The "0", is it an obstacle?

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Introduction

The reference paradigm of this work is included on *Ricerca in Didattica* [2], [33].
A Mathematics didactic research should pay attention to investigate:

- The epistemological representations concerning the mathematical content;
- The historical-epistemological representations concerning the mathematical content;
- The behaviours study attended by the students regarding the experimental activities proposal.

In the passage from the arithmetic thought to the algebraic one an important role consisted on in-depth study of the so-called epistemological obstacles.

We are aware that the individuation of the obstacles is a very huge research field and it requires different confirmation, supported from historical-epistemological and experimental investigation, from the setting-point of didactic situations that over the same obstacles. For that reason, in this note, we are limited to consider the relative obstacle to the *"zero"* and to try and find any possible causes.

This work come from different experiences of the single authors; particularly those ([5], [6], [16], [17], [18], [32], [33]) and to contents of the GRIM Seminar, March 1996, concerning the *"zero"* like obstacle.

We are added in *Appendice 1* some errors list appeared on the "Il Pitagora" Magazine, given that their contents are the key points for the issues of this work, and also because they point out the students and the teachers conceptions in an historical period end of the 1800 to the beginning of 1900. These concepts are very useful to understand today's conceptions.

Those errors originate from knowledge acquired for other purposes and adapted to different problems. In that sense they are considered, like Bruosseau say, "obstacles" to learn the mathematics.

Surely that obstacles could be defined didactic, in the terms that they depends on, in general, of the didactic transposition, or they could be defined with a didactic origin, in the terms that are tied up to the communication moment coinciding with the epistemological obstacles, that has an important role on the knowledge.

This work concerns some considerations about the epistemological representations, based on some historical -epistemological traces, and a first experimental phase relating to a preparation of a pre-test (see *Appendice 2*)

1.1 Why the "zero"? Does it exist a zero question?

The "zero" is a sign/symbol very interesting and singular that cause the thought, It produce paradoxical ideas, making reflection on different field furnishing ambiguous answers. It is closely connected to the idea of "Nothing", "Anything", "Number", " Virtual World" "Variable", "Greatness". The introduction of the *"zero"* on the practical mathematics, like that of *"escape point"* in the prospective art and like that of *" imaginary money"* in the economic exchange, can be view semiotic equivalent to the same meaningful configuration; indeed the first is a sign between the number signs, the second is an image between the images and the third is a coin between the coins able to *produce a level change* in the learning and acquaintance process transforming something that in an inferior level is purely *operational* in something *structural*, in object to a superior level.

An idea of the "zero" complexity it has given from the fact that, like mathematical sign, it introduced with difficulty in the European culture; indeed, being closed to the

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"nothing" it became difficult to understand it like a *sign on the signs*, *that is a meta-sign*, with the meaning of point out, derived form a syntax included on it, the absence of some other signs. It is, on the other hand, a *name* that points out a *number* too. This double appearance has allowed the zero to be serving like *ambiguity place between an empty character and character meanings the void*.

The zero represents the non-presence of the numbers 1,2., 9 and at the same time it produce the whole progression endless of the whole.

For Rotman [30] the understanding of the zero role causes a semiotic closing that is playable from the algebraic variable in the XVI century, formed around the notion of *not meaningful presence* of some signs.

What could it be told with the help of the sign *"zero"* that could not it be told without it?

1.2 The semiotic closure of the Zero

At the end of the XVI century S. Stevin (1548-1620), Dutch mathematician, in his treatise "The Dime", he pleaded the extension of the Indian numeration system from the decimal finite to infinity expressing *big wonder for the creative power of the zero*, for the way in which it build an infinity of number signs.

He threw back the classical idea of number, that is it must point out always-definite things; otherwise from, (i.e. Klein [12]): "*Plato*" that he speaks about numbers with *visible and tangible bodies…. that is, counting sheep, horses…*these processes give numbers of *sheep* and of *horses* and from *Aristotle* that speaks about *abstractions* but always linked to specific concrete collections. In each case for them the number is an assemblage of unity.

