INNOVATIVE PATHS FOR APPROACHING RATIONAL NUMBERS FROM THE STRUCTURAL POINT OF VIEW

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This proposal faces the question of the approach to rational numbers from the structural point of view taking into account the main didactical knots in the passage from lower to upper secondary school, also linked to the historical development of this ordered field. After giving a frame of our cultural motivations, we trace the guidelines of an one in-progress research of middle-school didactic innovation on this topic. In particular we present some results of activities experimented in the classrooms, together with some of the pupils' productions, synthesis of classroom discussions on the construction of particular pieces of knowledge, which show their way of facing the questions posed. We conclude by underlining the incidence of teacher' beliefs in the development of the research.

There are some epistemological divergencies between the modern, structural vision that concentrates in middle school the introduction of the various numerical ambits, rational numbers in particular, and the old, consolidated teaching tradition that deals with fractions from a merely operative point of view, without even getting to the concept of rational number as class of equivalent fractions. In this tradition, the approach to the operations aims at determining their result for particular couples of fractions and it almost never arrives at making explicit its laws of correspondence in general. The comparison of fractions, moreover, is usually carried out by passing from the decimal representation of the quotient between numerator and denominator (which of course is often approximated), but this procedure, unfortunately, doesn't allow to move to the comparison of fractions in general terms.
On passing from lower to upper secondary school the study is extended to algebraic fractions, which are initially seen as quotients of natural numbers expressed in general terms and in multiplicative form and always from an operative point of view. This passage, that usually is the first approach to algebra at all, is rather complex and there are many studies that highlight the pupils' difficulties and errors on transforming simple algebraic fractions through passages that show a lack of conceptualization (Fishbein e Barach 1993). The didactical praxis highlights a lack of stability in the knowledge achieved; in the best cases the pupils use calculus rules correctly but they separate the operations with fraction expressions from the meanings.
This must be framed within the question of the skills that secondary school teachers presume and often expect the pupils to have, and in middle school teacher's awareness of the fact that these concepts need a long time for ripening both in activity (starting from primary school) and in theory. We agree with Lolli (1996, p. 39) as to the fact *that within the pupils' mathematical growth, the progressive extension of the numerical system is a rather dramatic experience, which happens so quickly and carelessly, though probably with good reasons, that it gets scarcely absorbed and needs to be recollected and meditated later on.* As far as the rationals at secondary school level go, their later reconsidering hardly ever happens, which originates remarkable lacks of knowledge in the pupils.
There are many studies on the approach to fractions at primary school level (beside the classic study by Streefeland 1993, we pinpoint the more recent ones by Olivier & Al. 1996, Moss & Case 1999, Pitkethly & Hunting 1996, Newstead & Murray 1998, Vaccaro 1998), and others concerning its interlacing with decimal numbers (Bonotto 1996 and previous ones), or centered on the geometrical representation (Chiappini et Alii 1999), less frequent are the studies on fractions from a strictly numerical point of view (Barash & Ronith 1996, Tirosh 1997), and those on rational numbers from an algebraic-structural point of view are almost absent.
The typology of these studies reflects, through the learning problems investigated, the nature of the teaching itself, which turns out to have no balance among the intuitive, algorithmic and formal aspects

