About Students' Understanding and Learning the Concept of Surface Area

Cinzia Bonotto
Department of Pure and Applied Mathematics, University of Padua, Italy, bonotto@math.unipd.it

Abstract The research we are about to describe compares two methods of teaching the concept of surface area. In a fifth-grade elementary-school class, we analyzed a traditional method, based on the acquisition and application of the formula for the calculation of an area. In a fourth-grade class, we applied a more innovative method based on "mathematization" of real-world experiences. We consider that it is important to immerse children in a classroom culture that focuses on the importance of realistic mathematical-modeling activities, i.e., of both real-world based and quantitatively constrained sense-making (Ruesser & Stebler, 1997). For this to happen, two changes are necessary: one is in the teacher's attitude to mathematics, the other in the classroom socio-math norms in the sense of E. Yackel and P. Cobb, 1996. Our studies, of which are going to present a paradigmatic example, take into account these factors.

N.1 Introduction

The Research Centre in Mathematics Education at the University of Padua (IT) has been working on the ideas expressed by the Italian Programs for the primary schools and has been developing new approaches to the mathematical concepts that were the objectives of the curriculum. In particular since 1993/94, it has been implementing some elementary school activities to recover in the classroom contexts knowledge and techniques usually developed outside school.

Having recognized that individuals usually acquire greater competence in mathematics reasoning outside of school than inside, we propose to introduce in school some of the conditions that make out-of-school learning more effective.

Traditional classroom teaching often seems to favor the rift between classroom and real-life experience. The tendency to exclude real-world knowledge, and therefore related considerations, from the scholastic environment (cf. J. Lave, 1995; B. Greer; 1997, K. Reusser, 1997) has been observed not only among pupils, but also on the part of teacher-trainees. One study carried out by L. Verschaffel, E. De Corte and I. Borghart, 1997, showed that in their scoring of tests, teachers-in-training considered wrong those answers that were drawn from realistic considerations and were not based on standard procedures.

These observations are particularly significant with respect to the concept of surface. Outside of school, any child is capable of recognizing a real surface. The child distinguishes between it and a line or a number, considering it as a plane space. In school, though, pupils tend to identify surface and area, that is, considering surface only at a numerical level. We think that one reason for this may lie in the fact that not much importance is given to the concrete act of measurement. In the case of surface, the fact that measuring does not take place impedes children in their understanding of the meaning and role of square units. It is also an obstacle to comprehending why a formula is used to calculate a surface area, while an instrument is used to measure a line (cf. C. Bonotto and M. Maddalosso, 1997; M. Basso, C. Bonotto and P. Sorzio, 1998).

The importance of taking direct measurements came out clearly in a study carried out by T. Nunes, P. Light, J. Mason, 1993. Children were asked to evaluate which was the bigger of two non-superposable surfaces, and they were allowed to use measuring instruments like a ruler or area unit (small square blocks). The results showed the strong correlation between the number of correct answers and the application of a measuring strategy based on a count of area units. Thus, the connection between the unit of measure used and the surface that was measured became clear. Another interesting observation comes from research done by L. Outhred and M. Mitchelmore, 1992, which highlights the difficulties children have in finding the right relation between measurement and comparison of different figures. Not just the use of an area unit is important. It is also important to be able to recognize the way the units are disposed: if the formation is analyzed in terms of rows and columns, one is able to explain the formula which is in fact used to calculate the area of a figure.

In Italy's current New Ministerial Program for elementary schools, the two activities of measuring and comparing lengths, area and period of time are considered part of the same subject. The two, however, require different levels of abstraction. Comparison involves two or more figures and is performed through the analysis of the figures' aspect and perhaps some of their properties. Measurement involves observing a single figure against a given sample unit. We speak of direct measurement when the dimensions are determined by counting (e.g., using squares as a first step in approaching the formula), of indirect measurement when the formula is applied, that is, when the size of a figure is derived from the measurement of linear dimensions (cf. C. Marchini, 1999).
In the research described here, we compared two methods of teaching the concept of surface area. We examined a traditional method, based on the acquisition and application of the formula in a fifth-grade elementary class; we ourselves, in a fourth-grade class, introduced a more innovative system based on "mathematization" of out-of-school, real-world experiences. While the fifth-graders simply learned to apply the formula, the fourth-graders learned to derive it, counting how many area units (small squares) there were in a real surface which they could easily handle (a sheet of paper). This article shows the different results obtained from the point of view of comprehension of the new concept.