For Stevin the unity was a number like each other; the *arché* of the number was not the unity but the zero; the zero is the own origin of the number: as like for the geometry the point generate the straight line as the zero gives origin to the numbers.

Stevin gives a semiotic interpretation of the number transferring the lack of reference of the zero, that is his lack of *positive content*, to *all* the numbers

For Stevin the numbers are signs that *take meaning in relationship to other signs*, and *the creation of a meaningful infinitely long system* it was the first approach to the continuous real uni-dimensional and it had a big importance for the XVII century mathematics and for the following one to arrive at the ideas of the real number from Cantor and Dedekind.

Stevin used an algebra with an endless summation; for example 0,3333... meanings
3 (0) +3 (1/ 10) +3 (1/ 100) +3 (1/ 1000) *ad infinitum*, using a language where in the same moment there are numbers determined, possibly unknown, but fixed (constant), and there are not numerical indefinite entities ("variables").[2].

Today we consider *the variable like a sign*, whose meaning is in relationship to other signs necessarily absent, inside of an algebraic expression, signs that constitutes the domain, and like *meta-sign*, given that it points out the virtual presence, potential, possible but not real of a sign.

The mathematics essential idea of the variable, due also to F. Viète, 1540-1603, (contemporary of Stevin) is that: the variable is like an *indefinite number with which it could be calculated like if it had determined.*

1.3 What is the connection between the zero and a variable like meta-sign?

The zero works in a dual way: it moves between his inside role, like number between the numbers and his external role, like meta-sign that gives beginning to the activity of the subject that it count.

The same things happens for the algebraic variable, internally it can be manipulate like object, sign between the signs in formulas, it is treated like numbers sign according to a

[2] - C. Singer in [31], pag. 206, claim to S. Stevin the merit to introduce the decimal system to the fraction representation.

common syntax (added, multiplied.) *externally points out the possible presence, but not real of signs of number*.

This duality is mediate from a new mathematical subject, *the algebraic subject*. It has the ability of mean the absence of the subject that counts, the shift of a *real* presence to a *virtual. The distinction between the algebra and the elementary arithmetic, it stays quite in the distinction between the subject that counts and the algebraic subject that performs the calculation staying autonomous and arithmetically aware of oneself.*

The variable is a sign for the signs that they could be products from whoever counts it. The *zero* also being a connected sign with the idea of *nothing*, of *empty*, of the place in which *nothing is*, represent: the origin of the calculation, the trace of *whoever counts*; in other words of who causes the sequence of the numbers.

The algebraic subject performs an operation of closing on the infinitive proliferation of the signs of number, which they come in be with the *zero*.

1.4 The "zero" is an epistemological obstacle?

Surely we could affirm that the *"zero"* it is an epistemological obstacle because:

1. It is a *errors catalyst*, in practice a very effective test for really estimate how much and which acquaintance (arithmetic-algebraic) it have stayed transposed in know from part of the student;
2. It is historical in the sense that the diffusion of the *"zero"* in the western culture happens only in the 1202 with the "Liber Abbaci" of Leonardo Pisano and it is used essentially like numeral, while for consider it like number to all the effects, we need to wait at the end of the 1500; it is rooted the conviction according to the Greek and the ancient and classical world they generally don't know the *"zero"* (The discussion however is open) [3].
3. Like "error" exists and it persists in time.

For Brousseau too the *"zero"* it is an epistemological obstacle given that:

1. It does not reside in the "formulation" of the acquaintance institutionalise but in the representation that the subject communicates to assure the operation and the comprehension of the knowledge.
2. It is a *knowledge* that is acquired when we take conscience of his role in her begun to point of the know and in the passage from the arithmetic to the algebra.

The *"zero" is present in the arithmetic* and it is possible affirm that it doesn't constitute a problem both for the operations of addition and subtraction; that they is included in the "common sense," and also for the multiplication operation sight like repeated addition: in fact, the equality $n{\cdot}0=0{\cdot}n=0$ comes easily approved going on to the common language *nothing plus nothing equal to nothing*; while for the operation of division, the *division by zero* it doesn't presented to the students given that it does not make sense.

