Computer aid in graphical interpretation of ODE

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Abstract. Modern computer software can effectively improve mathematical education. In particular, it concerns ordinary differential equations. Here advanced notions and methods are in use, time-consuming calculations are necessary to get agraphical interpretation and numerical solutions. Both last aspects are specially important when we can not obtain an exact, analytical formula for the function satisfying given initial problem. In this case a direction field can help in the justification whether a numerical solution is correct enough. A direction field is also useful when we investigate the uniqueness of solutions. In paper at hand we present how the systems DERIVE (from Soft Warehouse Inc.) and WinPlot (by Richard Parris) may be applied to problems mentioned above in the way which increases independent student activity.
Key words: computers in mathematical education, ordinary differential equations

1. Introduction

No doubt, since Newton and Leibniz there is no higher mathematics and physics without differential equations. They are basic in the formalized description of the universe [10]-[11]. Thus there is very important to dominate and deeply understand their nature, to feel the idea behind the concept of differential equation. A crucial point here is the notion of a direction field. The importance of this concept is justified not only by the fact that almost every book on differential equations (e.g. [5]-[7], [9], [14], [15], [16]) discusses it. The role of the direction field is indispensable if an ODE is not solvable analytically and numerical methods (e.g. the classical Runge-Kutta algorithm) have to be applied. Then, specially in cases of ill-posed problems, the coincidence between the numerical solution and slopes forming the direction field is a proof to accept or reject results numerically calculated. Obviously, the hand sketching of such a field consumes a lot of time, the assistance of computers is here very effective.

We deal with any function f defined in an area D contained in the real plane R_2 equipped with the rectangular co-ordinate system Oxy. We distinguish aset D_p of points in D. In most cases they are nodes of regular rectangular net, so the index p is apair of two numbers, m and n, saying how many different abscissas x_j and different ordinates y_k build the set D_p. Then we write $D_p = D_{m,n} := \{ (x_j, y_k): j=0..m, k=0..n\}$.

First, in Section 2, we give basic definitions. In Section 3 we report a sample lesson on direction field for an ODE with one singular point. In the next section we investigate an ODE with no singularities is investigated, here the aspect of the DF-examination of a numerical solution is considered. In Section 5 we deal with an ODE of the second degree. Otherwise than in previous sections, where the computer algebra system DERIVE (from Soft Warehouse Inc.; see e.g. [4]) is applied, here we use the program WinPlot (by Richard Parris; see e.g. [13]). At last, we formulate conclusions underlining the advantages of CAI (computer assisted instruction) in the area of ODE.

2. Basic definitions

A ***direction field*** (DF) of a differential equation

(2.1) $$y' = f(x, y)$$

(generated by the set $D_{m,n}$, named often as a *canvas*) is the set

(2.2) $$\{ f(x_j, y_k) : (x_j, y_k) \in D_{m,n} \}.$$

A direction field may be (and usually it is) interpreted geometrically. To each point $(x_j, y_k) \in D_{m,n}$ we may assign the angle $\alpha_{j,k}$ such that

(2.3) $$\tan(\alpha_{j,k}) = f(x_j, y_k)$$

and then we can draw a linear segment (called a ***direction***) sloped with this angle. Every such a segment indicates a direction along which a solution y of the equation y’ = f(x, y) locally goes. In other words, every solution y to the equation y’ = f(x, y) fits to these directions. Immediately from the form of considered equation we see that this fitting is of a tangent nature: a linear segment visualizing the direction lays on the tangent line to the curve y= y(x). When enough number of linear segments is drawn (i.e. the net $D_{m,n}$ is dense enough), one can often see trends in the solution curves. This allows a graphical analysis of solutions. Moreover, one can plot a numerical solution to see does it fit the slopes forming a DF.
Let’s say that in some realization a DF is called a slope field (no arrows are drawn) or a vector field (arrows are on).