that in their connections create mathematical knowledge (Fishbein 1994).
The study that we present, on the contrary, aims at this particular balance and at highlighting the conceptual aspects. The path, that lasts three years, starts from a series of activities on fractions, also through real-life situations, which show the different acceptances of fractions (fraction as division, as part of a whole, as composition of operators, as relation between quantities), and then concentrates on fractions in purely numerical aspects. It aims at generalization on one hand (shifting from fractions to algebraic fractions) and at approaching the structural aspects in rationals.
Our studies, in the frame of the theoretical model of algebra teaching/learning formulated by Arzarello *et al.* (1995), are based on the hypothesis that an early approach to the use of letters allows to face elementary questions about fractions from a general point of view, so that the pupils can develop highly flexible, effective and transparent conceptual models as to: rational numbers, order and operations among rational numbers in general terms.
One specific aim is to induce the analysis of the meanings that different representations of simple numerical or algebraic fractions convey, so as to avoid - or at least limit - stereotyped behaviour or classical errors on transforming them. From a conceptual point of view, we want to bring the pupils to the awareness of: 1) how to recognize equivalent fractions[1]; 2) how to compare fractions without resorting to the decimal representation; 3) the reasons that are at the basis of the general definitions of addition and multiplication (through questions such as their being independent from the representing fractions, the embedding of natural numbers into the non-negative rational numbers); 4) the calculus simplifications produced by reducing a fraction to the minimal terms and, specifically for addition, by resorting to the minimum common divisor of the denominators; 5) the existence of opposite and reciprocal for each rational number different from zero (and through these concepts up to the operations of difference and division); 6) density and Archimedean but incomplete nature of the order structure and of the laws of monotony.
The teachers themselves chose to insert these activities into the traditional didactical path, which reflects the historical one (from naturals to non-negative rationals and eventually to all rationals) even if this choice implies the changing of the object 'rational number' (in the symmetrization there is the substantial changing of each class of equivalent fractions) and consequently possible bigger difficulties in the comparison of fractions expressed in literal terms for the distinction between positive or negative values of the denominators.

The general methodology in the researches

The methodology approach belongs to the so-called "Italian model for innovation" (Arzarello-Bartolini Bussi 1998) and generally speaking it is developed through the following steps carried out with the teachers-researchers involved: i) joint study of research literature on the theme chosen and creation of research hypotheses; ii) planning the experimental activity needed to verify the hypotheses in their essential points and a-priori analysis of the potential difficulties for the pupils; iii) joint analysis of the protocols produced during experimentation (pupils' productions, reports of construction discussions, of evaluation discussions, etc.); ivi) selection of documents considered to be meaningful in order to testify thinking paths, behaviours and difficulties in the pupils; vi) elaboration of the results obtained and reflection on the processes that have determined them.
In the study we decided to compare constantly addition and multiplication in order to force the pupils' attention not only on the analogies, but also on the differences among the two. One of our hypotheses (see Malara & Iaderosa 1998) is specifically that thanks to this comparisons it is

1 At most, the pupils manage to conceptualize how to pass from one fraction to an equivalent one, but they can't say in general when two fractions are equivalent. This happens also at higher school levels.

possible to avoid transfers and improper mixtures of properties from one structure to the other, which give vent to several and persistent errors.[2]

Aspects of the experimental work

As to the research the topics we have already experimented concern the four points mentioned; some of the activities connected are reported in table 1, some other regard the problem of defining in general terms the operations and their legitimacy as to the corresponding ones in natural numbers through collective discussion. Particular attention is given to exercises in which the pupils are asked to express a given fraction as the sum of two fractions. Such exercises make them see the numerator of a fraction as the sum of many terms, and the fraction itself as the sum of as many fractions with the same denominator, so as to

table 1

Examples of the activities given
Seventh and eigth grade 1. Build fractions equivalent to the give ones, according to the indications expressed by the scheme $(\leftarrow\ \rightarrow)$ $\leftarrow \frac{15}{25} \rightarrow$; $\leftarrow \frac{10 \cdot b}{14} \rightarrow$; $\leftarrow \frac{3 \cdot 2.\cdot a}{14} \rightarrow$; $\leftarrow \frac{18 \cdot k}{2 \cdot a \cdot 15} \rightarrow$ 2. For each couple of fractions in brackets, decide which is minor, and explain why: $\left\{\frac{15}{17};\frac{13}{19}\right\}$; $\left\{\frac{19}{36};\frac{11}{24}\right\}$; $\left\{\frac{235}{352};\frac{115}{176}\right\}$
Eigth grade 3. We ask you to build equivalent fractions for each of the following fractions; find out whether there are any difficult cases and why they are difficult: $\frac{5}{7}$; $\frac{2 \cdot 11}{5 \cdot 9}$; $\frac{p}{k}$; $\frac{3 \cdot a}{12}$; $\frac{5+4}{11}$; $\frac{a+b}{2}$ 4. Following the strategy you prefer, compare each couple of fractions and explain which is the minor one $\left\{\frac{3}{4};\frac{5}{6}\right\}$; $\left\{\frac{11}{24};\frac{17}{36}\right\}$; $\left\{\frac{2a}{17};\frac{3a}{22}\right\}$
Seventh and eigth grade 5. Replace the letters with the possible numerical values that make the following equalities true: $\frac{k+1}{4} = 2$; $\frac{8}{k+1} = 4$; $\frac{4}{5} = \frac{12}{k+2}$; $\frac{15}{18} = \frac{k}{k+1}$; $\frac{k}{3} \cdot \frac{4}{m} = \frac{5}{6}$; $\frac{2}{3} \div \frac{c}{d} = \frac{6}{21}$

