A Non-Standard Arithmetic Structure as an Environment for Students' Problem Solving and an Introduction into Structural Thinking[1]

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1. Introduction

In the recent years, problem solving has become one of the most important activities of school mathematics, the main reason probably being that it "places the student in the role of actor in the construction of his/her own knowledge" (Grugnetti, Jaquet, 1996). There has been a considerable body of research concerning its use in teaching mathematics (see e.g. Frank, Lester, 1994), the one which is particularly close to our approach is Yusof, Tall (1999) which explores the impact of this technique at the university level.

In our contribution, we will present one context, which we call restricted arithmetic, which is, in our opinion, suitable for problem solving both at the university and secondary levels. First, a series of tasks/problems will be presented and then the context of restricted arithmetic will be explored as to its value both for problem solving activities and for facilitating the transition from elementary to abstract mathematics. This transition can be characterised as a didactic reversal (Gray et al., 1999), i. e. "constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects", which "causes new kinds of cognitive difficulty" and has been given considerable attention in the present research in mathematics education.

2. Restricted Arithmetic[2]

Next, we will introduce so called restricted arithmetic which is, in fact, an analogy with "ordinary" arithmetic (e.g. the arithmetic of integers) into which the operations of addition and multiplication are transferred from ordinary arithmetic via the operation of reduction.

A non-standard arithmetic structure $A_2=(\underline{A}_2,\oplus,\otimes)$ consists of the set $\underline{A}_2=\{1,2,...,99\}$ of 99 natural numbers (which we call *z*-numbers[3]) and two binary operations *z*-addition $\oplus$ and *z*-multiplication $\otimes$ defined as follows: $\forall\ x, y \in \underline{A}_2,\ x\oplus y=\mathrm{r}(x+y)$ and $x\otimes y=\mathrm{r}(xy)$ where r: $\underline{N}\text{->}\underline{N}$ we call *reducing mapping*.

The reducing mapping can be simply presented on the set of all three digit numbers ABC and four digit numbers ABCD as r(100A+10B+C)=r(A+(10B+C)), r(1000A+100B+10C+D)=r((10A+B)+(10C+D)). For example 73⊕69=r(142)=1+42=43, 81⊗90=r(r(7290))=r(72+90)=r(162)=1+62=63.

Students are first introduced to the operation of reduction, *z*-addition and *z*-multiplication as shown above. Then they are asked to solve some tasks. Some of them will be presented here and when appropriate, we will comment on them. The tasks were only divided into sections for the purpose of this paper.

Note: Not all problems are solved by each student. Many tasks can be formulated in various different ways (as shown below) and students are asked that question which, at the moment, serves best his/her needs. The way of a student's work and his/her choice of tasks vary according to his/her preference and level of knowledge. For example, a student who has already discovered the additive identity element will choose different tasks and use different solving strategies than a student without this knowledge.

It is obvious that this way of presenting problems puts teachers under a different type of pressure than traditional teaching. They are facilitators, rather than instructors. They have to monitor each individual's progress, monitor class discussions, react to students' different

[1] The contribution was aided by the grant GAÈR No. 406/97/P132.

[2] The author of this non-standard arithmetic structure is prof. Milan Hejný.

[3] The prefix *z* means that it is an object of restricted arithmetic (the prefix comes from the Czech language).

hypotheses, be careful not to disclose new knowledge to students prematurely, etc. It is in full correspondence with the principles of constructivism (see e.g. Noddings, 1990, Rice, 1992).

3. Tasks

3.1 Reduction

- Solve the following tasks until you feel comfortable with reduction: r(134), r(605), r(2731), r(1481), r(4605), r(1481), r(7777), r(100), r(1020),....
- By reducing which numbers do we get number 6 (18, 34, 99)?
- Find a graphic representation of numbers in A_2.

A graphic representation, both on a circle and a number line, could lead to the discovery of the additive identity element and/or additive inverses (”negative numbers” on a number line).

3.2 Addition and multiplication

- Solve the following tasks until you feel comfortable with the operations: 6⊕60, 38⊕25, 68⊕97, 99⊕35, 73⊕49, 35⊕99, 25⊕38, 54⊕46, 6⊗9, 4⊗48, 2⊗23, 33⊗5, 13⊗6, 99⊗18, 21⊗12, 59⊗10, 85⊗99.
- Write a word problem for one of the previous calculations.

Tasks contain number 99 (i.e. the additive identity element) in different roles.

3.3 Additive and multiplicative linear equations

- Solve the following linear equations: x⊕17=99, x⊕61=4, x⊕6=92, 25⊕x=36, 99⊕x=13, 66⊕x=66, 2⊗x=40, 2⊗x=1, 2⊗x=99, 3⊗x=30, 3⊗x=1, 3⊗x=99, 3⊗x=45, 14⊗x=91, 13⊗x=45, 6⊗x=3, 93⊗x=3, 50⊗x=5, 6⊗x=45, 3⊗x⊕2=83, 5⊗x⊕10=5.
- Let a, b, c be given z-numbers. Solve the following parametric equations: a⊗x=1, a⊗x=33, a⊗x=99, a⊗x=c, x⊕b=c, a⊗x⊕b=c.
- Classify additive and multiplicative linear equations according to the number of solutions.

