

Processes of mathematization in a learning environment combining devices and computational tools

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Abstract

In this article we present the design and the pilot run of a rich learning environment that lays emphasis on the mediation of tools. In this environment two artifacts are functionally combined: a device in which the basic ways of measuring a height have been incorporated and the computer through which students can study a dynamic simulation of the device. The device was a product of a combination of functional characteristics of devices that have emerged historically. The discovery of how to use the instrument arises through a process of mathematization, which is completed in two phases, out of which the first pertains to the similarity of right triangles and the other one to the trigonometric tangent. The research highlights the distinct contributions of each of the tools in the process of mathematization and aims to assess the effect on the learning process of their concerted presence. The research ends with a schematization articulating connections between the two stages, which we present in the final table.

Riassunto

In questo articolo presentiamo il progetto ed una esperienza pilota inserita in un ricco ambiente culturale che enfatizza la mediazione degli strumenti. In questo ambiente sono combinati funzionalmente due manufatti: un'apparecchiatura nella quale sono stati incorporati i differenti modi per misurare un'altezza ed il computer attraverso il quale gli studenti possono studiare una simulazione dinamica del congegno. Tale strumento era un prodotto di una combinazione di caratteristiche funzionali di strumenti che sono emersi storicamente. La scoperta di come usare lo strumento sorge attraverso un processo di matematizzazione che è completato in due fasi, la prima fase concerne la similitudine di triangoli e la seconda la tangente trigonometrica. La ricerca accentua i contributi distinti di ognuno degli strumenti nel processo di matematizzazione e punta ad accertare l'effetto sul processo di apprendimento della loro presenza concertata. La ricerca si conclude con una schematizzazione che articola i collegamenti tra le due tappe che vengono poi presentate nella tabella finale.

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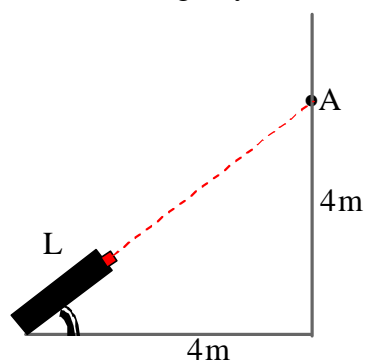
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Starting point of our research project.

Three years ago we conducted a research project at the Educational Research Center of Greece to study how well students were able to utilize their mathematical knowledge. A list of questions was given to 9th, 10th, and 11th grade students investigating skills related to their ability to apply school knowledge in every day life situations. The research project took place in a random sample of 90 students from four different schools in Athens.

Among others, there was also the following question:

“The image shows a projector and its spot A on a wall. In order to move the spot to double the height, you should:



- a) double the angle
- b) place the projector closer to the wall
- c) make the angle 18° wider
- d) make the angle 40° wider”

Students were asked to provide a short gloss on the reasoning followed to arrive at their answer. Judging by the results of the answers, the main interest was concentrated on the following findings:

- a) 60 % of the 9th grade students chose answer A. At 10th and 11th grade, the percentages of students that chose answer A were 42% and 43,5% respectively. (19 students chose the right answer, that is 22% of the total number of the students.)
- b) 6 students of the sample recognized that the concept of trigonometric tangent is relevant to the question, but only 2 of them chose the right answer.

The students’ difficulties observed here mainly stem from their tendency to spontaneously attribute linear relations to any magnitudes. The linear relation looks appropriate in many cases as a phenomenal tool of description and linear models are reinforced by their frequent use. (Freudenthal 1983 p. 205)

The misuse of linearity in non-linear situations – sometimes referred to as the ‘illusion of linearity’ (or proportionality) – is a ‘classic’ misconception, probably one of the oldest in the literature of mathematical thought, de Bock, van Dooren, Janssens, Verschaffel (2002) and might be characterized as a classic epistemological obstacle.

Another source of difficulties is that the trigonometric tangent function, even though it has been taught exhaustively in the 10th and 11th grade, is not recalled in the context of the particular problem. The way in which this function is introduced to the students seems to lead to a symbol with a meaning defined mainly within the context of mathematical symbols, since most times it is understood as the ratio of the sine and cosine. Such a representation of the concept is linked very little with the practical needs that gave rise to it and sidesteps the immediate functional characteristics where it could come alive ⁽¹⁾. In any case, the vast majority of students providing the right answer did so intuitively through mental experiment.

