

Graphic representations and algebraic expressions: an example of software mediation

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Summary Among the possible didactical uses of Computer Algebra Systems, using them as a help for the formulation of conjectures and as a tool for their verification is particularly interesting. *Pupils often, when they analyse a function from an algebraic point of view, don't pay attention to the coherence of the calculations and results with its graph. In particular, this attitude is evident when a problem is proposed in a way that highlights the algebraic aspect, for example when the equations depend on some parameters. With the use of symbolic and graphic software the students can see these two aspects at the same time. They can compare the role of the parameters with the graphic representation, formulate conjectures based on the observation of the graphs and verify them graphically and algebraically. Starting from these observations, we proposed, to a sample of 16 year old students attending a scientific high school, an activity dealing with second degree functions depending on parameters, using the software Derive for Windows (version 4.09). We would like to present some observations about the role of the interaction student-computer both in the analysis of the situations and in the discussion of the results.*

Introduction

Among the various possible uses of technology in the teaching of mathematics, the use of Computer Algebra Systems (i.e. software for the symbolic processing of calculations and the drawing of graphics) has begun to take hold in the last few years as an aid to the didactics of various mathematical disciplines: algebra, analytical geometry, analysis, linear algebra, probability, statistics.

In effect, the notable potential of such systems allows a broad versatility of use and permits the comparison of situations with complex calculations which are not easily managed by the students with pen and paper. Moreover, the graphic possibilities allow the placing side by side of the analytical expressions and the representations of the curves studied and this offers very interesting starting points.

What's more, it is evident that the growing diffusion of this software poses problems of various types for the teacher in that it is clear that the abilities required and encouraged by C.A.S. are different from those which are useful for carrying out a study with pen and paper.

In my opinion, of particular interest among the possible didactical uses of the software for symbolic manipulations and graphics design is its use as a stimulus for the formulation of conjectures and as a tool for their verification.

The study which is referred to here is the result of research on the teaching-learning of algebra and aims to examine the interaction between the graphic aspects and algebraic aspects in the study of analytical geometry. Many studies on algebraic thinking have brought to light the centrality of the problem of the contemporaneous handling of the symbols and of their meanings. Analytical geometry, since it requires a constant interaction between the geometric aspect of a problem, algebraic translation and geometric re-interpretation of the results obtained algebraically, can be a particularly suitable environment for encouraging a semantic control of the transformations and of the algebraic symbolism.

Moreover, the scholastic use of analytical geometry causes one to lose sight of the connection between these aspects and brings the pupil to resolve the problems algebraically, without worrying about the coherence of calculations and results with the assigned geometric situation. This is manifested, in particular, when a problem is proposed in a form that favours the algebraic aspect and in which the equations contain parameters. The use of symbolic and graphic software can allow the pupils to keep the two aspects present at the same time, examine the rôle of the parameters with respect to the graphic representation, formulate conjectures on the basis of the observation of the graphics and verify them graphically and algebraically.

In this presentation, I would like to display and comment on an activity on second degree functions dependent on parameters which was proposed to a third year scientific high school

class with the support of the software Derive, version 4.09 for Windows. Several observations will be presented which emerged from the analysis of the relative protocols both from the exploration phase of the problems and from the discussion phase, highlighting the rôle of the interaction with the computer at different times. I then intend to consider possible developments of the same activity and to discuss the possible didactic issues.

The choice of the type of software and the version used essentially depends on the fact that, at least in Italy, such software is among the most diffuse and is not very expensive. Plus, there is a version of Derive which is currently available in programmable hand calculators and therefore to a greater extent diffuse. The considerations which follow, however, are not tied to the software used, given that syntactical aspects were not taken into account and that analogous didactical activities can be carried out with any C.A.S.

Work method

The activity that I would like to describe was carried out by proposing to the pupils a guided itinerary by means of work cards which requested a written response to several questions after having drawn some graphics with the help of the software. To do this the pupils worked in pairs at each computer and supplied, to the assigned cards, a single response per couple.

The essential points at this phase of the work are therefore:

- drawing of graphics with the computer
- observation of the graphics with the guidance of the questions
- written responses in agreement with the partner

The cards are then collected and analysed by the teacher who, in the next encounter, proposes a guided collective discussion of the protocols of the various groups.

When necessary, the verification of the conjectures is conducted on two levels:

- algebraic (by hand or with Derive)
- graphic (with Derive)

These phases of work (work at the computer and compiling the cards, discussion, verification) are repeated for each group of cards which introduces a successive level of learning or a new problem; keeping present the conclusions of the previous phases.