create a basis of experiences which limit and go against the enacting of wrong habits of automatic simplification, such as cancelling b in (a+b)/b. As to the questions reported in table 1, we indicate some pupils'behaviours, mainly of seven grades, which indicate lacking, even if not so wide spread, conceptualizations. There are:

- pupils who in the passage from a fraction to equivalent divide the two terms by a natural number which is not common factor of both;
- pupils who reveal some uncertainty in facing divisions or multiplications of the two terms, when they are expressed in factors (for instance a pupil writes:
$\frac{2 \cdot a}{5} \leftarrow \frac{:3}{:3}\quad \frac{3 \cdot 2 \cdot a}{15}\quad \frac{\times 2}{\times 2} \rightarrow \frac{6 \cdot 4 \cdot a}{30}$);
- pupils who, in case of presence of a literal factor, either they stop themselves (for instance a

[2] The frequent error consisting in the cross simplification of two rationals, for example the simplification of 10 with 5 in the sum 10/17 + 1/5, can be limited through the simultaneous comparison between addition and multiplication of two rationals, even in general terms, guiding the pupils' attention onto the reason why this simplification works only with multiplication.

pupil writes beside $\frac{3\cdot a}{12}$ "I do not know a"), or abandon it on semplifying (for instance a pupil writes $\frac{5}{7} \leftarrow \frac{:2}{:2} = \frac{10\cdot b}{14} \rightarrow \frac{100\cdot b}{140}$), or they substitute it with a number (for instance a weak pupil writes $\frac{3\cdot a}{12}$, a = 2, specifies the value $\frac{6}{12} = \frac{12}{24}$ and adds "if there were the numbers the thing would be ok").

In the construction of equivalent fractions only few eigth grade pupils operate multiplying numerator and denominator by a same literal term, but they forget to leave off the case that can represent the nought. Several pupils reveal a pour autonomy in the use of parentheses (for instance a pupil writes : $\frac{a+b}{2}\frac{\cdot 4}{\cdot 4} \rightarrow \frac{a+b+b+b+b}{8}$).

Table 2

Silvia' analysis of the question 3 of table 1
a) By "difficult" I mean "to reason on it". Yes, I have found some cases in which I reason on them with examples: $\frac{5+4}{11}$ and $\frac{a+b}{2}$ their level of difficulty is similar then there is not one more difficult then the other one. b) In the two "difficult" cases to me I was a little bit uncertain if I could add a parenthesis or not, then I tried with some examples for verifying the right execution (or operation) and they are: $\frac{5+4}{11} \rightarrow \frac{5+4\cdot 3}{11}$ I wish to find an equivalent fraction that is three times the given one. But now I have a doubt: In these years of middle school I have learnt that the multiplication has to be made before the others. Then, coming back to the example I have: $\frac{5+4}{11}\ \frac{5+4\cdot 3}{11\cdot 3} \rightarrow \frac{5+12}{33}$ According to the rules it is right but not for finding an equivalent fraction. $\frac{5+4\cdot 3}{11\cdot 3} \rightarrow \frac{5+12}{33} \rightarrow \frac{17}{33}$ $\frac{(5+4)3}{11\cdot 3} \rightarrow \frac{9\cdot 3}{33} \rightarrow \frac{27}{33}$ The results are different, but the right one is $\frac{27}{33}$, because I have to multiply 5+4. The same thing is worth for $\frac{a+b}{2}$. Now $\frac{(a+b)\cdot 2}{2\cdot 2} \rightarrow \frac{(a+b)\cdot 2}{4}$ is right because I have to multiply $a+b$. I have to consider as if it were an unique natural number.