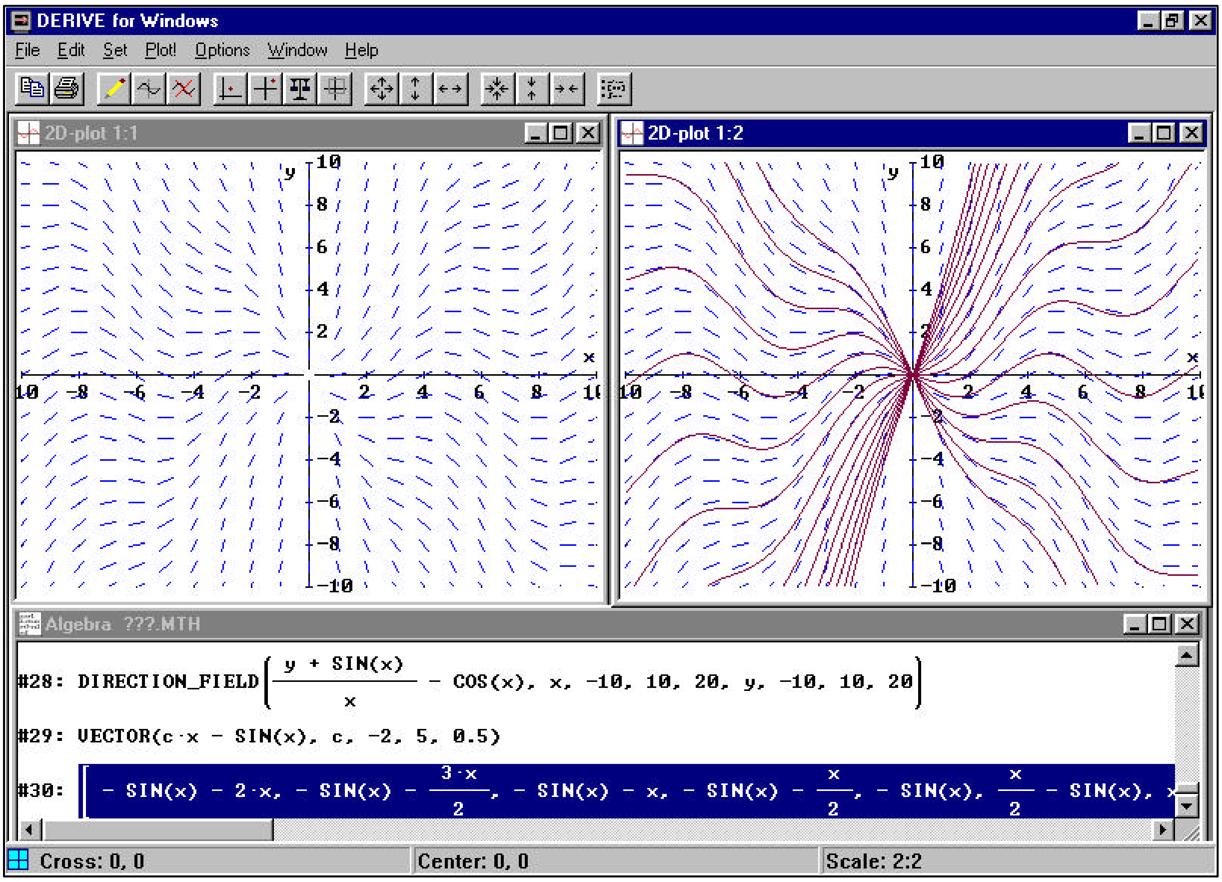

Fig.1. Direction field of the equation y’ = {y + sin(x)}/x – cos(x)
and graphs of fifteen particular solutions of this equation (Section 3)

3. Sample DERIVE lesson on direction field

Typical student’s exercise runs as follows: a student assumes the general solution

$$y = y(x,c)$$

of an ODE, next (s)he gets the derivative y’(x,c) and substitute a constant c eliminated from the general solution. This work completed, the student may produce graphs of the direction field and particular solutions.
For instance, assuming

(3.1) $$y = c\cdot x - \sin(x)$$

we derive

(3.2) $$c = \{y + \sin(x)\}/x$$

and

(3.3) $$y' = c - \cos(x) = \{y + \sin(x)\}/x - \cos(x).$$

That’s why we deal with the equation

(3.4) $$y' = \{y + \sin(x)\}/x - \cos(x).$$

Now we produce its direction field. At last we plot some particular solutions. It means we plot graphs of functions obtained by fixing the value of the constant c.

Operations which within DERIVE produce figures as Fig.1 and Fig.2 are described, e.g. [8] and [3].

All operations described above are easily issued if there are handy highlight (sub)expression already displayed and we copy them (via hot keys F3 or F4) into the editing line.

