Exploratory Math Modules for Classroom Practice Through Manipulation

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Abstract

The mathematics education community finds itself in the midst of a peculiar time where a constant evolution of mathematics curriculum is present. Within the recent reform efforts, and after about ten years of publishing and attempting partial implementation of the NCTM *Curriculum and Evaluation standards* different calls of opposition have emerged ranging from a call for "reform from within" to calling for a return to "Back to Basics." In either case, the main concern here is to find ways/approaches to develop curricula that are not too extensive and are within the expertise of the K-12 teachers. In this paper a series of modules, based on exploration and hands-on manipulation, will be outlined. An exemplary module will be presented focusing on exploring basic spatial properties of 2D shapes through shape-decomposites and shape-composites formations. In particular, interrelationships among shapes (whole-to-whole), shapes and their parts (whole-to-part), and among the parts of shapes (part-to-part) will be the focus of the module.

Introduction

Frequently students exhibit a variety of levels of geometric understanding. They display this divergence of understanding in many different ways. Some claim to already know how to prove a given proposition; others may not even have a clear idea what a polygon is. Likewise, there are those who are doubtful about under what conditions a parallelogram is a rectangle, whether or not a parallelogram is a trapezoid, and so on. In the midst of this classroom confusion, and in spite of the many written documents addressing the issue, one may well ask why it still persists. Are some students simply more "mathematical" by nature, and others simply lacking mathematical understanding? Whatever an answer might be, the main question remains: Is there an effective way/approach that is both straightforward and uncomplicated to deal with this phenomenon?

Pierre Marie van Hiele (1989) identified five levels of reasoning which students go through in dealing with geometric concepts and figures. The van Hiele research indicates that these levels (identified as Levels of Thought Development in Geometry) are not biologically achieved during a person's maturation; they can only be achieved by instruction and should be learned in their proper order: Levels 0, 1, 2, 3, and 4. The progress through these levels seems different from that of Piaget levels. As we know from psychology, the Piaget Levels of Cognitive Development occur naturally and progressively during biological maturation. A description of van Hiele five levels can be found in several NCTM publications such the *Mathematics Teacher* (see Shaughnessy & Burger, 1985, p. 419-28) and the *NCTM 1987 Year Book* (see, M. Crowley, 1987, p. 1-16). A description of these levels may be helpful. Level 0: students identify and operate on shapes in their global appearances (holistic); Level 1: students recognize shapes by their properties (part-whole); Level 2: students recognize relationships among properties and shapes (part-part; and, whole-whole); Level 3: students understand the deductive reasoning process; Level 4: students can work in different axiomatic systems.

The NCTM's *Standards* stated that the *spatial sense* is an intuitive feel for the individual surroundings and the objects in them. To develop spatial sense, children must have many experiences that focus on geometric relations; the direction, orientation, and perspectives of objects in space; the relative shapes and sizes of figures and objects; and how change in shape relates to change in size. In particular, the *Standards* state that when children investigate the result of combining two shapes to form a new one, predict the effect of changing the number of the sides of a shape, draw a shape after it has been rotated a quarter or a half turn, or explore what happens when the dimensions of a shape are changed, they acquire deeper understanding of shapes and their properties. These type of activities promote spatial sense (1989, p. 49).

The purpose of this paper is to create a piece of curriculum (a module) for the classroom practice that would meet the NCTM's recommendations described above and to offer experiences through exploring basic spatial properties of 2D shapes utilizing shape-decomposites and shape-composites formations. In particular, interrelationships among shapes (whole-to-whole), shapes and their parts (whole-to-part), and among the parts of shapes (part-to-part) will be the focus of the module.

Modules

The mathematics education community finds itself in the midst of a peculiar time where a constant evolution of mathematics curriculum is present. Within the recent reform efforts and after about ten years of publishing and attempting to partially implement the NCTM *Standards* different calls of opposition have emerged ranging from a call for "reform from within" (Bosse, 1998) to calling for a return to "Back to Basics" (Mathematically Correct web site, 1998). In either case, the main concern here is to find ways/approaches to develop curricula that are not overly extensive, and are within the expertise of the K-12 teachers.