The results motivated us to conduct research in 2 directions. First was the historical investigation of the obstacle related to the non-linear correlation between the angle and

the length ⁽²⁾. The second involved the construction of a rich learning environment and the investigation of the ways whereby this environment mediates the understanding of the trigonometric tangent.

The learning environment combines on the one hand the capabilities of new technologies, and on the other hand the utilization of historical devices that are connected to the mathematical object where we have observed difficulties in students' understanding.

The pilot research project that we conducted attempts to trace the special dynamic that rich learning environments can acquire, if they are designed with the aim to bring out the difficulties and the course of understanding and, ultimately, to constitute the researcher's magnifying glass.

Theoretical considerations

The design of the environment that we selected lays particular emphasis on the way various components are linked in order to mediate the communication of the participants. These components are to initially the mathematical symbols, software, notation, and the use of a device, utilized for the measurement of height.

According to Vygotsky (1978), *artifacts* and *psychological tools* such as symbols serve as cognitive tools, expanding and extending thought and knowledge, mediating our practical and intellectual activities. In Vygotsky's early distinctions, he considers devices as entering in an intense interaction with the subjects that handle them. Devices are extensions of human capabilities and far more than this since, by changing the environment through the introduction of new activities, they exercise inevitable influence in the way that the subjects that handle them think (an idea that was strongly supported subsequently by McLuhan (1964) as well).

The above opinions constitute general psychological views that do not take into account mathematical education and mathematical notation. In an approach followed by a number of researchers Mariotti (2001), Artigue (2001), Sfard (2002), mathematical symbols are placed next to devices and their relationship to them is investigated. Specifically, Sfard (2002) views symbols and artifacts as inextricable elements of mathematical learning. However, it is widely understood that students have difficulty in the use of mathematical symbols while solving a mathematical problem. Devices as well as symbols require a complex familiarization process and often a reference field to functions in lived reality.

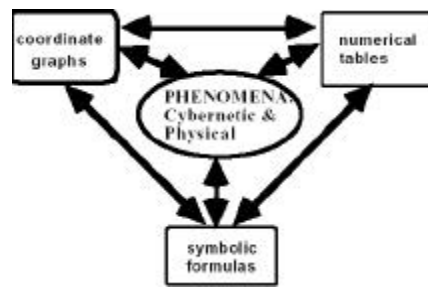
The inability of the students taking part as subjects in the research project to connect the situation in the problem with the trigonometric tangent may be due to the context within which the $\tan x$ symbol acquired its meaning for them. This context becomes apparent when we look inside the Greek Lyceum (Senior High School) schoolbooks: it is defined exclusively by the mathematical notation system for symbolic, numeric, and graphic depictions of mathematical functions. Kaput refers to a number of research projects conducted that show the inability of students to understand the meaning of a function when it is studied strictly in terms of graphs, formulas, and tables and apart from referential anchors in students' experience and he suggests that:

the mutual constitution of meaning for not only the notations and links among them, but for the phenomena and situations that they may be used to model, is insufficiently rooted in authentic student experience.

Kaput (1996)

Phenomena are not simply connected with mathematical depictions of the function, but constitute the core of the meaning which lies in sensory motor experience and which in

turn is in a dynamic bidirectional correlation with the representations that are attributed to it. The above are condensed by Kaput in the following diagram:



Designing a learning environment

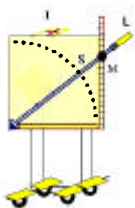
In the context of our theoretical hypotheses we have designed a learning environment through which we investigate the way that students use devices and understand the concept of similarity of right triangles and of the trigonometric tangent.

In line with Mariotti (1997), our designing choices concerning the features of a ‘field of experience’ involve the following:

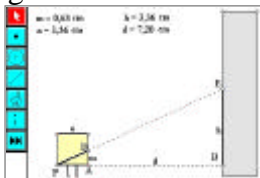
- The presence of concrete and semantically pregnant referents (external context) for performing concrete actions where dynamic mental experiments are carried out.
- The presence of semiotic mediation tools (including excerpts from historical sources), chosen from the cultural heritage with the aim of introducing the mathematical idea of theorem. Those two features are embedded in a device described below.
- The construction of an evolving student internal context, rooted in dynamic exploration. This feature relates to the use of a dynamic computer simulation of the device.