1. Tasks contain number 99 in different roles.
2. Linear equations represent, in fact, the core of the initial work in A_2. During their solutions, the need to discover and/or define the concepts of identity elements, inverses, zero divisors, the operations of subtraction, division, etc. arises.
3. The last two tasks should lead to the classification of linear equations with respect to the number of solutions which is rather interesting and different from the classification in ordinary arithmetic.

3.4 Additive identity element

Students should be able to identify additive and multiplicative identity elements, whilst solving tasks in 3.1, 3.2 and mainly 3.3 without the need of any explicit question. But if they do not do so, they can be asked one of the following questions.

- Is there any number in A_2 with similar properties as 0 in the domain of integers?
- Solve quickly the following problems:17⊕99⊕25⊕13, 28⊕19⊕80⊕99, 23⊕31⊕11⊕88, 33⊕66⊕22⊕24
- Find as many ways as possible for expressing number 99 as the product of two z-numbers.

3.5 Some properties (commutative, associative, distributive laws)

- Find out if it holds: for all z-numbers a, b, a⊕b= b⊕a. If so, prove.
- Similarly, a⊗b= b⊗a; for all z-numbers a, b, c, a⊕(b⊕c)= (a⊕b)⊕c; a⊗(b⊗c)= (a⊗b)⊗c; a⊗(b⊕c)= a⊗b⊕a⊗c.

Students do not often feel the need to verify these properties, they transfer them as obvious from ordinary arithmetic. Some of them do so later, when they encounter something which they initially believed to be true in A_2 but which was, in fact, false. Otherwise, a teacher should cast doubt on their validity.

3.6 Other operations

See the note for 3.4.

- Define the operation of subtraction in A_2.
- Find additive inverses of *z*-numbers.
- Check the divisibility rules for numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 33, 99, etc. in A_2.
- Find multiplicative inverses of *z*-numbers.
- Define the operation of division in A_2.

Numbers which are zero divisors contrast with the other z-numbers (as in the classification of linear equations).

3.7 Zero divisors

– For which *z*-numbers does the equation $a\otimes x=m$ have more solutions? What are the properties of such numbers?

- For which *z*-numbers: $a\otimes b=a\otimes c$ iff $b=c$?

For most students, this is their first encounter of zero divisors, i.e. a concept which is missing in ordinary arithmetic. Our experience shows that students are intrigued by it and want to explore its properties.

3.8 Odd and even numbers

- Define odd and even *z*-numbers.
- Determine some propositions concerning odd and even integers and verify them for *z*-numbers.
- Define *z*-primes and the decomposition of *z*-numbers into the product of *z*-primes.

3.9 Squares, square roots, quadratic equations

- Find all *z*-squares, i.e. all *z*-numbers of the form x^2, where x is a *z*-number.
- Draw a diagram of all *z*-squares and the appropriate square roots. Are there any patterns?
- How should the *z*-number a be given so that the quadratic equation $x^2=a$ had (i) only one solution, (ii) two solutions, (iii) three solutions in A_2? Is there any other possibility?
- Solve the following equations in A_2: $x^2\oplus 2\otimes x\oplus 1=99$, $x^2\oplus 98\otimes x\oplus 93=99$, ..., $x^2\oplus a\otimes x\oplus b=99$, $3\otimes x^2\oplus a\otimes x\oplus b=99$, $11\otimes x^2\oplus a\otimes x\oplus b=99$, $a\otimes x^2\oplus b\otimes x\oplus c=99$.
- Find out if Vieta's root theorem holds in restricted arithmetic.

The investigation of squares and square roots leads to very interesting results (particularly when a diagram is drawn). The theory of quadratic equations is in restricted arithmetic much more complicated than in ordinary arithmetic. Some notions are the same (e.g. the number of roots of quadratic equations depends on the discriminant), others are different (e.g. the classification of quadratic equations as to the number of solutions).

3.10 Sequences, powers

- Find out what the sequence of z-numbers 2^0, 2^1, 2^2, 2^3,… looks like. Find out the 101st element of the sequence.
- In the previous task, we found out that $2^{100}=34$. However, there is no z-number 100. Is there a mistake?
- Does it hold that $2^{50}\otimes 2^{50}=2^1$?
- Does it hold for all z-numbers a that $a^{99}=1$?
- Investigate the sequence of z-numbers n^0, n^1, n^2, n^3,… for (i) n=1, (ii) n=10, (iii) n=4, (iv) n=25, (v) n=3.

Powers of z-numbers form another rich theme within restricted arithmetic. It is easy to start exploring (to calculate powers of different z-numbers, distinguish them according to the length of the period, etc.) and investigations can lead up to the concept of group, subgroup, Lagrange's theorem, cyclic group, Euler's theorem, etc.