The primary school pupils take confidence with the zero numeral and they are able to reach a good manipulation using the abacus, and to the decimal or polynomial presentation of a number, but generally doesn't come in evidence them the peculiarity of the *"zero"* as regards the other numbers and in succession, when it gets them in the division, it *is implicit* that the divisor is higher than zero.

In this way in the students it is created a knowledge which, the *"zero"* does not a problem; in fact, he is represented from them like nothing, void in accord with their *common sense*, while it is an image, a sign out of the *natural language*.

The zero is essentially understood from the *primary school students*, like an operational process and *not structurally* like object.

In the algebra field, sight like generalisation of the arithmetic[4], when it is used

[3]-[30], [34]

[4]P. Boero: "t algebra organise itself beginning from the signs of the four arithmetical operations and from that of equal, with the relative rules of use " in L'Insegnamento della Mat. e delle scienze integrate , vol.15,n.10

without consider his peculiarity, the *zero becomes a problem* since in this new context it is necessary, *it is a constitutive brick* for the syntactic construction of the same, It engages a fundamental role in the algebraic structures, a *pregnant meaning* in the discussion and in the equations solution, it *loses the contact* with the reality and the interlacement of the two languages: the natural one and the formal one, it become cause of tension and source of difficulty.

The *"zero"* that causes, generally, correct answers in the arithmetic context, it furnishes thick false answers in the algebraic context instead [5]. *The passage from the arithmetic to the algebra was possible, only when the "zero" has culturally receipted in the double valence of sign and of meta-sign. To explain what it meaning the zero like sign it is necessary must mean the other signs, it points out the absence of "things" and the absence of "signs". Therefore the "zero" has a privileged law to the attention.*

2.1 Approaches to the number concept on the primary school

Analysing the approaches to the number in the primary school, We can highlight that they try *to do recover the concrete experiences did from the children in their world,* trying to have a character formative and *trying to valorise the productive thought of new discoveries, to create not much the ability of "to do" as to form the ability of " to arrive to do, understanding what it do."*

The initial approach to the concept of number naturally it must be in harmony with the model that the child already has inside of oneself, he must not use an artificial language, but he must follow the natural processes rather, and stimulate the abilities logic-cognitive of the child for don't revert on the *structural maximalism,* come back in the 1960-70, characterised from a certain type of formalism and from the premature use of the symbolism, or in the *programmatic minimalism* of the previous period, that preserving as well to the arithmetic a principal place, sometimes exclusive, in the programs of the elementary school written from the 1860 onwards, it considered *practice* only, that is having like unique objective the learning of the four operations for the resolution of *practical* problems [8], [14], [26].

The idea of the natural number is complex and requires therefore "an approach that use different point of view (ordinality, cardinality, measure, set theory, recursiveness etc.)" [3] and it is acquired after long interiorization from part of the child.

It on the other hand the concept of number is an abstract concept which arrived through an evolution of the logical thought happened in various step [1], [7], [10], [20].

The approach to the number is the development of a didactic itinerary that uses a definition of number according to a certain mathematical theory; for Pellerey [27] the more remarkable approaches for the construction of the concept of number are:

1. Approach of the set theory
2. Ordinal Approach
3. Recursive Approach
4. Based Approach on the measure

In many didactic plans for the the first cycle of the primary schools, there is a kind of *pluralism* of approaches [11], in general they recover the different aspect of the number, approaching to him from different point of view, recovering the extra-scholastic patrimony of the child, that He already found to unconscious level, a strategy that even if not systematic, it succeeds functional to his demands.

The approach of the set theory, it is based on the definition of number of Frege and Russell and it use the language of the *ingenuous Theory of the whole*.