Based on these features, we consider that the basis of our environment should consist of two different kinds of tools:

- 1) A make-shift device which has been constructed based on the functional characteristics of a device for the measuring the height of a remote object, the Quadrant ⁽³⁾ and consists of square. At one corner (down left), a pointer S has been attached which is able to rotate and on the pointer a small laser pointer L was affixed. At the top of the square two spirit levels I have been placed indicating whether the square is absolutely level. The pointer S moves on a graduated bar M, while one is also able to measure the angle with respect to the horizontal level.



- 2) A simulation of the device in a computational environment, making use of the geometrical software sketchpad.



Through the simulation the students are able to recognize the characteristics of the device and its operation on the screen, to perform measurements of the dynamically variable magnitudes and to tabulate values, and finally to examine possible relations among the magnitudes, making them appear in their graphs.

The student had to invent a way to measure with the device the height of a remote object.

Specifically, they would use the device in two cases. In the first case, the pointer was stationary and the students were able to only move the device back and forth since

it had wheels. In the second case the device was stationary, but had a goniometer attached through which the students could measure the angle of the pointer to the horizontal level. This specific modification of use effectively changes the context of negotiation, that is the point of view. This is an important moment for putting in place mathematical ideas; it constitutes a significant moment for the characterization of an epistemological obstacle, Spagnolo (1999).

The general research question which concerned us from the beginning during the planning stage was the ability of students to form essential correlations of the various representational contexts that are involved, what Hershkowitz & al (2003) calls the ability to *interweave* the various representatives.

The pilot research

The settings

Initially we conducted an observation in the Psychico College where, after consultation with the teachers, the students worked in the environment that we have already described. In the room (computer laboratory), there was only one device; as for the students, they studied the simulation of the device in the computer divided in groups. This form of observation was necessitated by the setting of the particular school, since the worksheet aimed to teach the similarity of right triangles in a period of four instructional hours. The particular setting brought out difficulties that indicated to us out a more organized structure for the rest of the research project.

The next part of the project was conducted in the 26th Gymnasium (Junior High-School) of Athens and four student of the 8th grade participated, which were involved for a period of eight instructional hours outside the school program.

The choice of students was random and relied on the expression of interest by the students themselves. It should be noted that three of the students had already been taught, in class with their fellow students, the concept of the similarity of triangles in the course of the regular curriculum of the school, while the fourth student (T) stated that he had missed some of the classes when this particular topic was taught by the professor.

The data of the research were collected from notes and a video recording, which had been preceded by a short interview with the students. In the presentation that follows we include the most characteristic conversations.

The course of the research project

The project took the form of a semi-structured and directed process in which the researcher participated.

The basic structure of the activity involved two phases. In the first phase, the student looked for a way to use the mobile device for the measurement of height. In this phase the research interest focused on the mediation of the two devices and the way in which students associated the multiple representations that were produced by the devices.

The students initially came into contact with the device and examined its parts and subsequently they were asked to think in which way the device would measure the height of a remote object.

Then, the students created a static geometric figure to represent the given conditions of the problem; they were also asked to locate the stable and variable magnitudes that come into the problem.

Following, they were asked to decide if there is some Mathematical concept that would be appropriate for the particular conditions of the problem.

Furthermore, the students engaged with a graphic representation of the device on the computer created with the sketchpad software application. Here, the students had the ability to dynamically modify the figure, to take measurements of the variable magnitudes, to compile a table of values, and through it to create a graph of points in a Cartesian system of axes. The arrangement of points was negotiated with the students.

The possibility of the formation by the students of the concept of function that would condense the values that resulted from the software into a mathematical formula was investigated on the assumption that the transition to formal symbolic renderings constitutes a high level of abstraction and could be characterized as the final goal of a mathematization process.

In the second phase, the students were engaged with the device, in which they had the ability to rotate the pointer and to measure its slant.

After they concluded that they were unable to measure height without the use of the similarity concept, they studied a simulation on the computer, performed measurements of the angle and height, and ascertained that their graphic representation was not a straight line any more, but a curve.

It should be noted that the simulations were ready-made and that the students simply recognized in them the basic functional characteristics of the device.

First phase

- *Dealing with an unknown tool*

The students noticed some parts of the device, but without making any essential correlations. In particular they considered the device as a means of measurement, stating, “it measures the flatness of the space”. They may have been led to this conclusion by the presence of the spirit levels placed on top of the device.

