appear.

In the experiences described, the two problems proposed to students were new for them, because the subject 'proportionality' had not yet been presented in class by the teacher: different solution strategies emerged spontaneously and a comparison discussion was necessary.

Hypothetical reasoning was much more frequent during the collective discussion than on the protocols, with a frequent and interesting use of 'change of data'. This means that the comparison of different strategies forced argumentations in favour or against a strategy, therefore promoting the development of conditional forms.

The most interesting argumentations from the mathematical point of view (the 'mature' use of 0 and of large numbers, of examples and counter-examples) were present under the form of hypothetical reasoning, which seems therefore the most adequate in discussing mathematical models in problem solving.

Furthermore, these types of argumentations were proposed by medium-high level students, but they had great influence on schoolmates. Even if it is not very clear from the quoted students' interventions, it happened often that someone re-proposed the same types of hypothetical reasoning made by schoolmates, especially in reference to the use of zero described before.

It is surely not possible to conclude that problem solving activities allow the acquisition of hypothetical reasoning: this type of reasoning is already mastered in particular by students who propose the most efficacious argumentation. But we believe that the described didactical situations could be considered good opportunities to refine, extend and develop a fruitful use of hypothetical reasoning.

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