

# **The representation of mathematical concepts in primary mathematics textbooks: a focus on multiplication**

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This paper centres around an analysis of primary mathematics text books from five countries. The focus is on the similarities and differences between the way in which images of mathematical operations are represented in text books. Particular attention is given to the representation of multiplication. The results of the study indicate that the English text analysed placed less emphasis on mathematical structure and the linking of mathematical representations than the texts from the other countries studied. In particular, whilst in the English texts the images often seemed to be distractors from the mathematics, in other texts the images were used to focus explicitly on mathematical principle/concepts.

## **Introduction**

In this paper primary mathematics text books from England, France, Hungary, Singapore and the USA<sup>1</sup> are compared. The view is taken that within a particular country text books reflect the dominant perspectives about what mathematics is, the mathematics which citizens need to know, and the ways in which mathematics can be taught and learned. In other words what appears in a mathematics text book does not appear by chance. It is influenced by the multifaceted aspects of an educational culture. In this way mathematics text books provide a window onto the mathematics education world of a particular country.

The view is taken that pupils' construction of knowledge cannot be separated from the external representations of this knowledge. These external representations include pictures, icons and such mathematical symbols as tables, graphs and arithmetic symbols. They also include objects such as fingers for counting and representations which are developed for pedagogic purposes. These symbolic objects are transformative, in that they enable a person to do something which he/she could not do alone. For example tallies on paper support an individual to count a large number of objects; the long multiplication algorithm enables an individual to multiply numbers together which would be very difficult without such a paper-based algorithm. This perspective derives from the work of Vygotsky (1978) and Wertsch (1991) and emphasises that human action is mediated by 'technical' and 'cognitive tools'. 'Technical' and 'cognitive' tools are social in that they are the product of sociocultural history and are always used first as a means of influencing others and only later as a means of influencing oneself (Vygotsky, 1978).

There has been very little analytical research which has focused on the ways in which mathematics is presented in text books. This may be because many mathematics educators support what Voigt (1998) calls the folk belief that tasks, questions and symbols of mathematics lessons have definite, clear-cut meanings. Or it may be because of a view that text books should not be used by "good" mathematics teachers.

Recent work of Santos-Bernard (1997) suggests that children do not necessarily read and use illustrations in the same way as they are read by authors and teachers. She analysed the approach of pupils in both Mexico and the UK to illustrations in mathematics texts and found that low attaining pupils have particular difficulty in interpreting texts in ways which the author intended. Pupils can also over-interpret the information presented in an illustration, or alternatively ignore pictorial information when it is pertinent to the problem being solved. Low attaining pupils in particular found it hard to extract appropriate information from two

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<sup>1</sup> This chapter derives from one of two parallel QCA funded projects which compared primary mathematics texts from England, France, Hungary, Ireland, Japan, the Netherlands, Singapore and the USA. The other study was concerned carried out by Geoffrey Howson and compared texts from Canada, Ireland, The Netherlands, Japan, Switzerland and England.

sources – illustrative and textual. Further they were far more likely to misinterpret pictorial information than other pupils. In this sense particular illustrations can cause significant distractions to the pupils in pursuing a given problem. The work of Voigt (1998) has also drawn attention to the fact that ambiguity and negotiation of meaning are essential features of mathematics classrooms. “Every true-to-life empirical situation (given as a story, picture, text, etc) can be mathematised in various ways, depending on one’s interest” (p 208). He presents considerable evidence that children interpret real-life pictures in text books in many different ways although he says that “many authors of textbooks as well as mathematics teachers seem to assume that such pictures have unambiguous mathematical meanings and represent tasks that have definite solutions” (p. 196).

### **Overview and Analytical Framework**

The full study (see Harries & Sutherland 1998, 1999) from which this paper derives, involved a factual analysis of the content of each text book, an analysis of the texts for potential meaning construction and learning both within and across the chosen countries, an analysis of the style/purpose of the teachers’ guides, and a comparative analysis of the development of the conceptual area of multiplication and division across each scheme. One of our aims was to focus in detail on the ways in which mathematical ideas are represented in text books, and for the purpose of this paper attention is restricted to the introduction of multiplication and division. During the presentation examples will be shown from text books in Singapore, Hungary, France and England. We do not claim that the text books<sup>2</sup> analysed in some way represent all the text books which might be used. However it is interesting to note that as with the TIMSS analysis of classroom processes (Steigler and Hiebert 1997) we found that when we analysed two text book schemes for a country there were considerable similarities between these texts. Table 1 presents an overview of the organisational characteristics of the text books for each of the five countries in the study.