3.11 Other tasks
Fibonacci's sequence, Pascal triangle, factorial, arithmetic and geometric sequences, algebraic structures (groups, subgroups, algebraic structures with two operations), etc.
Restricted arithmetic presents students with more and more tasks/problems and besides, they can also pose their own problem on the basis of their knowledge of work within the domain of integers.

4. Restricted arithmetic as a problem solving context and/or a bridge between elementary and abstract mathematics

Let us summarise main features of restricted arithmetic.

- Students pursue mathematical explorations.
- Some new ideas are involved which cannot easily be assimilated into the learner's existing knowledge. New cognitive structures must be constructed.
- Students experience constructive doubt and conflict regarding mathematical issues (contradictory results, results that did not make sense, solutions which contrasted with what they had initially expected) (Borasi, 1994).
- Students are forced to analyse in detail those activities which they have considered obvious and automatic so far (e.g. solving linear equations, basic operations) to see how they can be transferred (or if they have to be modified) to A_2.
- Restricted arithmetic creates a learning environment in which students themselves formulate the problems and questions they want to study. It is a source of a variety of different tasks at different levels of difficulty.
- Restricted arithmetic is not a ready-made product (as mathematics is very often introduced to students), it is an open context which students must explore for themselves. The teacher him/herself does not often know if a hypothesis formulated by students is valid[4] and has to " tackle a problem in front of the class to show that even mathematicians do not produce neat solutions at first. This encouraged students to feel less reluctant to make conjectures which might prove to be wrong on the possible route to success" (Yusof, Tall, 1999).
- The variety of tasks in A_2 provides an opportunity for the individualisation of learning mathematics. Slower learners can choose as many different tasks on the same level of difficulty as they need, while more able students can proceed to more difficult problems and both groups reach a sense of achievement.
- The structure of A_2 can be expanded easily. We have tried to use the structure of A_1, which contains only 9 numbers 1, 2, ..., 9, the operation of reduction is defined as the sum of digits and the operation of addition and multiplication as in A_2. On the other hand, students themselves suggested the use of the structure of A_3, in which, by analogy, the reduction is defined as the sum of three ciphers at a row.
- New concepts are introduced and defined (preferably by students themselves) when they are needed (e.g. when solving additive linear equations they need subtraction which has not been introduced to them yet and for subtraction they need inverses). Thus, the concepts are not introduced formally and are seen in their mutual relationships and in a context different from ordinary arithmetic. The traditional approach to university algebra, (at least in our country) is based on a well known sequence: primitive notion, axiom, definition, theorem, proof, illustration. Our experiences show that this often results in formal understanding of concepts (with little

[4] The author herself was presented with restricted arithmetic in a way described above. She had to explore it for herself and this process has not ended yet. This fact seems to have a profound effect on the motivation of her students.

knowledge of how to apply them). We believe that a number of fragmented examples as used traditionally leads to the loss of connections among concepts in the students' mental structure.

- Restricted arithmetic is a context from which any semantics has been removed (and thus it gets nearer to abstract mathematics). Numbers in A_2 can no longer be understood semantically as quantities, but rather structurally as objects which can only be manipulated according to given rules. As students can use their experience with ordinary arithmetic of integers when exploring A_2, it can serve as a bridge between elementary mathematics of counting and manipulation and abstract mathematics of definitions and proofs.
- Students tend to rely on their images from number theory when studying and applying group theory (see also Hazzan, 1999). Thus, they see number 0 as a universal model of the additive identity element (the neutral element is understood in a semantic way (i.e. nothing) rather than in a structural way), or negative numbers as universal models of additive inverses. These associations do not hold in A_2 and our experiments have shown that A_2 is an appropriate context for corrupting this widely spread wrong image (it serves as a kind of a counterexample).
- The advantage of A_2 over modular arithmetic (modulo 99 in our case) lies in that (1) A_2 cannot be found in any textbook (as far as we know) and students must only rely on their experience with ordinary arithmetic, (2) no preliminary theory is needed before working in A_2. Note: Students usually cannot see the connection between A_2 and arithmetic modulo 99, notwithstanding the fact if they had already been introduced to modular arithmetic (thanks to the way the operation of reduction is introduced). It is our experience that because they think that they discover something really new which cannot be found in textbooks, they feel strongly motivated.
- Last but not least, the important result of work in A_2 is that students often feel motivated to study abstract algebraic structures (which can help them in exploring A_2).

5. Conclusion

Problem solving is particularly important for prospective mathematics teachers (with whom our experiments have been done) for several reasons, one of the main being the fact that when solving problems, they get plenty of opportunities to make conjectures, pose and prove or disprove their hypotheses, to communicate with their colleagues, they experience difficulties similar to those met by students in the class, they learn the importance of evaluating the process instead of the result, etc. (see also Boero, Dapueto, Parenti, 1996).

Restricted arithmetic has proved to be a rich context for mathematical explorations. It remains to be investigated in more detail what its other merits as to the development of mathematical knowledge and abilities are.

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