The number is sight like the representation of a class of equipotent whole and not of this or of that particular object, recovering in this way the cardinal appearance.
the zero in this approach is represented from the class of the *whole with no element*, the class

[5]In [19] Mariotti show the necessity to make the student thought over meaning of division by zero again.

of the empty whole, the *zero*, like number is not "nothing", "empty" but *a concept with his precise determination.*
For the student instead the *"zero"* it is intimately connected to the idea of nothing, of empty, *of what it is not, of what there is not*, instead, for example, 2 is connected to the idea of *what there is,* he/she/it/you of what is: a couple, a pair, a double, a duo; the association to the empty whole makes appear the meaning of the "zero" different from that of the other numbers. *"The zero is a quantity that disappears, that there is not, the zero is the sign of the absence of quantity"* [19]. *The zero is as intimately connected to the idea of nothing, of empty. In set theory approach often is pointed out also the ordinal appearance of the number,* each number occupies a his precise place in the numerical sequence, departing from an initial element (zero or one) it is gone on, after have done a correspondence one to one, implementing the whole with a different element, developing as also the *recursive thought* (iterative), based on the function that associates an element the his following.

The recursive thought could be acquired exploiting *natural sequences* that the children already have, also like the chancing of the day and of the night, of the seasons or *personal experiences*, like those of the eat and of the sleep, of the cardiac pulsation.

The child builds sequences in his mind *not numerical sequences* linked to the action and *the zero has lived like absence of something: of movement, of sound and so forth.*

The *recursive approach* allows to build the whole of the natural numbers through the *verbally count* and it is referred, as like, to the ordinal approach to the axiomatic theories of the natural numbers.

In the measure-based approach, the natural number has *connected* more than to an object quantity of a whole *to the quantity of measurement unit* that could be found measuring a certain quantity.

This approach presents some difficulty, *the Greek* thought about the number like measure of everything's, they were used to identify *measure* with *count*, as like the child that tries to understand for instance, how many glasses could be filled with a bottle of water.

The child doesn't have idea of the need to the invariability of the champion used for the measure, for which the calculus could result not correct.

Measurement, in fact, it is more complex that count, since, fist of all, needs individualise an ownership as regards which compare the objects (i.e.: length, ability, etc.), apply therefore to a measure *relative* respect to a *champion* for see how much times she is contained in the objects for spend so that the measure to the number.

In the didactic activity for the approach through the measure could be used *rulers or numbers in colour* for visualise that the unity is contained a number of times precise in an other number.

In this approach the zero could not be taken in consideration since doesn't make *sense* measure void quantity or to consider measures sample void.

In general a geometric approach doesn't take into special consideration the void quantity that appears more or less hidden in the study of the geometry.[6]

2.2 An epistemological "reflection" on the role of "zero" in the axioms of Peano, Padoa and Pieri.

Peano's formulation is still largely used in the education of teachers. In *"Arithmetices Principa nova methodo exposita"*[7] of 1889 G. Peano describes Arithmetics as a *hypothetical-deductive system*, based on four primitive concepts (number, unity, successor and equal) and nine axioms (four of which regarding equality). Later on, beginning in 1891[8] and in the five (official) editions of *"Formulario Mathematico"* (1895-1908) the primitive

We relate in *Appendix 5* a C. Ciamberlini's article, [4] on the null quantities in geometry.
[7]- art.16,[22]
[8]- art.37,[23]

concepts are reduced to three (number, unit, successor and from 1898 [9] on number, zero and successor) and the axioms to five (see *Appendice 3*).

In these works Peano uses the symbol N (sometimes N_1) to designate a positive integer number and the symbol N_0 to designate an absolute integer number, and he says in *"Aritmetica Generale e Algebra"* [25] the number zero to be a "non natural" number.

Peano's axiomatic systems are enough for the arithmetic of positive or absolute integer numbers, being the axioms independent. A. Padoa è [21] noticed that, even though Peano's postulates are independent, the system of primitive concepts is dependent from the postulates. This because from them you can deduct the definition of the number (0 or 1) that is not successor of any other. He assumes as primitive concepts only number and successor and as primitives the following:

P_d1) The *successor* of a number is a *number*.

P_d2) Two *numbers* having the same *successor* are the same number.