A module is a specific teaching-learning environment consisting of a sequence of spatial hands-on manipulations using shape-decomposing and shape-composing operations. The shape-decomposing operation is based on the Theory of Dissection, for details see Eves (1972). The shape-composing operation is based on the basic transformations or motions which are primarily translation (slide), rotation (turn), reflection (flip), or any combination of them. The transformations are described in several middle school contemporary text books such as Alexander, et al. (Mathquest 7, 1988); Ebos, et al. (Math In Context 9, 1993); and Knill et al. (Mathpower 9, 1994). The combination of the two operations, shape-decomposition and shape-composition, is the prime concept here. Mathematically and in a Euclidean sense, this combination allows us to **extend** the concept of congruence from **rigid shapes congruence** (same intact shapes) to congruence by pieces (different shapes) thus opening a wider range of interesting possibilities. The writer has been continuously working on some of these possibilities for about a decade and has found an appreciation at both the classroom practices and professional gatherings. For details see Rahim (1986); Rahim and Sawada (1986 & 1990); Rahim, Sawada, and Strasser (1996) and Rahim and Olson (1998).

Several Modules have been developed on this combined operation focusing on 2D shapes covering triangles and quadrilaterals. Below is a detailed discussion of a module based on a square shape region.

The Square Module

- Consider the shape of a square drawn on a plain sheet of paper.
- Cut off the square region.
- Choose the midpoint of one of its sides.

- Join the midpoint with one of the opposite vertices by a line segment.
- Decompose (dissect) the square along the line segment and denote the two resulting pieces as 1 and 2. The resulting shape-decomposition is illustrated in Figure 1.

Dissecting ABCD leads to an inquiry of identifying the shape of each of the resulting pieces, 1 and 2, as well as their properties (part-to-whole relationships). This inquiry would create an opportunity to raise questions on the type of the triangle and the type of the trapezoid shown in Figure 1 (properties of shapes).

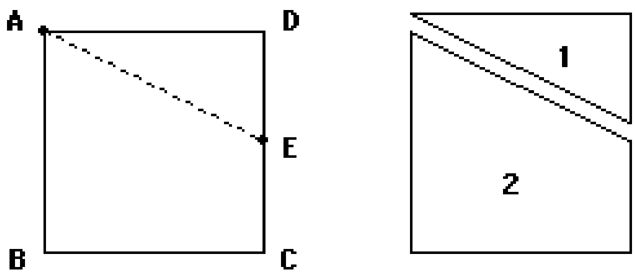


Figure 1

Progress into the Module Through a Variety of Activities

Activity 1: Composing a Right Triangle Region

Place the two pieces, 1 and 2, on the desk and compose the original square region. Apply as you wish, a slide, turn, or flip, or a combination of them on piece 1, 2, or both so that, without overlapping the pieces, you would compose a triangular shape. Do not lift any of the pieces completely off the desk.

A composed triangular shape is shown in Figure 2a.

Figure 2a presents a situation focusing on basic aspects and properties of the motions and the shapes involved - the original square, the resulting shapes, 1 and 2, and the resulting composed shape. For example:

- Justify that the resulting shape is a right triangle.

This would bring into classroom practice a situation focuses on the required reasons for the points F, I, H and the points G, J, H to be collinear and that the composed triangle is a right angle (see Figure 2a). Such requirement, in turn, would lead to recall basic properties of the three motions. Below (Figure 2) is a highlight of such properties:

- Highlight the basic properties of the motions used.