The students received their initial geometrical representations when the laser pointer was activated.

R: Can you tell what it is?

T: This is part of a pen; this is the aerial of a car (points to the pointer of the device)

B: This looks like an angle (points to the device pointer)

T: Ah, right! It does look like an angle

.....

R: Notice the numeration here (means graduated bar A)

T: It shows the size.

B: It shows one of the vertical sides of the right triangle.

.....

R: Furthermore, the device can move back and forth since it has wheels.

T: Is it makeshift?

B: (thinks) We need to know one of the vertical sides, don't ... we need two sides or one? (to himself)

Note that for student (B) some representations of the right triangle emerged after the laser was activated and the device was moved.

Student (T) was still engaged with the material features of the device and he recognized geometrical features via verbal communication with (B).

- *A static geometrical model*

The students have just drawn a geometrical figure of the device on their work sheets:

B: I found it! This (he shows the ‘small’ triangle) and this (he shows the ‘large’ triangle) are proportional.... Because they have a common side, they have two angles... rather, they do not have a common side, they have two angles that are equal because of the way they are drawn and one common angle

R: Therefore, can you describe a practical way to calculate the height of the room?

T: Do we know how far the laser reaches?

R: Why do you ask this?

T: Because if we set it, let’s say, there (he points behind the device) and then you say that the laser has 10 meters (he means ‘range’). Then if we see that the laser light barely touches the wall, we could estimate that the hypotenuse is about 10 meters long, and knowing the length of the floor, we also can figure out the height using the trigonometric numbers of the angle.

Here the students follow different procedures to calculate the height via right triangles. Student (B) has rendered the problem in a suitable mathematical structure, which could be said to constitute a prototype of measurement⁽⁴⁾ via the similarity of right triangles, and thus he works with two triangles. Therefore he selects and adapts the model of similarity of right triangles to the structure of the problem.

It seems that student (T) has not mathematically associated the ‘small’ triangle with the ‘large’ one; therefore, he prefers the trigonometric approach. He insists on the employment of one triangle (the large one) suggesting that it would be useful for the problem to calculate the hypotenuse. Student (B) then raises the objection that this would waste ‘precious time’. This is an indication of the importance that he has attributed to the researcher’s requirement to minimize the steps to the solution of the problem.

Finally, having recognized the similarity of right triangles, the students used the ratio $h/d=c/b$ to calculate the height h of the room.

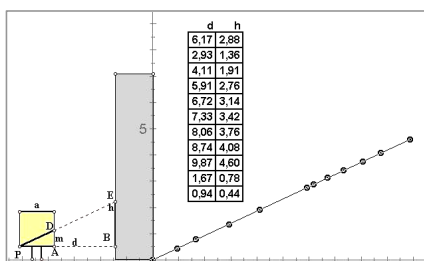
- *A dynamic model of the device*

Then the students manipulated the simulation of the device on the screen, defined the main variables (distance d and height h), and created a table of their values.

When the students had placed number pairs on a Cartesian system via the software, collinear points became apparent.

The Researcher (R) asked students to give an interpretation of the collinearity of the points.

Episode



R: Was this something we expected?

B: Yes, since the quantities are proportional.

R: What do you mean?

B: Each quantity will vary as much as the other ... maybe... I am not sure, but if the function is plotted as a straight line ... it is something that happens only in the case of proportional magnitudes.

T: Yes, I've heard it from our Physics professor when he taught us graphs.

.....

R: If they are proportional, what is the relation between them?

(The students are at an impasse; (B) looks at the table of values and then claims that the magnitudes are not proportional because the difference in values is not fixed (!).

R: Then we should find out what is the relation between them.

B: Maybe we should take into account their ratio?

R: How did you come by this?

B: Since it is not their difference, then it must be their ratio.

R: Is there any possibility that the ratio does not have any bearing?

B: (After a little bit of thought, he moves the simulation of the device on the computer screen). No, because as the triangles move ... they remain the same, therefore they have the same ratio.

Now, the students engage with the software through multiple representations of the mathematical theory that pertains to the device (device simulation, table of values, graph). We have evidence, first that the device mediates in the interweaving of different representations and in this way the students establish new associations between the concepts of similarity, ratio, proportion. We should note the initial inability of student (B) to link the table of values to proportional magnitudes and the stable ratio. The associations occurred only through the dynamic manipulation of the simulation of the device. The effect of the former activity was to create an association between the similarity of triangles and the stable ratio.