In table 2 we report a protocol of a girl who, after an initial confusion for the missing use of the parentheses, carries out an interesting analysis of the situation which testifies the style of work in the class, which is centered on the explication and control of the processes and on the meanings of the numerical or algebraic writings.

The questions of the comparison of rational numbers turned out to be the most difficult one and the pupils have preferred using previously known strategies (comparison of the correspondent decimal numbers, recourse of graphical representation) rather than to the new ones (reduction to the same denominator, crossed multiplication). There were pupils who have expressed the comparison by referring to the differences between numerator and denominator (for instance a pupil, for comparing the pair $\left\{\frac{19}{36};\frac{11}{24}\right\}$, makes a wrong graphical representation and writes 36-19 = 17 e 24-11=13, the he adds "it is smaller $\frac{19}{36}$ because more pieces remain"). There also have been personal strategies, for instance a seventh grade pupil who, for comparing the pair $\left\{\frac{3}{4};\frac{5}{6}\right\}$does the comparison of the pair of their complements to the unit $\left\{\frac{1}{4};\frac{1}{6}\right\}$ and says that $\frac{3}{4}$ is smaller than $\frac{5}{6}$ because $\frac{1}{4}$ is bigger than $\frac{1}{6}$. For comparing $\left\{\frac{2a}{17};\frac{3a}{22}\right\}$ some pupils substitute a same number to the letter a and compare the correspondent fractions. The problematic nature of the results on the comparison has been object of discussion among teachers and coordinator of the research and it is being currently analysed.

Question 5 was proposed some months after the previous activities in order to assess the incidence of these with reference to the control of the meaning of the given equalities, about it we observed that the less able pupils have enacted intuitive strategies. The questions where the denominator was expressed as addition have resulted more difficult for low/middle-level pupils. About the questions involving the operation of multiplication and division several students have resorted to the inverse operations, for instance in front of $\frac{k}{3}\cdot\frac{4}{m}=\frac{5}{6}$ the pupils have written: $\frac{4k}{3m}=\frac{5}{6}$; $\frac{5}{6}\div\frac{4}{3}=\frac{k}{m}$; $\frac{5}{6}\cdot\frac{3}{4}=\frac{5}{8}=\frac{k}{m}$. In this case we have also observed that the pupils do not suceed in doing a relational reading of their results, for instance some pupils have gone on writing $k=5$ and $m=8$ without considering the possibility that by multiplying these values for a same natural positive number others could be obtained.

An example of question faced in collective discussion: the construction of the definition of addition between rational numbers.

For the collective construction of the operations between rational numbers, the following points have been very productive: i) working on the equivalent fractions; ii) resorting to the comparison with decimal representation (for looking for or giving a legitimacy to the hypothesised definitions); iii) the guide-idea of controlling the preservation of the results of the operations in the passage from natural numbers to rationals. For question of room here, we limit ourselves to giving a classroom activity synthesis (teacher L. Gherpelli) about the construction of the definition of the addition (what has happened has given the opportunity to highlight a very subtle and unforeseen aspect of the question).

The teacher starts posing the question of what $\frac{3}{4}+\frac{2}{5}$ can be. A pupil states that adding respectively the numerators and the denominators the sum would give $\frac{5}{9}$; changing the representations of the fractions, for instance considering $\frac{6}{8}+\frac{4}{10}$ it would be $\frac{10}{18}$, which is equivalent to $\frac{5}{9}$ and the definition works. The teacher poses in discussion the correctness of this idea in the class. The pupils appear convinced. Then she writes on the blackboard $\frac{3a}{4a}+\frac{2a}{5a}=\frac{5a}{9a}=\frac{5}{9}$ trying to force the pupils to recognize the particularity of the situation. But there is still agreement on the reasoning of the classmate. Then the teacher led the pupils to observe that multiplying by 3 both terms of $\frac{2}{5}$ one obtains $\frac{6}{15}$ and that the addition $\frac{6}{8}+\frac{6}{15}$ will give $\frac{12}{23}$, result which should be equivalent to $\frac{5}{9}$. The fractions are compared and the pupils verify that they are not equivalent. This episode gives the pupils the opportunity to understand that, even if the rule works on multiplying all the terms of the two fractions for the same number, for having general value it has to work in general for fractions which can be obtained from the given one multiplying their terms for different numbers.