(i) The measures of the parts of a shape are invariant under each of the motions - the shape is preserved;

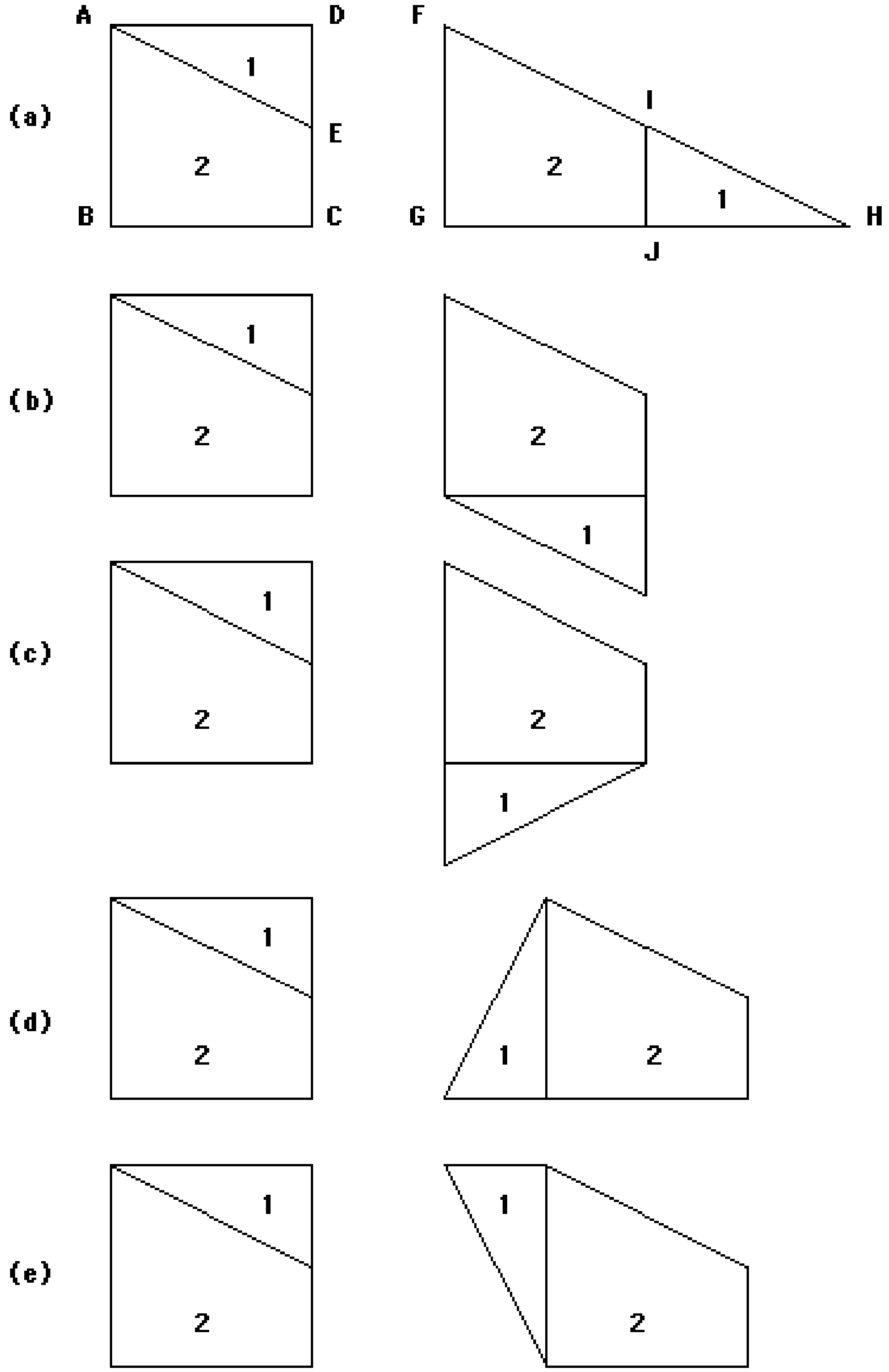
(ii) in general,

1 under the motion of a slide, the corresponding sides of a shape and its image are parallel;

2 under a 90° rotation, the corresponding sides of a shape and its image are perpendicular;

3 under a 180° rotation, the corresponding sides of a shape and its image are parallel;