N.2 Framework

In common teaching practice, connecting classroom mathematics activities with reality generally is done solely through word problems. Word problems represent the interplay between mathematics and reality, and they are actually the only example of realistic mathematical modeling and problem solving used in school. And yet, during the past decades, a growing body of empirical research (e.g., Freudenthal, Greer, Reusser, Schoenfeld, Verschaffel and De Corte) has documented that word-problem solving as practiced in school mathematics hardly matches the idea of mathematical modeling and mathematization.
Instead, we deem that is important to immerse children into a classroom culture that focuses on the importance of activities of realistic mathematical modeling, i.e., both real-world based and quantitatively constrained sense-making. For this to happen, two changes are necessary: one in teachers' attitudes to math, and one in the norms of student-teacher relationships, or socio-mathematical norms, in the sense of E. Yackel and P. Cobb, 1996, or K. Gravemeijer, 1997. Our studies, presented here by way of a paradigmatic example, take into account these factors. It is characterized by the use of selected cultural artifacts, objects that incorporate mathematical elements that a person encounters in everyday situations (a receipt from a supermarket, or the label on a notebook, etc.; cf. Bonotto, 1999 and 2001). Material like this is particularly meaningful because through it, students learn to analyze and interpret the reality around them in mathematical terms. In our experiments, pupils were introduced to mathematics in such a way as to make it easier for them to move from situations in which math is normally used to the underlying mathematical structure, and back, from the mathematical concepts to the real-world situations, according to "horizontal mathematization" in the sense of Treffers (1997).
The use of cultural artifacts can serve further purposes as well. With some changes, like the partial removal of data, the objects can become real mathematization tools capable of

- creating new mathematical goals
- developing new mathematical knowledge, as a stepping-stone to launch, at a first stage, new concepts
- providing pupils and students with a basic, hands-on experience in mathematization.

In this new role, the cultural artifact can be used to introduce new mathematical knowledge through those special learning processes that Freudenthal, quot., defines as "*anticipatory learning" or "learning by advance organizers"*. In the research described in this paper, the cultural artifact we chose to use was the cover of a loose-leaf ring-binder containing sheets of

graph paper.
The objective, then, apart from improving the effectiveness of mathematics education, is also to present mathematics in a new light, by changing both teachers' and pupils' common behaviour and attitudes towards school mathematics.

N.3 First study

Subjects, material, research methods

The first study was carried out in the second quarter of the 1998-1999 school year. It took place in the fifth grade of a primary school (Trebaseleghe, Padua), with 20 pupils. The class had already dealt with the concept of surface and had used the game of Tangram; square measures had only just been briefly mentioned.

Three experiments were led, each lasting about two hours. The teacher of logico-mathematical areas was present throughout, along with two researchers. Each session used photocopies of the cover of a rectangular ring-binder of a kind currently sold in stationery stores. Each pupil was given a photocopy. The protocols containing the children's answers were given to the researchers at the end of each experiment; all the discussions were recorded.

In the first experiment, after a brief introductory discussion during which the class read the data supplied on the label of the binders, the children were asked to answer a few questions in writing. Two of the questions were:

1) *"What do you think the surface of each sheet in the binder is?"*
2) *"If you lay out, in any way you wish, all 90 sheets, do you think they will cover an area of* $1m^2$*?"*

Each child was given the same photocopy one month later. At this point the next two experiments were carried out. In the second experiment, we especially wanted to bring out visual comparison, using the answers the pupils had given to the previous questions as a starting point for a general discussion. Having analyzed the photocopy from a mathematical point of view, the children were given the possibility of reviewing the answers they had given the preceding month and of correcting any mistakes or incongruencies in them.

The last experiment came one week later still. Here we wanted to look at the coverage, or filling in, with area units. The pupils were given the same photocopy and asked to answer one question in writing:

3) *"How many squares whose sides measure 5mm can you draw on one sheet? Write how you plan to figure this out."*

The answers given were the basis of a further discussion involving the whole class according to a methodology described in Bonotto, 1999.