Second, the graph activates associations that lead to the linear function. This is indicated by the fact that after the above activity they reach an equation by which they can calculate the height h when the distance d is known and they note on their papers $h=(c/b).d$ (1).

Comment: In effect, this last association constitutes the conceptual bridge which leads from the ratios to the general form of the linear function $y=a.x$. However, the data do not indicate this transition, which would constitute the final generalization. The students used type (1) to the end as a calculator through which they could perform measurements. The important thing, however, is that when the students summarized the results of the activity in the end, they said that the association of the similarity of triangles with the linear function occurred through the graph and not through the formula. From the above, we could hypothesize that the graph was what enabled the association of the geometrical space of the similarity of right triangles with the algebraic space of linear functions. Another hypothesis that requires clarification is that the geometrical nature of the software determined to a considerable degree the mode of transition.

Second phase:

- *Mental experiment*

The students had to measure a height using the stationary device with the rotating pointer. They had an additional option as they could use a goniometer, which was attached to the pointer.

During the next stage, the process that was followed had a structure analogous to the previous one (with respect to mathematization). In particular, they defined the two main

variables of the new problem and they manipulated a simulation of the new situation on the screen looking for a possible relation between those variables.

At this point, the researcher asked students to think about the problem of the projector, which is described in the introduction.

R: If we want to double the height, what do we have to do to the angle?

T: To double it ... I don't know

R: What do you say B?

B: (thinks for some time) I think we should focus on the ratio, and through the ratio we will see what to do with the angle.

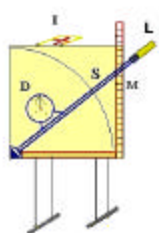
R: Can the height double, if the angle is doubled?

T: It can.

B: (looks at the simulation and the device and thinks) ... No, it cannot, because if it becomes a 45° angle and then it is doubled, it does not make a triangle.

It is not clear if the previous equations provided him with an insight that the ratio can help him in the calculation of the angle. We notice that the answer is arrived at by the student after a mental experiment, and therefore he avoids the “illusion of linearity” obstacle.

- *New ways for measurement*



The discussion that followed concerned the way in which students could measure a height with the new version of the device.

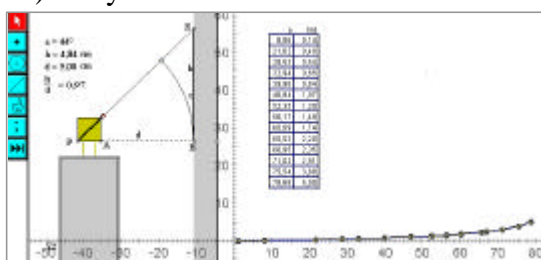
The students did not face any difficulty and concluded that for every height, they should calculate the new ratio c/b and then utilizing formula (1), they could calculate the height.

Then the researcher introduced the limitation that they can only use the goniometer. Here students faced some difficulty since they were trying to reach a solution outside the context they had created earlier. The researcher, recognizing the impasse, presented to the students the mode with which this problem was dealt with in Medieval times and they concluded that they needed to build a table, in which the first column would contain the values for angle a and the second the corresponding values for the ratio c/b . The use of the computer was deemed necessary.

- *A dynamic model of the device*

Now, the students were faced with a new simulation and they manipulate the software as a medium through which they could study the new problem conditions. Given the previous experience with the software and the device, they immediately applied a concrete strategy.

- 1) They determined the two variables of the problem (angle and height).
- 2) They tested the capacities of the new simulation on the computer and they assigned correspondences between the functions of the device and the software.
- 3) They used the software in order to measure via the simulation the angle a and the ratio c/b .



Because magnitudes c and b were not available on screen the students used the ratio h/d claiming that the two ratios are equal. This table does not show an explicit relation between the variables; however, the

students claimed that they could now calculate any height by using the angle

4) Very soon the students addressed themselves to the graph that displayed not collinear points, but points that lay on a continuous normal curve. (The formal curve was not apprehended by students automatically as a function, but they noticed the break in linearity).

R: Are the magnitudes proportional?

T-B: No

B: This is what we have already said ... that if you double the angle, the height does not double.