4 under a 270° rotation, the corresponding sides of a shape and its image are perpendicular;


(a)
A
D
F
I
E
B
C
G
H
J
1
2
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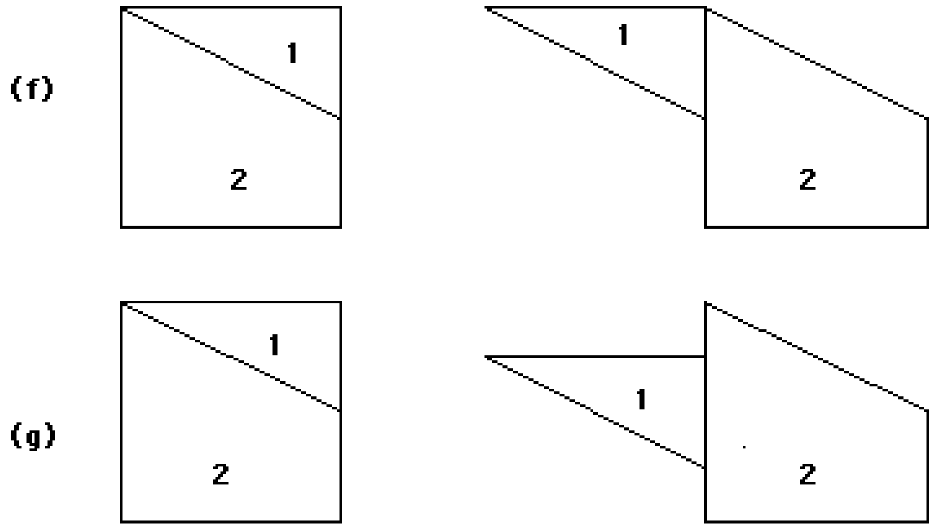

Figure 2

5 under a reflection, the line of reflection is the perpendicular bisector of the line joining any point of a shape and its corresponding image;

(iii) the relationships among the motions such as

1 two consecutive reflections about two parallel lines make a slide; and

2 two consecutive reflections about two perpendicular lines make a 180° rotation; and so on.

The following is an exemplary justification that the resulting shape shown in Figure 2a is a right triangle:

In Figure 2a,

a 180° turn of piece 1 to the right (clockwise) about the point E

---> ABCD is transformed into the shape FGH;

FIH is a straight line [half turn];

GJH is a straight line [DE = EC, < C = < D = 90°, & a half turn]

---> the shape FGH is a triangle

---> _ FGH is a right triangle [< G = 90°].

Another point for discussion would be to explore whether or not the areas of the square ABCD and the triangle FGH are equal; and whether the perimeters of the two shapes are necessarily equal.

Activity 2: Composing a Parallelogram Region

Reassemble pieces 1 and 2 back into the original square shape. In a similar way of Activity 1, compose a parallelogram shape. Try to justify that the resulting shape is a parallelogram.

A composition of a parallelogram shape is shown in Figure 2b.

A justification that the resulting shape in Figure 2b is a quadrilateral is required first; then a discussion of the conditions for a quadrilateral to be a parallelogram would be made similar to that of Activity 1. In each case throughout the above compositions there could be more than one way to arrive to the desired composition. For example, in Figure 2b, starting with the square shape one may apply a slide downward on piece 1 or alternatively a slide upward on piece 2 and the desired composition follows. A combination of more than one motion is another possibility: for example, starting with the original shape, make a triangular shape as in Activity 1 and then turn piece 1 to the right a half turn about J.

The question of the relation between the area of the two shapes would be raised here. Correctly recognizing the relationship between the two areas is based on the understanding that nothing has been lost or added to the amount of the total size of the original shape by the shape-decomposition and shape- composition operations.

Activity 3: Composing a Trapezoid Region

Starting with the original shape, use pieces 1 and 2 to compose a trapezoid shape. Justify that the result of your composition is a trapezoid.

Figure 2c shows a possible composition.