Hypothesis

The aim of the first session of questions was to understand what the pupils meant by surface. The objective of the following two sessions was to analyze how and to what extent, in real-life situations, the children managed to apply the knowledge they had acquired through traditional teaching methods.

First of all, our hypothesis was that this kind of teaching, where the application of a formula is what counts, promotes the children's tendency to identify the concepts of surface and area, and that this way of teaching leads pupils to think that the only valid way of determining the size of a surface is applying the formula. Moreover, we suspected that if one presented the size of a surface as the product of two linear dimensions, the meaning and the real representation of square measures would not be made clear.

We therefore expected the fifth-graders to consider surface purely as a number, mechanically associating a square unit of measure to that number by applying the formula, and mnemonically remembering the rule for going from one square measure to another.

N.4 Second study

Subjects, material, research methods

This study, too, was carried out in the second quarter of the 1998-1999 school year. It took place in the fourth grade of a primary school (Trebaseleghe, Padua), with 22 pupils. In this class the concept of surface area had not yet been dealt with, although the children had played with the game of Tangram when the teacher was teaching them about perimeters. It should also be noted that in this class other mathematical concepts had been broached starting from out-of-school situations. Thus, the children were used to a certain kind of "mathematization" of real life, that is, to dealing with situations in their daily lives from a mathematical perspective.

Four experiments were carried out, at one-week intervals, each lasting two hours. The teacher of logico-mathematic areas was present throughout, along with two researchers. The protocols containing the pupils' answers were handed over to the researchers at the end of the experiments; all of the discussions were recorded.

In the first experiment, each child was given a photocopy of the cover of a loose-leaf ring-binder containing graph paper, such as are currently sold in stationery stores. We deliberately erased the information on the binder label pertaining to the size of the graph-paper squares, so as not to complicate the instructions we wanted to give. Having briefly read the label, the children were supposed to answer three questions in writing:

1) *"To you, what does 15x21 mean?"*
2) *"What unit of measure do you think 15x21 is written in?"*
3) *"Choose and write the side-lengths of the little squares, which aren't written on the label. Then tell me how many of these little squares you need to fill up one sheet of paper."*

The class examined the answers and discussed them. During the discussion, rectangular sheets of different formats – A3, A4, A5, and A6 – were used. Then the children were asked to do the following exercise for the next week: cover the blank surface of the photocopy with 1cm squares.

This exercise was to be the basis of the second experiment, in which we wanted to introduce coverage with area units. In this second experiment, four questions were asked, although not simultaneously; a question was asked only when all of the children had answered the preceding one. The first three concerned the sheet that the children had filled with little squares.

1) *"How many little squares did you use to fill in the page? Write how you counted them and why you did it that way."*
2) *"What did you need the little square for? What does the little square mean to you?"*
3) *"To measure the surface of the sheet, could you simply have used a ruler? Explain your reasoning."*

The last question concerned another rectangular sheet that had been distributed to the class, filled in only partially with little squares.

4) *"Quickly tell me the surface of the part of the sheet that is covered by little squares."*

In the third experiment, the children had to answer three questions about two rectangular sheets they had been given. The sheets were of equal surface and they were congruent; one was filled with squares with side of 1cm, the other with squares with side of 5mm.

1) *"Describe the two sheets."*
2) *"Calculate the surface area of each of the two sheets, look at the results, and comment on them."*
3) *"What does the little square represent in each of the two sheets?"*

In the fourth experiment each child was given a sheet of millimetric graph paper which served as the basis of a general discussion.

Hypothesis

The aims of the first question session were:

- analyze how the label was read
- introduce the comparison of surface.

In the following sessions, the aims were to:

- observe how the children analyzed the surface
- observe how clear the role of the little square as a unit of measure was to the children, and
- introduce the mathematical way of writing a square measure.

We wished to propose an alternative to the standard teaching method based on a drawing of a rectangle with a formula next to it, since we feel that this method leads to confusion in children's minds between the concept of surface and that of area. Our method consists in visually comparing real surfaces that are limited in size, so as to be able to estimate their areas and then verify the estimate by filling in or covering the surfaces with small squares and counting them. In our opinion, the visual comparison should make the meaning of "surface" clear from the outset. The second step was for the pupils to approach the problem of measuring a surface. Seeing that they could not use a ruler, and using area units, they would be able to measure a surface directly, without the notion of linear measures becoming distorted in the process.