The scanning of the phases of work highlights the double function of the software which is used as an instrument for conjecture and for verification by means of the alternate use of the graphics and algebraic windows.

Contents and examples of the work material

The work modality described has been applied to a study of second degree functions dependent on a parameter (sheaves of parabolas with axes parallel to axis y) which we will briefly describe.

The first case which is confronted is the easiest, that is, the one in which the parameter appears only in the coefficient of x^2 : $y=ax^2$

The pupils had previously carried out several lessons which allowed them to familiarise themselves with the software, in which they learned the necessary instructions for supplying values to a parameter from a minimum to a maximum with a predefined step and to represent the corresponding functions of the sheaf in a single graphic. They did this with linear functions.

Now the pupils are asked to trace the curve of the sheaf for a which varies from -5 to 5 in steps of 1 .

The graphic in figure 1 is thus obtained.

Observing the graphic, the students must then answer the following questions:

- 1) Why is a straight line also obtained?
- 2) How can you change the values of the parameter so as not to get a straight line? Can the step influence it?
- 3) What do these parabolas have in common?

4) How are they different?

Then the pupils are asked to draw the curve of the sheaf of the equation $y=ax^2-3x+1$ for a which goes from -3 to 3 in steps of 1 (fig. 2) and to answer questions which are almost identical to the preceding ones.

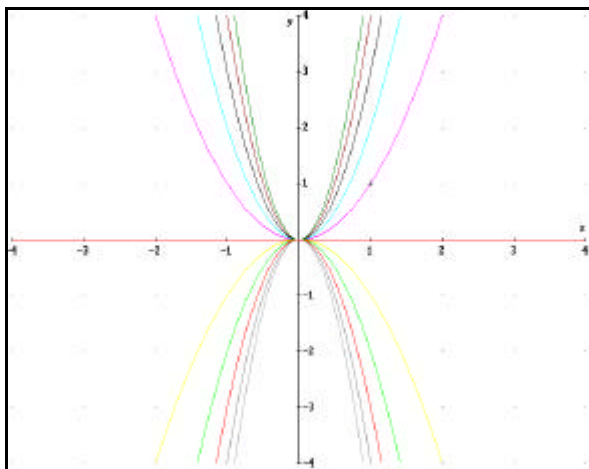


fig.1

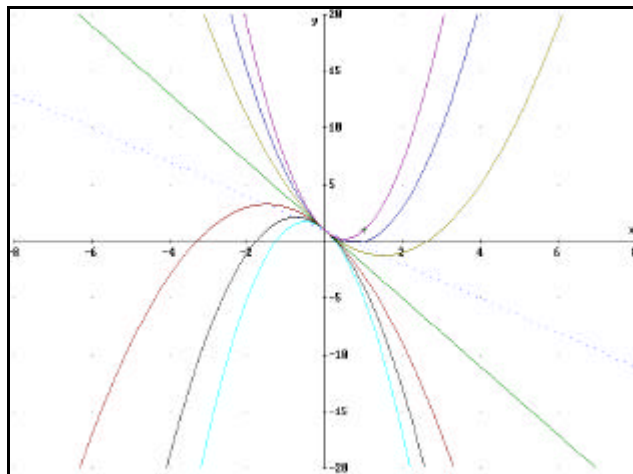


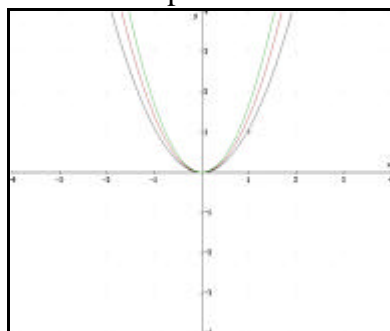
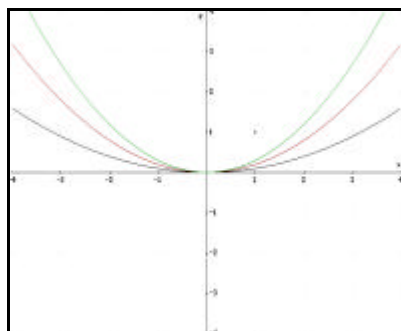
fig.2

The only other thing which is asked is to determine the equation of the straight line obtained for $a=0$ and to conjecture and verify how the vertices of the parabolas of the sheaf are arranged.

The objective of the questions is to direct the observation; to have them consider the ties between the values of the parameter and the parabolas which appear on the screen. The students can understand the influence of the values which the parameter assumes on the representation varying the steps, considering only positive values or only negative ones, erasing or adding parabolas.