P_d3) At least one *number* exists, that is not *successor* of any *number*.

P_d4) If a class (of *numbers*) contains a *number* that is not successor of any other and if the successor of any *number* of the class belongs to the class, then every number belongs to the class (complete induction principle).

As a consequence Padoa demonstrates that there's only one number that is not successor of any other, the following:

Theorem T1-Let x and y be two numbers for which x suc z[10], for every z, then x=y.[11]

Defining the unity or the zero as the only number that is not successor of any other, Peano's axiomatic system for N and N_0 respectively (see Appendice 3), is equivalent to Padoa's.[12]

M. Pieri in 1907 [28] proposed a modification to Padoa's axiomatic system, to be able to substitute the principle of complete induction with a simpler proposition.

Like Padoa he assumes as primitive concepts number and successor, and formulates the following mutually independent postulates:

P_i1) At least one number exists

P_i2) The successor of a number is number

P_i3) Two numbers that are not successors of any number are always equal.

P_i4) In every non-illusory class of numbers there is at least one number that is not the successor of any number of the class (Minimum Principle).

P_i4) is commonly known as the Minimum Principle as Pieri so defines it in [28], page 450:

"...this existential judgement is to be preferred to d PD4 as easier: and I am not referring to evidence; as it is about the common knowledge that among the numbers (integers, positives or null) of an existing class there must be one that is not bigger than any other."

The axiomatic system of Pieri is then equivalent to that of Padoa and as a consequence to that of Peano (see Appendice 4).

This equivalence has lead to the indifferent utilisation of Peano's axiomatic system or of Pieri's. At times the 'Principle of complete induction" is used and at times the 'Minimum Principle", even though there's not a complete correspondence between them. The 'Principle of complete induction" is equivalent (see Appendice 4) to the following propositions: a) There exists only one number that is not successor of any other number, b) In any non-illusory class of numbers there exists at least a number that is not successor of any numbers of the class (Minimum Principle).

The axiomatic systems of Peano have different models, which are there are different sets of objects where it is

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[9]- art.99,[24]

[10] -successor of z

[11] -Dim: Let us assume that is x?y and that I is the set of all numbers different from y, therefore $x \in I$ and $\forall t \in I$ suc$t \in I$, then for P_d4) I includes every number and y is not a number, but this is absurd.

[12] -In fact the axioms P2) and P_02) coincide with P_d1), P3) and P_03) with P_d2), P5) and P_05) with P_d4), while $P1) \wedge P4)$ and $P_01) \wedge P_04)$ imply P_d3) and vice versa.

possible to interpret the primitive concepts in such a way that the axioms be satisfied. For example:

$N=\{1,\ 2,\ ...,\ n,\ ...\}$	where "1" is the "unity" and $n'=n+1$ the "successor". (1.1)
$N_0=\{0,\ 1,\ 2,\ ...,\ n,\ ...\}$	where "0" is the "zero" and $n'=n+1$ the "successor". (1.2)
$P=\{2,\ 4,\ ...,\ 2n,\ ...\}$	where "2" is the "unity" and $(2n)'=2n+2$ the "successor". (2.1
$P_0=\{0,\ 2,\ 4,\ ..,\ 2n,\ ...\}$	where "0" is the "unity" and $(2n)'=2n+2$ the "successor". (2.2
$M=\{n,\ 2n,\ 3n,\ ..,\ kn,\ ...\}$	where "n" is the "unity" and $(kn)'=kn+n$ the "successor". (3.1
$M_0=\{0,\ n,\ 2n,\ ..,\ kn,\ ...\}$	where "0" is the "zero" and $(kn)'=kn+n$ per "successor". (3.2
for $a\neq 0$, $A=\{1/a,\ 1/a^2,\ ..,\ 1/a^k,\ ..\}$	where "1/a" is the "unity" and $(1/a^k)'=1/a^k\cdot 1/a$ the "successor". (4.1
For $a\neq 0$, $A_0=\{1/a^0,\ 1/a,\ ....,\ 1/a^k,\ ...\}$	where "$1/a^0$ " is the "zero" and $(1/a^k)'=1/a^k\cdot 1/a$ the "successor". (4.2
for $b\neq 0$, $B=\{b,\ b^2,\ ....,\ b^k,\ ....\}$	where "b" is the "unity" and $(b^k)'=b^k\cdot b$ the "successor". (5.1
for $b\neq 0$, $B_0=\{b^0,\ b,\ ....,\ b^k,\ ....\}$	where "b^0 " is the "zero" and $(b^k)'=b^k\cdot b$ the "successor". (5.2