This way, in our opinion, it would become much clearer to the pupils

- that comparing different surface sizes does not mean only comparing the numbers one gets by calculating their areas
- that it is therefore necessary to distinguish between a surface area and measuring that surface
- that visual comparison can be verified by real measurement of the surfaces
- what square units of measure are and why they are used
- what the real meaning of the formula $b\times h$ is.

N.5 Comparison between the two studies and open questions

Through the experiments we conducted we were able to observe how the surface of a rectangular figure is considered by pupils who have simply learned the formula for calculating an area and pupils who have been taught a different procedure to arrive at the size of a figure.

The latter procedure consisted in visual comparison to start with, followed by numerical calculation, i.e., covering the surface and counting concrete, square units of measure. Two activities were thus being highlighted: visual comparison and direct surface measurement. The children showed that they clearly understood the significance of comparison. All of the children first looked at and estimated the surfaces at hand, then verified their estimation by superposing one on the other. One difference between the two classes, however, did appear. While the fifth-graders simply compared the sheets as they had been asked to, and stopped there, the fourth-graders went further, correctly comparing their sheets with another real surface that they themselves normally use (their report-card notebook). They considered comparison to be an activity that allows one to determine which figure occupies more space. This is undoubtedly due to the kind of teaching they have received since their first year in school, a teaching method that has always taken into consideration the children's daily lives.

But, then, what do the children understand by surface measurement? For the fifth-graders, who were already familiar with the notion of surface, measuring a surface meant obtaining a number, and applying the arithmetic operation defined by the formula, without analyzing or estimating the figure first. From the classroom discussions it became evident that there was a difference for the children between the way one evaluates a figure in school and the way one estimates a real surface outside of school. This led to an inconsistency between the answers they gave when they thought of the sheet as a rectangle and their answers when they simply observed a sheet of paper. For these pupils it was very important to arrive at an answer to the problem, and it did not necessarily matter if their answer was coherent with the figure they were studying or not.

In the fourth grade, where we introduced the concept of surface by having the children first visually compare real surfaces, then measure them directly, there was never any incoherence between the initial evaluation and the final calculation. In the experiment described in this article, by observing the two rectangular sheets of graph-paper, the children were able to deduce that the sheets were of equal surface size. They were also capable, however, of explaining the different numerical results they had gotten when they counted the small squares on each sheet or surface. For these children, to measure a surface therefore means to count area units, which can change from one surface to another. With traditional mathematics teaching, the surface is identified with a number, probably without the students' understanding the reason why the size of a surface is calculated arithmetically, while the measurement of a line is done with a concrete measuring tool.

This difference between the two classes is probably due to a different approach to the concept of surface area, which leads to a different way of thinking about square measures. The fifth-grade children consider them abstract entities; for these pupils the entities are the product of two equal lengths (consider the difficulty they had in using the small squares as square units of measure). Square measures have a more concrete meaning for the fourth-graders who, given a real sheet of paper, could see for themselves that it was not possible to measure its surface with a ruler. The children's explanations of the rule for going from one square measure to another were also different: for the fifth-graders it is just a rule to learn by heart, while for the fourth-graders it is a consequence of the relation between the actual area units used.

Furthermore, knowing how the little squares were disposed, as a quick way of counting the fourth-grade pupils chose the procedure of multiplying the number of squares they had drawn along the two sides of their sheets. For them, too, the area of a rectangle is calculated by multiplying the lengths of its sides, but contrary to the older children, they knew how to get to the formula and explain it.