Intuition and verification which is the locus of the vertices are not simple at this level of knowledge and must be brought up again in the guided discussion.

It can then be suggested to the pupils to draw parabolas of the sheaf $y=ax^2$ having them assume, for the parameters, “small” values (for example from 0.1 to 0.3 with increments of 0.1), values between 1 and 2 or “big” values and having them compare the graphics obtained so as to draw further indications on the rôle of the parameter.



In an analogous way, the study of second degree functions is then proposed in which the parameter appears only in the coefficient of x .

One of the examples treated is $y=-2x^2+bx$

The questions posed in this case are completely analogous to the preceding ones and suggest observations on

- concavity
- absence of straight lines in the sheaf
- correspondence between values of the parameter and parabolas
- symmetries

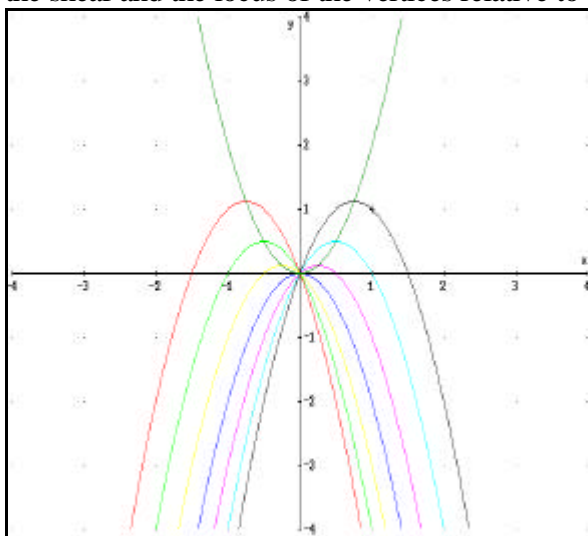
Moreover, conjectures are requested about the locus of the vertices.

The pupils' work on this point and the subsequent comparison of strategies and results was one of the most interesting and constructive periods of the task.

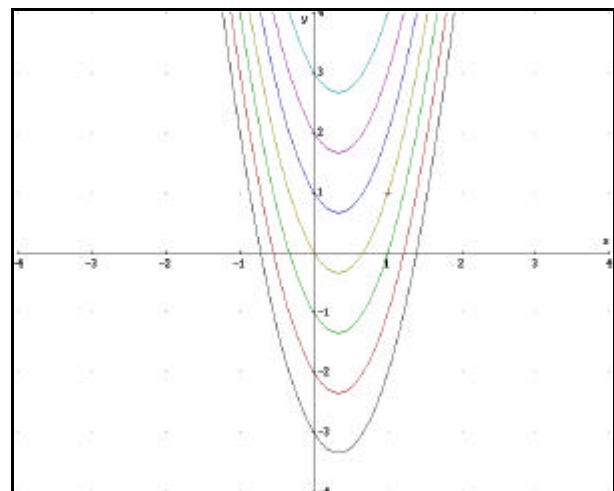
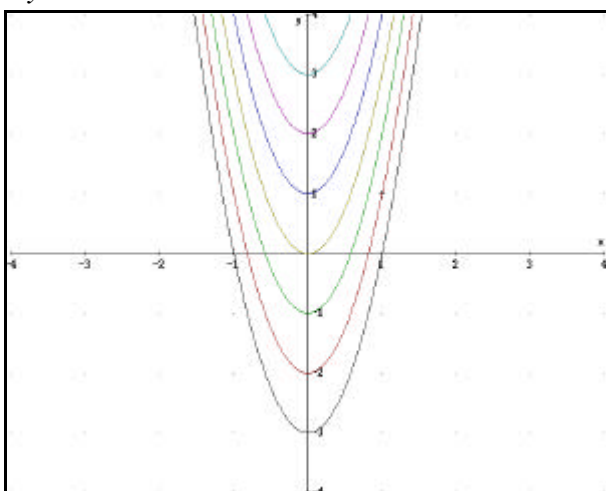
The children's procedures can be divided into at least two typologies:

- those who tried to utilize geometric graphic modalities
- those who followed algebraic procedures.

It is particularly interesting to watch the reasoning processes of the first group, that is, of those pupils who based themselves on the geometric-graphic aspects. A first observation, common for many, was that the locus of the vertices must be a symmetric curve with respect to the axis y and passing through the point $(0,0)$. This is an important observation, generally not present in those who look for the locus of the vertices with purely algebraic procedures. Intuition or perhaps a not completely aware use of the analogy then suggested to some to try to determine, by attempts, an adequate parabola for resolving the problem. Obviously, for these pupils, it was more difficult and less spontaneous to place a verification of an algebraic nature beside intuition and the graphic verification. In the following figure, the parabolas of the sheaf and the locus of the vertices relative to the problem in question are reported.