The first three models are additive, and the elements are in arithmetic progression of *reason* equal to the respective units; the last two models are multiplicative, and the elements are in geometric progression of *reason* the respective units.

In the examples given it is of significance to note that all elements of the sets considered can be obtained by means of the repeated application of the successor operation. This is the basis of the axiomatic system of Peano, and it always depends on the *unity* never on the *zero*, being this indifferent to that operation.

Comparing the two axiomatic systems of Peano and the relative models, it is to be observed that it is possible to pass from one to the other by simply exchanging the words "unity" and "zero", both in the primitive concepts and in the axioms. *This exchange*, however, *is by no means enough*, to define the operations of addition and multiplication.

These operations, in the example (1.1) and (1.2), will be dually defined respectively by:

i) $n+1=n'$ ii) $n+m'=(n+m)'$	i) $n.1=n$ ii) $n.m'=n.m+n, \forall n,m\in N$ (1.1)
i) $n+0=n$ ii) $n+m'=(n+m)'$	i) $n\cdot 0=0$ ii) $n\cdot m'=n\cdot m+n, \forall n,m\in N_0$ (1

The $N_0(+,x)$ structure is richer than the structure $N(+,x)$ as it contains a neuter element for both operations respectively; the neuter element of the multiplication operation behaves in a "natural manner" with regard to the addition operation[13]. Whereas the neuter element for the addition in (1.2) has a *"non-natural behaviour"* with respect to the operation of multiplication as $n\cdot 0=0\ \forall n \in N$, that is *"zero destroys"* every natural. It follows that the addition neuter element doesn't have a "natural behaviour" with respect to the operation of division, inverse of the multiplication.

In a similar way the observations for (1.1) and (1.2) can be extended to all models presented.

[13] That is it builds natural numbers, indeed n+1 is a new number and so $n+m\neq n\ \forall n\in N$ with fixed m.

In their works on the natural numbers Dedekind[14], Cantor[15], Peano[16], Landau [15] have excluded the null class and the null numbers.
Differently from Peano's, the axiomatic systems of Padoa and Pieri can be used indifferently, both to introduce natural numbers and to introduce the absolute integers. To this purpose it *suffices to consider as number that is not successor of any number, the unity or the zero.*

Historically, N and N_0 have not been considered indifferently, even though nowadays the seeming "resemblance" is so much part of mathematics teaching, *that with N the set of natural numbers is indicated, without specifying if the zero is included or not.*

4.1 Final Observation (conclusion).

To overcome obstacles, in general, ad hoc non-didactical situations are built. This might not be necessary for the *zero*, as, during normal scholastic courses, the zero is introduced, according to its different aspects (figure, void set, neuter element...), with different didactic strategies that discuss again the assimilation of the concept.

In a test many students of first year of Mathematics replied correctly to the question: " *2/0* and *0/2* are both meaningful expressions in R? ". Interesting are the different motivations given from students of different types of schools.
The students of scientific secondary schools or technical institutes have motivated with: 1) $2/0=\infty$ or 2) $\lim 2/x=\infty$ The students of classical secondary school have motivated with: 3) *there is no inverse for x=0.*

From the many correct answers it can be derived that the zero doesn't represent a catalyst for errors anymore, and the systematic learning has happened through the powerful means of analysis and algebra. The wrong motivations 1) and 2) show the students uneasiness towards a division for zero.
The motivation 3) reflects the natural assimilation of the abstract concept of field and not the (perhaps) natural numerical development or enlargement of the scholastic programs.