At that moment the problem of the definition of the addition is still open. Then the teacher recalls the question of the coherence with the decimal representation: $\frac{3}{4}+\frac{2}{5}$ has to be as 0,75+0,4=1,15 which is $\frac{115}{100}$, and she suggests to investigate on the relationship of this rational number with the others represented by the addends. The pupils have a moment of confusion. The teacher suggests that they simplify the fractions, the pupils indicate $\frac{23}{20} \leftarrow \frac{:5}{:5} = \frac{115}{100}$, then she proposes to look for a relationship between $\frac{3}{4}+\frac{2}{5}$ and $\frac{23}{20}$, at this point a pupil goes to the blackboard and writes $\frac{3 \cdot 5}{4 \cdot 5}=\frac{15}{20}$, $\frac{2 \cdot 4}{5 \cdot 4}=\frac{8}{20}$ another pupil observes that 15+8=23. Then collectively they observe that $\frac{3}{4}+\frac{2}{5}=\frac{(3\times 5)+(2\times 4)}{20}$, this way each fraction is multiplied for the denominator of the other. For the teacher in this situation, the utility of the previous activity on the plural representations of natural numbers through what the pupils get the custom to express a number as arithmetical expression of others has been evident (Malara and Gerpelli, 1997). Then the teacher asks the pupils to determinate the sum $\frac{2}{3}+\frac{5}{16}$. The pupils observe that $\frac{2}{3}$ is as $\frac{8}{12}$ or as $\frac{32}{48}$, $\frac{5}{16}$ is as $\frac{15}{48}$ then their sum is $\frac{32+15}{48}$; still their observe that $32 = 2\times 16$ and that $15 = 3\times 5$ and the previous rule works. After other trials the teacher and the pupils arrive at formulating the general rule $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$. Finally the teacher proposes to find a confirmation of the rule considering the addition of two natural numbers and sees what happens if these naturals are seen as rational numbers. She writes $3+2 = 5$, $3=\frac{6}{2}$, $2=\frac{10}{5}$, the pupils write that $\frac{6}{2}+\frac{10}{5}=\frac{6\times 5+10\times 2}{2\times 5}=\frac{(6+2\times 2)\times 5}{2\times 5}=\frac{10}{2}=5$; but also in general $\frac{3a}{a}+\frac{2b}{b}=\frac{3a\times b+2b\times a}{a\times b}=\frac{5a\times b}{a\times b}=5$.

Conclusive considerations

The research is still in progress and it is too early to speak of the definitive results. But a surely positive one is the fact that the teachers have recognized the productivity of this new ways of looking at rational numbers. In particular they underline that: i) using letters allows metacognitive teaching on the properties and algorithms of the operations; ii) beside allowing the pupils to distinguish the number from its representation, working on the multiple representations of a number in the ambit of natural numbers is very important; because, here too, it makes them easily accept to see equivalent fractions as different representations of the same rational number; iii) pay attention to the structural aspects allows an easy approach to two aspects which, didactically speaking, are rather delicate: a) widening the concept of number; b) widening the numerical ambit.

Other questions, related to the comparison of rational numbers, have to be reconsidered and deepened. Particularly the conceptualization of the relationship between the method of reducing at the same denominator and that of doing the crossed multiplication and also the question of how the value of a fraction varies in correspondence of the variation of its numerator and/or its denominator. Moreover there is still a controversial question, as to which the teachers and the coordinator of the research have different positions, about the characterization of the equivalence classes. The teachers are not convinced about the opportunity to pose this question to the class, even restricted to particular cases, in spite of the simplicity of the concepts in play and the experiences of the pupils in the use of the letters and the substitutions.

This highlights two important questions about the realization of the research of didactical innovations: the necessary times of reflections of the teachers to arrange the new points of view, the incidence of the convictions of the teachers in the selection of the activities to pose in the experimentation.

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