The study of the case, apparently simpler, in which the parameter appears only in the known term is also interesting. Let's consider the two examples represented in the figure: $y=3x^2+c$ and $y=3x^2-2x+c$

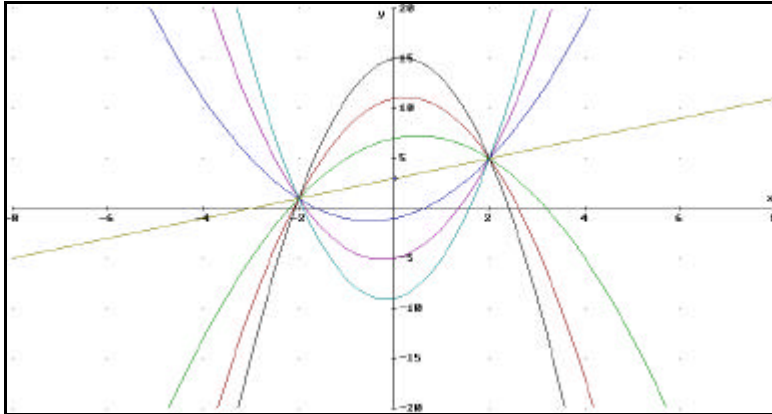


It appears evident, from an algebraic point of view, that each sheaf is made up of parabolas which are obtainable one from the other by translation. Nevertheless, the children were astonished when observing the graphic. As a matter of fact, if the pupils were asked to foresee the one that would appear on the screen and to draw a sketch, it could be noted that their drawings, without the help of the computer, tended to maintain constant the width of the

“alley” between one curve and the other instead of the distance between the ordinates of the points with the same abscissa.

Then, more general situations are proposed in which the parameter appears both as coefficient and as known term. By means of comparison with the preceding examples the idea that it is possible to classify all the situations that can be presented when you have a set of parabolas whose equations depend on a single parameter of the first grade, if there is none, one or two points common to all the parabolas of the sheaf is introduced:

$$y=kx^2+x-4k+3$$



One can arrive thus at the study of the most general situation which can be written:

$$y=ax^2+bx+c+k(a_1x^2+b_1x+c_1)$$

and using the equation between the brackets to identify the points common to all the parabolas of the sheaf. As a matter of fact, if, for a value x , the equations between the brackets is annulled, you will have, whatever k is, the same value of y .

Final verification

At the end of the task an attempt at verification with pen and paper was proposed to the pupils of the themes which were treated in the work unit carried out with the support of the software.

In the verification, complex calculations are avoided to help the pupils concentrate their attention on the procedure, in line with all that was carried on during the work. Among the questions asked of the pupils, there was also, in a few cases, how the exercise might have been resolved using the software. The objective of this question was not so much the verification of the mnemonic awareness of the necessary instructions, but the verification of the comprehension of the essential procedures and, of that, what can be asked of the software to do and how.

The results were quite good especially from the aspect of better comprehension of the problematic situation proposed and a less mechanical use of algebra.

Final observations and possible developments

The software was easily used by the pupils. The possibility of obtaining the graphic representation of the various situations rapidly and easily was utilized spontaneously

- in the formulation of conjectures
- for the verifications.

We note, on the other hand, the substantial refusal on the part of most of the pupils to use the software for carrying out algebraic calculations which they know how to do easily by hand.

Difficulties were revealed in the management of the collective discussion also because, to be able to discuss, it is necessary for everyone to be able to observe the same graphics. For this a projector was used.

The investigation could be extended to other curve sheaves, in particular of polynomials and to accustom the pupils to understanding the ties between the coefficients and the trend of the curves.

We reaffirm, in fact, that the objective of this task is, in particular, that of giving meaning to algebraic manipulation and to keeping constantly present the interpretive graphic aspect which often, on the contrary, is completely neglected.

At the end, it should be noted how the pupils, in the course of the work, have positively changed their behaviour both as regards collaboration in the work at the computer and their participation in the collective discussions and in their attention to the graphic aspect, to the connection of the graphic aspects to the algebraic ones. With respect to the procedures supported by observation and conjecture and then verified algebraically, only a few remained anchored to a greater “trust” in the algebraic calculations and in the formulas learned.

Obviously, there remain numerous questions on the effective abilities acquired by the pupils and a possible balance between old and new skills and knowledge.

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