The *zero* is a good example to show that its elimination happens (if it happens) if it becomes, like for the *"mathematicians"*, an element of their *"common sense"*. This happens, we think, for students not attending a scientifical Liceo (to be verified).

This article has tried to give an answer to why the zero is still a catalyst of errors now as 100 years ago. The diagnosis is easy: *"it is an epistemological obstacle"*; difficult is the cure, as it needs a different action. In different context, in fact, the *zero* assumes different meanings, logically accepted even if notwithstanding common sense, producing almost always exceptional cases in the theory considered. (see geometry:...analysis: infinity, physics: abstract models)

The *zero* needs an awareness in the teacher that from elementary school has to underline its distinct role from the other natural numbers. It will be surely necessary *to emphasise* every time the peculiarity of the *zero*, not inserting it only in limit cases, but in the theory (see null segment, null angle, null solution,..)

Bibliography

[1] Barrow J.D. *Pi in the Sky Couting, Thinking, and Being,* 1992 Oxford Univ.Press [trad.Cannillo T, *La luna nel pozzo cosmico*, 1994, Milano, Adelphi ed.S.p.A].

[2] Brousseau G. *Fondements et méthodes de la didactique de mathématiques*, Recherces en didactique des Mathématiques, 1986, Grenoble ed. la Pensée Sauvage, Vol.7.2.

[3] Camarda S.-Magro M.T.-SpagnoloF. *Il concetto di numero* , 1982, Firenze Giunti Marzocco.

[4] Ciamberlini C. *Le grandezze nulle nella Matematica elementare*, 1920, Torino, Saggi di Didattica Matematica, Raccolta di scritti vari, G.B.Paravia & C.pp.94, 96.

[5] Di Leonardo M.V.-Marino T.-Spagnolo F. *Alcune osservazioni didattiche ed epistemologiche sul postulato diEudosso-Archimede ed il metodo di esaustione,* 1994, Bologna La Mat. e la sua didattica, ed.Pitagora, n. 1,

[14] R. Dedekind, *Essenza e significato dei numeri, Continuità e numeri Irrazionali*, ed.A. Stock 1926, Roma;
[15] In [20], page 26.
[16] G. Peano says *non natural* the number *zero* in [25], in the notes 1.2 e 1.4 page 29 and 1.5 page 30.