A discussion analogous to that of Activity 1 can be made here focusing on the basic properties that a quadrilateral is a trapezoid. Differences among the newly composed shape and the parallelogram in Activity 2 would be highlighted. It is of most important to highlight to the students whether or not a parallelogram is a trapezoid and vice versa. These lattice type interrelationships among the family of quadrilaterals are often unclear to students.

In a similar fashion, the rest of the activities may be introduced.

Activity 4: Composing a Quadrilateral without Parallel Sides

Starting from the square shape, use pieces 1 and 2 to assemble a quadrilateral with no parallel sides.

Figure 2d shows a composition.

Activity 5: Composing a Pentagon

Starting from the original shape; assemble a five-sided shape, a pentagon. Show your reasons why your shape is a pentagon.

Figure 2e represents a possible composition.

Activity 6: Composing a Hexagon

Reassemble the original shape; compose a shape of a hexagon.

Figure 2f shows a possible composition.

Activity 7: Composing a Heptagon

Reassemble the original shape; compose a shape of a heptagon.

Figure 2g illustrates a composition.

Activity 8: A Return Journey

In the above activities, the original square shape was set up to be the starting point. In this activity, one may choose, for a change, to continue from the newly composed figure onward. In a sense, we would have a journey in the space of polygons. But then we may get tired and think of returning back home after all; thus the journey would then have to be a return one.

Throughout the following steps, make a journal of your actions, and a full record of what you would have used.

1 Using the pieces 1 and 2, reassemble the square shape; and then compose a right triangular shape;
2 starting from the triangular shape, compose a parallelogram;
3 from the parallelogram, compose a trapezoid;
4 from the trapezoid, compose a quadrilateral with no parallel sides;
5 from the quadrilateral, compose a five-sided figure - a pentagon;
6 from the pentagon , compose a six-sided shape - a hexagon;
7 from the hexagon, compose a seven-sided shape - a heptagon, and finally;
8 assemble back the original shape.

Figure 3 shows a resemblance of such a journey.

Exercise 1

Review the results of the above compositions. Try to find out the maximum possible number of sides for a shape that you can assemble using pieces 1 and 2 with no overlapping. Further, try to find an algebraic rule for the maximum number of sides that a composed shape can possibly have.

Exercise 2

Throughout the following shape-decomposition and shape-composition, you may use horizontal, vertical, or oblique dissection with no overlapping and the area is preserved:

1. Start with a right triangle piece of plain paper; try to decompose it into whatever pieces you wish (try two pieces first) so that you would be able to compose a rectangle. In so doing, you may end up with a square rather than a rectangle depending on the characteristics of the right triangle you have; if so, identify the specific right triangle that will yield a square.
2. Start with a parallelogram piece of plain paper; try to decompose it into pieces so that they can be composed into a rectangle.
3. Do the same as in (2) with a right angle trapezoid piece of paper.
4. Challenge: do the same as in (2) with a non-right angle trapezoid piece of paper; (you may need to decompose it into 3 pieces).
5. Challenge: try to find a way to transform any rectangle into an area equivalent square.

Conclusion

Most children, even before they begin school, can recognize and name some simple geometric figures. In North America, and more likely elsewhere, in early grades the curriculum does little to take children beyond this level because the inter-relatedness of simple polygons is rarely the focus of classroom practices. The problematic nature of modern school geometry has been identified earlier at various school levels (see Piaget, 1962, Wirszup, 1976, and Mayberry 1983). The second achievement results from the Third International Math and Science Study (TIMSS) released on June 10, 1997 and earlier in 1996 were also not encouraging. This unfortunate situation is due in part to the scarcity of a recognized mathematical process available to youngsters for expressing inter-relatedness among shapes. Therefore, this paper has intended to:

present a class of mathematical processes that are uncomplicated, within the domain of K - 12 teachers' expertise, and based on decomposing simple shapes into parts and composing the parts into new shape formations while keeping certain fundamental attributes of the figures intact, thus offering a process for developing spatial sense and understanding of basic properties of 2D shapes through exploring their interrelationships.