It is didactically desired to plot a function which *does not solve* the considered ODE. Usually, just after producing the direction field, a teacher says: „Let’s plot a particular solution, for instance with c = 0”. It means the students edit the expression −SIN(x). „Look”, the teacher says, „how this function perfectly fits to directions”. And (s)he may continue: „Now let’s draw the graph with an other value of c, for example with c = 1. Hence we edit the expression –1*x and we plot it”. This intentionally made mistake reveals that the graph of edited expression does not coincide with the directions. It makes the students remember and deeper understand what the direction field and solution to an ODE are.

Notice that the direction field appears that the resulted ODE (3.4) has the singular point at $x = 0$. It does not happen, e.g., when we deal with such equations as $y' = y - \sin(x) + \cos(x)$ and $y' = y + 3x^3 - x^3 + \cos(x) - \sin(x)$ derived from the general solution $y = c\cdot\exp(x) + \sin(x)$ and $y = c\cdot\exp(x) + x^3 + \sin(x)$, respectively (comp. [2]).

4. Direction field examines a numerical solution

Let’s consider the equation

$$y' = (2x - 4)/y^2 \ .$$

Its direction field produced in the DERIVE by the approximation of the call

DIRECTION_FIELD((2x–4)/y^2,x,0,5,0.5,y,–3,16,0.5)

is reproduced in Fig.2. There are also plotted three lines. They are produced by the approximation of the calls

RK([(2x–4)/y^2],[x,y],[0,1],h,5/h).

Both procedures, DIRECTION_FIELD and RK, are memorised in the unit ODE_APPR.MTH distributed jointly to the CAS DERIVE. The procedure RK realises the classical fourth order Runge-Kutta method and yields the sequence of values which (should) approximate the exact solution of the initial problem. There are plotted graphs obtained when we run with the step size h = 0.01, 0.001 and 0.0005. Some results concerning these values of h, as well as that corresponding to h = 0.1 and 0.0001, are listed in Table 1 (see [3] to compare the discussed case to solutions provided via Heun method implemented in the program Euler). It reveals that there is no correspondence between the step size h and the quality of numerical solution. Against a common feeling, not always the decrease in the step size h improves the output. In considered case the results produced with h = 0.001 and 0.0005 are completely erroneous (that with h = 0.001 keeps negative values, that with h = 0.0005 jumps drastically around the point x = 1), while that produced with bigger values of h (0.1 and 0.01) are fully acceptable. The time consumed by them is essentially shorter (some seconds against to 2 and 4 minutes used when the work is with h = 0.001 and 0.005). Moreover, the RK-solution returned with h = 0.0001 (25 minutes) is not essentially better than that produced in 1 and 10 seconds. We do not discuss here the reasons to that strange (and rather extraordinary) behaviour. We put our attention to the fact that the correctness of a numerical solution can be justified by the examination how far the graph coincides with the direction field.