pp.25, 37.
[6] Di Leonardo M.V.-Marino T.-Spagnolo F. et Alii *Considerazioni su alcuni articoli di Didattica della Matematica della rivista "il Pitagora"*, 1994 , Bologna La Mat. e la sua didattica, ed.Pitagora, n. 4 pp.383, 389.
[7] Enriques F. *Questioni riguardanti le Matematiche Elementari*, 1900, Bologna Vol.1, parte 8.VI, Zanichelli, (ristampa 1983).
[8] Ferrari M. *I nuovi programmi per la Scuola Elementare: ARITMETICA*, 1985 Paderno del Grappa L'inseg. della Mat: e delle Scienze integrate, vol 8, N 14, Agosto, pp.5, 15.
[9] Freudehantal H. *Ripensando l'educazione matematica*, (a cura di F. Manara) 1994.Brescia ed. La Scuola
[10] Gigli D. *Aritmetica generale*, Enciclopedia delle Matematiche Elementari (a cura di Berzolari, G. Vivanti e D. Gigli), Vol.I, parte I, 1930, Milano, Hoepli, pp.81, 212.
[11] I.R.R.S.A.E. Sicilia Servizio documentazione e informazione collana documenti, 1989, VIII, Area matematica- *Nuovi programmi per la scuola elementare*.
[12] Klein J. *Greek Mathematical Thought and The Origin of Algebra*, MIT Press, Cambridge, Massachusetts 1968, pp.46, 47.
[13] Kline M. *Storia del pensiero matematico*, 1991, Torino Einaudi.
[14] Krygoska A.Z. *Processi cognitivi e apprendimento della mat: nella scuola elementare-Le variabili del ragionamento matematico dei bambini all'età di otto anni*, (ed G. Prodi)1984 Brescia ed.La Scuola, pp.7, 18.
[15] Landau E. *Foundations of Analysis*, 1966 New York Chelsea Publishing,Company.
[16] Marino T.-Spagnolo F. "*Il Pitagora", Atti del convegno per il 75°compleanno di C.F.Manara*, Milano,10-11 Aprile 1991.
[17] Marino T- Spagnolo F. *Alcune considerazioni storiche su "IL PITAGORA"* *(Giornale di Matematica per gli alunni delle scuole secondarie)*, Comunicazione convegno storia della didattica, Milano aprile 1991. Catania,1991, Quaderni del Gruppo di ricerca didattica di Catania e Palermo, settembre 1991, pp.115, 118.
[18] Marino T.- Spagnolo F. *Gli ostacoli Epistemologici: Come si individuano e come si utilizzano nella ricerca in Didattica della Matematica*, L'insegnamento della. matematica e delle scienze integrate, vol.19B, n.2, Aprile 1996.
[19] Mariotti M.A. *Lo zero. E' un problema?*, 1986, Cagliari, L'educazione matematica, a.VII, Serie II, Vol.1,n.3, pp.263, 277.
[20] Natucci A. *Il concetto di numero e le sue definizioni*, 1923 Torino ed. Bocca.
[21] Padoa A. *Théorie des nombres entiers absolus*, 1902, Torino Revue de Math.Vol. VIII, ed. Bocca, pp.45, 54.
[22] Peano G. *Arithmetices Principia nova methodo exposita*, Opere scelte, U:M:I:, 1959, Roma ed. Cremonese, Vol.III, pp20, 55.
[23] Peano G. *Sul concetto di numero*, Opere scelte, U:M:I:, 1959,Roma ed. Cremonese, Vol.III, pp.80, 109.
[24] Peano G *I fondamenti dell'Aritmetica nel Formulario del 1898*, Opere scelte, U:M:I:, 1959, Roma ed. Cremonese, Vol.III, pp.215, 231.
[25] Peano G. *Aritmetica generale e algebra*, 1902,Torino, ed. G.B. Paravia & C.
[26] Peano G. *Sui libri di testo per l'aritmetica nelle scuole elementari*, Opere scelte U.M.I. .1959, Roma ed. Cremonese, Vol.III, pp.441, 446.
[27] Pellerey.M. *Quaderni OASI*, 1979, Troina (En), n.4.
[28] Pieri M. *Sopra gli assiomi aritmetici*, Opere sui fondamenti della Matematica ,U.M.I. 1980, Bologna ed. Cremonese, pp.449, 453.
[29] Prodi.G. *Analisi Matematica*, 1970, Torino ed. Boringhieri.
[30] Rotman B. *Signifying Nothing,The Semiotics of zero*, 1987, Stanford University Press Stanford, California, [trad. Faraone D. *Semiotica dello zero*, 1988, Milano ed. Spirali].
[31] Singer C. *A Short History of Scientific Ideas to 1900*, 1959, Oxford University Press, London, [trad. Tedeschi Negri F. *Breve storia del pensiero scientifico*, 1961, Milano ed. Einaudi].
[32] Spagnolo F. *Obstacles Epistémologiques: Le Postulat d'Eudoxe-Archimede*, (Tesi di Dottorato di Ricerca, Università di Bordeaux I (Francia), Luglio 1995. ed.Atelier National de Reproduction des Théses Microfiches (BP - 38040 Grenoble Cedex 9 - Francia) e in Quaderni GRIM di Palermo per la diffusione in Italia (Supplemento al n.5, 1995).
[33] Spagnolo F. *Insegnare le matematiche nella scuola secondaria*, 1998,Firenze,ed.La Nuova Italia.
[34] Vitali .R. *Lo zero presso i Greci*, 1987, Roma, Armando Editore, La mat. e la sua didattica, n.1, pp.42, 44.