The activity proceeded with the observation that there exist some proportional magnitudes on the screen and these are angle a and the arc t (in a circle where A is the center and distance d is the radius).

Finally, the students settled a question in the discussion concerning the significance of the arc in the simulation. The researcher had indicated the proportional relationship between the angle and the length of the arc. Furthermore, he made a point of the fact that the measured height is in one direction a tangent to the arc.

Discussion

The way in which the activity unfolded could be approached in terms of mathematization.

Freudenthal (1991) and Treffers (1987) distinguish mathematization into two different types of mathematical activity-horizontal mathematizing and vertical mathematizing. Horizontal mathematization involves going from the lived world to the world of symbols, while vertical mathematization involves moving within the world of symbols. As Freudenthal states, the difference between these two worlds is not clear-cut but these two forms of mathematization are of equal value.

In the activity examined above, horizontal mathematization was involved in the initial conditions of the problem that concerned the device. In order for the problem of using the device to transform into a mathematical problem and for the problem to be solved, the psychological instruments and the computer, the mediating role of which we should analyze, contributed definitively.

- **The device**

- i) It allowed for physical activity and sensory motor participation which provided it with a physical point of reference.
- ii) The transparency of the device allowed the students (*especially* B who had been taught the concept of “similar triangles”) to locate the essential mathematical model of the similarity of right triangles.
- iii) It allowed to a large extent for the theory that had been incorporated in tools similar to this one over the course of history to unfold, tools of which it constitutes a potential variant (the experimentation with variants of the device is continued in our research ⁽⁶⁾).

- **The software**

In the deployment of the software, we utilize with our design three of its capabilities: the simulation of the device, the graph, and the table of values.

- i) The dynamic handling of the simulation of the device transformed the static representation that students drew in their workbooks into a visual phenomenon. This consequently activated the schema of the similarity of right triangles.
- ii) The graph connected, in the first phase, the similarity of triangles with the linear function, while in the second phase it constituted an indication of the non-linear correlation of the variables. A general comment about the computer as a tool is that the students solved the real problem in the first phase without the presence of a computer. What the software created was the conceptual association of the geometric space of the similar triangles with the linear function.
- iii) The table of values, with the dynamic it affords, drastically departed in its use from the usual practice of students to handle proportions on paper. In the first phase, it created certain difficulties and consternation. The obstacle, however, was overcome with the help of the other two capabilities. In the second phase, students saw the table as immediately useful in the solution of the problem (that is the measurement of a height, utilizing the angle). Afterwards, the very table constituted in essence the functional expression of the trigonometric tangent since students did not arrive at the necessary condensation of the function into a formula ⁽⁶⁾. Finally, we could consider the table in the second phase as the means for the conceptual association of the similarity of right triangles with the trigonometric tangent.

The two phases of activity led to two different cycles of mathematization. In each of these cycles, we might distinguish three contexts that were in constant interaction.

The context of cognitive tools (symbols in the work book and later on screen) with which students attempted to structure the conditions of the problem and to provide a solution through the creation of an algorithm.

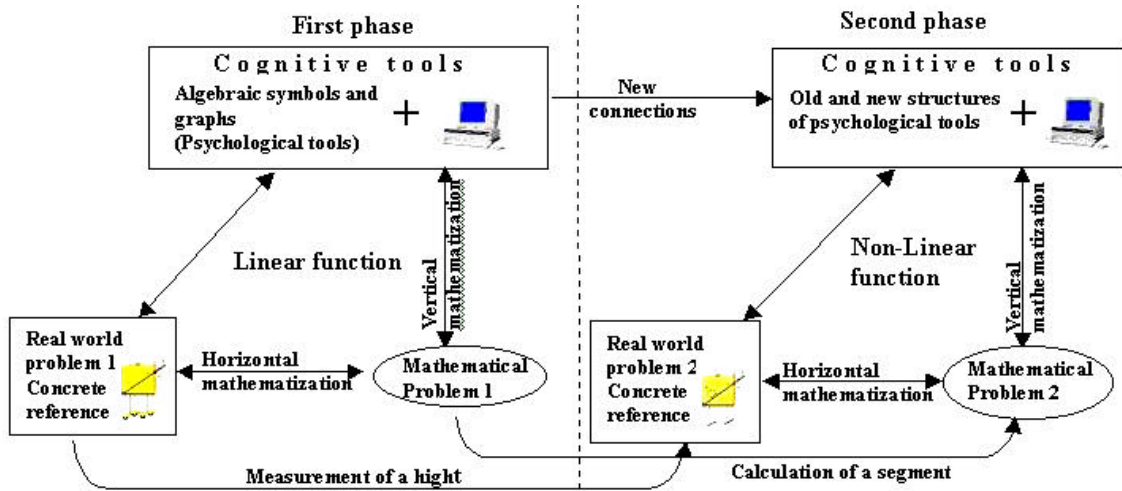
The context of the practical problem, part of which is also the device.