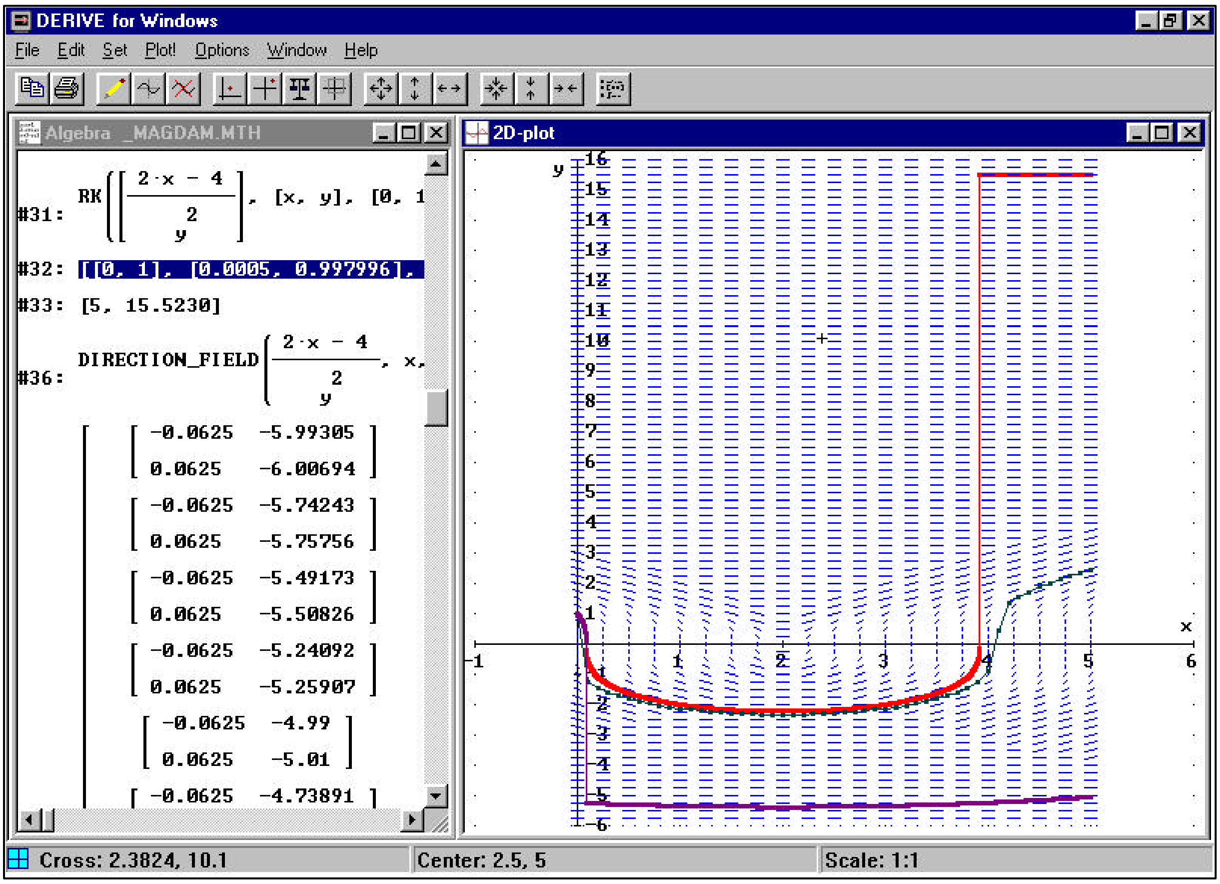

Fig. 2. Direction field to the equation y' = $(2x–4)/y^2$
and three solutions produced by the RK procedure (Section 4)

Table 1. Report on the work of the RK procedure

	step size h	number of steps	value at x=5	approximation time (in seconds)*)
1	0.1	50	2.45782	1
2	0.01	500	2.49435	10
3	0.001	5000	–5.04599	113
4	0.0005	10000	15.5230	246
5	0.0001	50000	2.5192	1518

*) on the IBM PC 133 MHz

From the didactic point of view it is advised to plot in the graph of the function

$$u:\ x \to (3x^2 - 12x + 1)^{1/3}$$

which solves the considered initial problem: $u' = (2x-4)/u^2$, $u(0) = 1$. The coincidence of this function to the directions (forming the direction field shown in Fig.2) is perfect. Naturally, this perfection is up to the accuracy of float-point arithmetic and the pixel resolution. There are just two elements which have to be taken into account when we justify the coincidence between a numerical solution and the direction field.