<b>Country</b>	<b>Characteristics</b>
Singapore	Each year 1-6 has 2 text books, 2 work books and 2 teachers’ guides – one for each half-year. From year 5 streaming takes place and there are separate books for stream.
Hungary	Each year 1-4 has two text books/work books. The front half is a text book and the latter half the work book. There is a teachers’ guide for each year.
England	Not year based – level based. For each level there is an extensive teachers’ guide, work cards or textbook, work book, problem solving resource, investigations, assessment material, resource pack, extension material
France	Each year has pupil texts, teacher’s guide for each pupil text with appendices for more manipulative activity. Classroom posters setting out in matrix form: knowledge or know-how to be taught; competencies to be learned and where these fit in to the overall scheme.

<sup>2</sup> In order to choose appropriate texts we followed the advice, experience and expertise of teacher educators in each country. The following is an overview of how these choices were made: England – the two most commonly used schemes; France, Hungary – two schemes chosen by an education expert; Singapore – the government text book scheme; USA – scheme as advised by experts at Chicago university.

USA	For each year there were pupil texts, pupil workbooks, assessment material, resource pack containing games etc., teachers' guide which is the kernel of the course and in which the teacher is given lesson plans for ~120 lessons, extension material
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Table 1: Scheme comparison across countries

The analysis has been framed by a consideration of:

- the nature of the images with which pupils engage as they read the text, this includes pictures, diagrams and symbols. Analysis of these images has been influenced by the following categories which derive from the work of Botsmanova (1972)
  - Object-illustrative images which illustrate objects in the problem but not the mathematical structure of the problem;
  - Object-analytical which reflect the mathematical structure of the problem
  - Abstract spatial diagrams which reflect an abstract numerical relationship.
- the ways in which pupils are introduced to links between mathematical concepts and the role of images in this respect;
- the relative emphasis on mathematical structure, mathematical processes and mathematical objects.

### Representing Multiplication

In all of the text books the notion of multiplication is first introduced as repeated addition of equal groups, and this evolves into representing multiplication as a 2x2 array. The Hungarian, French and English texts also use the notion function in the form of “jumps on a number line” to introduce multiplication. Whereas we are aware that there are arguments for the advantages and disadvantages of each approach (see Anghileri (1991) and Davidov (1991)) our focus is on analysing how the ideas are represented on the page of the text book. In the Singapore scheme the teacher's guide explicitly states that various representations are used to represent the idea of multiplication. For example a picture shows 3 groups with 5 pairs in each group. This image is then represented in words “there are 5 pairs in each group”, as a number sentence using symbols  $5+5+5$ ” and a hybrid using numbers and words “3 fives”. These representations emphasise mathematical structure and the links between the different ways of representing multiplication. The multiplication symbol is introduced a few pages later as “this is multiplication. It means putting together equal groups” Pupils are supported to see the equivalence of, for example  $5+5+5$  and  $3 \times 5$ , with these representations being placed next to each other on the page. The emphasis in the Singaporean text book is on whole/part relationships – multiplication is about putting together equal groups and division is the inverse of this. The teacher's guide emphasises this idea and uses object-analytical and abstract spatial diagrams when discussing how to introduce these ideas. Diagrams and pictures are very clearly focused on illustrating the mathematical concepts being represented, and the mathematical structure of these ideas. Decorative illustration is minimal.

If we compare the approach used in Singapore to that used in the English texts there are a number of differences. In the English text there is much less emphasis on mathematical structure and the images are mainly object-illustrative. When multiplication is first introduced images of equal groups are presented together with a symbolic representation such as  $4+4+4=.....$ . The multiplication symbol is introduced much later and instead an intermediate pedagogic representation is introduced ( $3(4)$  meaning 3 groups of 4). This separation of a standard mathematical symbol from a mathematical concept being introduced is a characteristic of the English texts whereas the Hungarian, Singaporean and French all

introduce the standard mathematical symbols at almost the same time as a concept is being introduced.