Direct measurement proved useful in acquiring the concept of surface area in all of its aspects. Moreover, it is a procedure the pupils frequently use, for example, to draw a figure of a certain size on a sheet of graph-paper. Since, in school, direct measurement is not considered particularly important, or it is downright neglected, it risks becoming a method that pupils think they don't have to use outside of school, thus sustaining the gap between schoolroom and non-schoolroom knowledge.[1]

The use of small squares as a first approach to the formula can, however, create some problems. In our experiments, we noted that the pupils often tend to express both area and linear measurements in terms of numbers of squares. This could cause confusion between the concepts of square and linear measures, especially when the concepts aren't dealt with in depth. Children must learn very clearly when it is necessary to use area units, what the meaning of area units is, what is meant by rectangular array, and the difference between a rectangular array and a line.[2]

Beyond the aspects discussed above, we believe the research we have presented here to be a useful tool to change attitudes with respect to mathematics, on the part of both pupils and teachers . The usefulness and accessibility of the discipline of mathematics, which many students find difficult and abstract, becomes all the more apparent when one enables children to draw new mathematical knowledge from the reality around them. One also helps overcome the rift between "schoolroom" and "out-of-school environment", giving greater value to the knowledge and strategies children possess in practice. As emphasized in other

[1] This was confirmed in a study we carried out among students in the second year of middle school, for whom measuring a surface exclusively meant applying a formula; these students then proved to have a number of difficulties related to the use of the small square as an area unit.

[2] This problem arose in particular in research we carried out in a third-grade elementary-school class that had not yet dealt with linear measurements. This study also highlighted the importance of intuition in knowledge acquisition.

studies, local strategies developed in practice are more effective than arithmetic algorithms, which are usually taught in school to give the students powerful general procedures that, in fact, are frequently useless in out-of-school contexts (Schliemann, 1995).

References

Basso M., Bonotto C. and Sorzio P.: 1998 Children's understanding of the decimal numbers through the use of the ruler. Proceedings of the 22th Conference of the International Group for the Psychology of Mathematics Education, South Africa (Alwyn Olivier and Karen Newstead eds), 2, 72-79.

Bonotto C.: 1999 Sull'uso di artefatti culturali nell'insegnamento- apprendimento della matematica/About the use of cultural artifacts in the teaching-learning of mathematics. *L'Educazione Matematica*, Serie VI, Serie VI, 1(2), 62-95.

Bonotto, C. : 2001 How to connect school mathematics with students' out-of-school knowledge, to appear in *Zentralblatt für Didaktik der mathematik*.

Bonotto C. e Maddalosso M.: 1997 Problematiche emerse dall'analisi di indagini sulla misura. Atti del 2° Internuclei Scuola dell'obbligo, Università degli studi di Parma, 59-63.

Freudenthal H.: 1991 *Revisiting Mathematics Education*. China Lectures. Kluwer, Dordrecht.

Gravemeijer K.: 1997 Commentary Solving Word Problems: A Case of Modelling. *Learning and Instruction*, 7, 389-397.

Greer B.: 1997 Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction*, 7, 293-307.

Lave J.: 1995 I problemi aritmetici. Il pensiero dell'altro (O. Liverta Sempio e A. Marchetti eds), Raffaello Cortina, 163-184.

Marchini C.: 1999 Il problema dell'area. *L'educazione matematica*, Serie VI, Serie VI, Febbraio, 27-48.

Nunes T., Light P. and Mason J.: 1993 Tools for thought: the measurement of length and area. *Learning and Instruction*, 3, 39-54.

Outhred L. and Mitchelmore M.: 1992 Representation of area: a pictoria perspective. Proceedings of the 16th Annual Conference of the International Group for the Psychology of Mathematics Education, 2, 194-201.

Reusser K. and Stebler R.: 1997 Every word problem has a solution: The suspension of reality and sense-making in the culture of school mathematics. *Learning and Instruction*, 7, 309-328.

Schliemann A. D.: 1995 Some concerns about bringing everyday mathematics to mathematics education, in L. Meira and D. Carraher (eds), Proceedings of the XIX International Conference for the Psycology of Mathematics Education, Recife, Brasil, 45-60.

Tierney C., Boyd C. and Davis G.: 1990 Prospective primary teachers's conception of area. Proceedings of the 14th Annual Conference of the International Group for the Psychology of Mathematics Education, 2, 307-315.

Treffers A.: 1987 *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction*. The Wiscobas Project, D. Reidel Publ. Co., Dordrecht.

Verschaffel L., De Corte E. and Borghart I.: 1997 Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems. *Learning and Instruction*, 7, 1997.

Yackel E. and Cobb P.: 1996 Classroom sociomathematical norms and intellectual autonomy. *Journal for Research in Mathematics Education*, 27(4), 458-477.