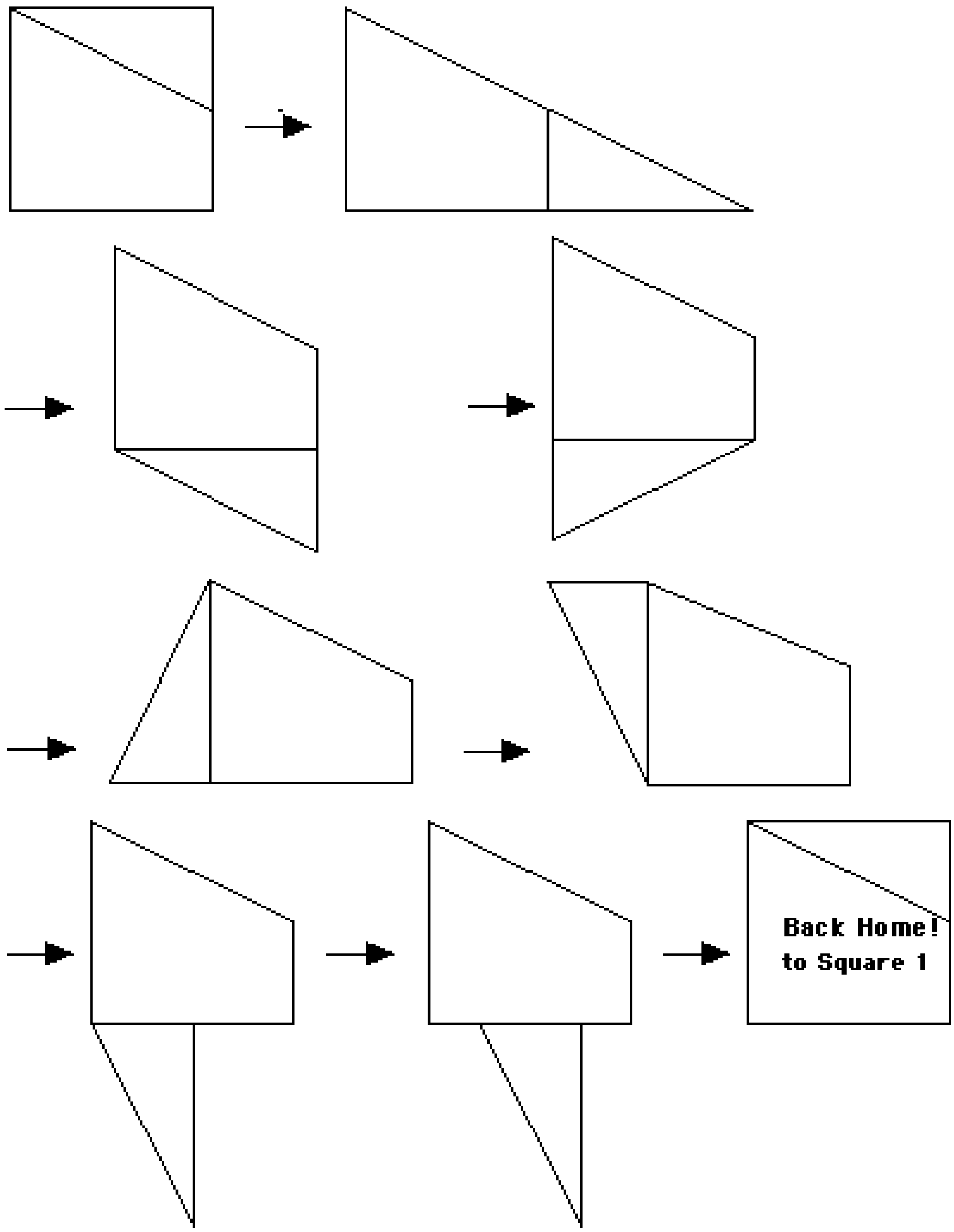

Figure 3

A Journey among Polygons

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