The notion of multiplication as an array is introduced in the English texts approximately a year after the first introduction of multiplication, as a way of introducing the idea of commutativity. In this page there is very little emphasis on mathematical structure and linking representations. The title of the page is “Block of Stickers” and this is where the emphasis seems to lie. This sharply contrasts with a similar idea in the French text in which the emphasis is clearly on the array and the commutativity of multiplication.

The difference in attention to the mathematical detail of how an idea is presented in English texts can also be seen in the images that are used to introduce the number line. The activity in the English text is called “space jumps” whereas a similar activity in the French text is called “the product of two numbers”. The English text emphasises a physical activity with an appropriate image whereas there is no such image in the French text and the activity is a more abstracted one of working on number relationships.

When multiplication is introduced in Hungary, the emphasis, as in Singapore, is on putting equal groups together to make a whole and splitting a whole up into equal parts (sometimes with remainders). As in Singapore multiplication and division are introduced almost simultaneously. Further different ways of representing multiplication are linked with pictures, words, and symbolic representations appearing together. Images here are almost entirely of the object-analytical type, with no use of purely decorative images. Words are used to explain diagrams/pictures and there is no use of talking heads.

A characteristic of the Hungarian, French and to some extent the Singaporean texts is the way in which pedagogic representations lead over time to the introduction of the standard algorithm. This contrasts with English texts where four years after the first introduction of multiplication pupils are presented with several pages of activities, which do not appear to have been organised in an organised way. They are presented as an “add on” or a “good idea” which is thrown in as a possible help but which is not prioritised. As indicated earlier the images that continue to dominate are object-illustrative ones, and these are prioritised over mathematical structure. As discussed elsewhere (Sutherland, 1999) the emphasis is on diversity of approaches and a particular algorithm is not being prioritised.

### **Concluding Remarks**

The theoretical perspective suggested that the images with which pupils engage will make a difference to the meaning they construct and hence the learning that takes place. The analysis suggests that the Hungarian, French, and Singaporean texts attempt, in different ways, to support pupils to develop mathematical meaning and to move along a chain of abstraction from a real situation, to images of the real situation, to mathematical representations of these images of real situations. Our analysis of the texts suggests that the English texts do not support pupils in the same way. In these texts there seems to be considerable emphasis on the use of object-illustrative images, presumably as a way of relating mathematics to “real situations”. These images seem to be more of a distraction than a help at times. This would be particularly true of low attaining pupils who are most likely to be distracted by an apparently incoherent use of images (see Santos-Bernard, 1997).

Other characteristics of the English text book scheme which are different from the Hungarian, French and Singaporean schemes are:

- a separation of the introduction of standard mathematical symbols from the first introduction of a mathematical idea;
- very little emphasis on mathematical structure;
- very little emphasis on linking mathematical representations.

Further in the English text book scheme there is no explicit reference to theoretical ideas of learning. This contrasts with the Singaporean scheme where the teachers' guide draws on the work of Bruner (1970) in order to discuss the pedagogic implications of working on specific content, and the French guides which draw heavily on the work of Brousseau (1999) and Vergnaud (1990). In Hungary there is a strong problem-solving context to the learning. This is reflected in the way in which concepts are explored through appropriately chosen problems often in the form of puzzles. There is an emphasis placed on building connections between different aspects of mathematical knowledge (Harries 1997).

There is an implicit assumption that mathematical notions such as multiplication are culture free. Our comparative analysis of the primary mathematics text books puts this assumption into question, and suggests that pupils from different countries are likely to construct different meanings of multiplication based on their experience of the way in which these notions are represented for them. In future research we will be interested in exploring such questions as "does a Hungarian pupil who has worked with a Hungarian text develop different understandings of multiplication from an English pupil working with an English text". Further we are interested in exploring how the computer can be used to both develop and link different representations of both number and number operations.

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