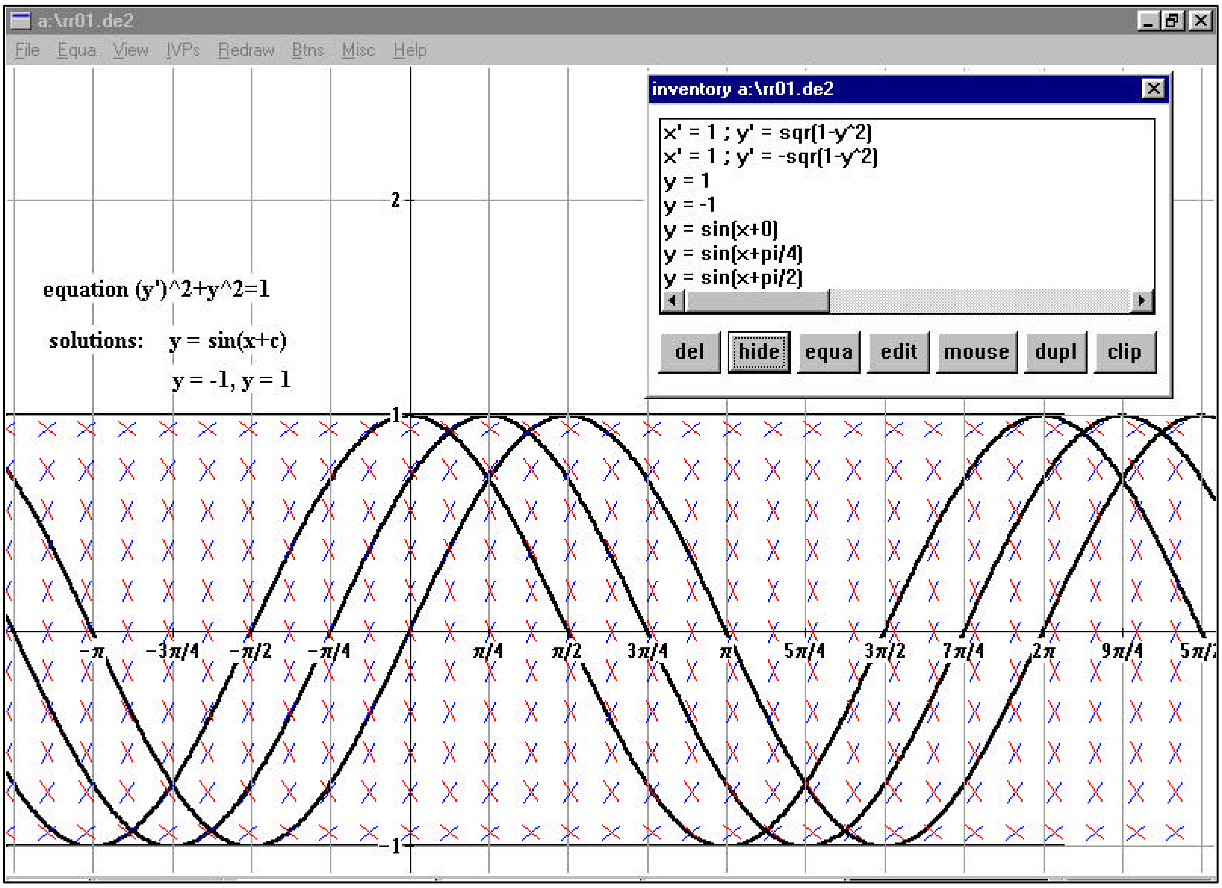

Fig.3. Slope field for ODE $\{y'\}^2 + y^2 = 1$ and graphs of some solutions (Section 5)

5. Investigating an ODE of the first order and second degree

Let us consider the equation [9]

(5.1) $$\{y'\}^2 + y^2 = 1$$

Here the derivative of the unknown function y appears in the square, so we have an ODE of second degree. A direction field is defined for equations of the form

$$y' = f(x,y),$$

so we have to derive such relations. In our case they simply are

(5.2) $$y' = \sqrt{1-y^2},$$

(5.3) $$y' = -\sqrt{1-y^2}.$$

That's why we complete direction field of the equation (5.1) is composed of two fields, namely that defined by (5.2) and that by (5.3). This interference appears in the crossing of directions (in Fig.3 it looks like crosses). At points where these directions coincide (i.e. at points having ordinates $y = -1$ or $y = +1$) any solution of the equation (mm.2) may pass smoothly into the solution of the equation (5.3). Obviously, this interlace takes place in the opposite direction: from (5.3) to (5.2). Moreover, at every point laying on the line $y = -1$ or $y = +1$ we can follow along any of them. On these two lines the Lipschitz condition does not hold, the relations $y = -1$ and $y = +1$ are singular solutions to (5.1).

Both equations (5.2) and (5.3) are analytically solved (by the separation of variables). Their solutions are defined on the whole real line and can be described by the one formula:

$$y = \sin(x + c).$$

Detailed description how to produce figures as Fig.3 can be found in [12] or [2].

6. Conclusions

Differential equations are essential (and, in the same time, one of the most difficult) part of higher mathematics and physics. That's why students have to get a solid, well-grounded knowledge on them. The nature of the subject makes that getting instructive examples is time-consuming. In particular, it concerns the notions of a direction field, the uniqueness and the acceptability of numerical solutions. We deal with these problems occurring in ODE of the first order and first or second degree. We treat them with the aid of computer programs (we take use of DERIVE and WinPlot, but there are other ones which are applied, see e.g. [6]). This help is effective not only in the aspect of the time, it contributes essentially to the deep understanding what a direction field is, what a singular point, particular and singular solutions are, which phenomena may happen when a numerical method is used. An additional advantage is than a student can individually explore the area, e.g. find singularities, exposure the unsuitability of applied methods (or a bad choice in the values of their parameters). After examining many cases (s)he may even undertake trials to formulate general conclusions. It intensifies a mathematical activity, what is the one of the most important aims in mathematical training [1].

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