The context of the mathematical problem which concerned the correlation of the variable quantities.

The horizontal mathematization in every cycle moved from the context of the practical problem towards the mathematical context of articulating and solving the problem. The vertical mathematization led to the creation of new associations between the learning tools, and therefore to new structures. Finally, the context of cognitive tools was in constant interaction with the context of the practical problem, as students attempted with the aid of these tools to impose a mathematical structure on the problem.

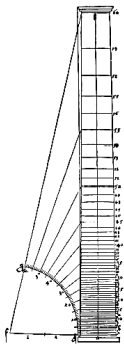
The transition from one phase to the next occurred through successive transformations. The transformation of the device from mobile to stationary produced a change in the quantities that vary proportionately, and at the same time the simulation also changed.

The two phases remained however in contact through the two unchanged requirements of the conditions of the problem: to measure the height of a physical object with the device and to calculate the length of a linear segment in a right triangle. The solution of the second measurement problem became possible with the utilization of the solution resulting from the first problem. The above description and the structure of the research activities are graphically depicted in the figure below.



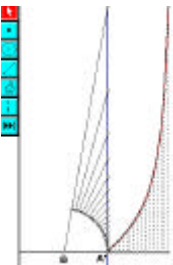
Notes

(1)



Alfred Dürer (1471-1528) examines in one of his studies the peculiar behavior of height (vertical side) of a right triangle when the opposite angle increases by equal measures (Eli Maor 1998 p. 151).

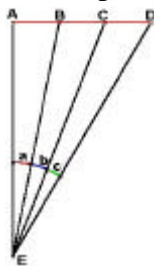
Here it is obvious that the intention of the painter was to determine specifically what Euclid had determined in general in proposition 4 of the *Optics*, and this is proved by the similarity of the two figures. Dürer had studied mathematics in Italy and understood that the laws of perspective had to be sought through ratios and measurements. What Dürer proves in this figure is the one-dimensional non-linear behavior of height when the angle varies.



Modern computational instruments allow us to develop in dynamic way Dürer's measurements on a plane placing one next to the other in distances equal to the arcs of the circle.

If we develop Durer's measurements in such a system, then we can observe the non-linear correlation of angle and height in two dimensions. Here, if we link the apices of the vertical segments, a curve will result, and this is the graphic depiction of the trigonometric tangent.

(2)

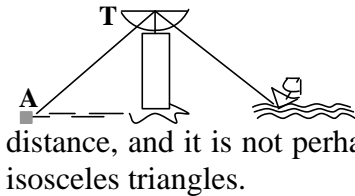


Perhaps the first systematic effort to study the relation between an angle and the height on the opposite side is statement no. 4 in Euclid's *Optics*: "Let us consider equal segments which lie on the same straight line. The ones viewed at greater distance appear smaller. "

Here Euclid proves that if lengths AB, BC, CD are equal then they appear from point E unequal and, specifically, that AB appears bigger than BC, which appears bigger than CD. This means that angle a is smaller than b which is smaller than c and, in modern terminology, the rate of change of the angle is smaller than the rate of change of the opposite length. In other words if we suppose that segment AB increases by equal lengths then its length is proportional to the trigonometric tangent of the angle through which we can see this segment from viewpoint E. It is characteristic that Euclid proves the non-linear

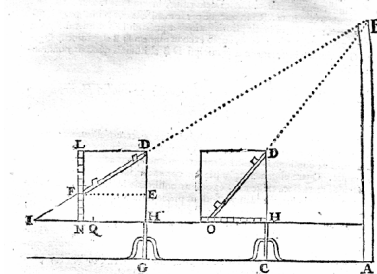
correlation between angle and opposite height using his basic conceptual tool, the ratio, which however does not allow the formation of some functional relation. The trigonometric tangent function emerges much later, in Euclid’s epigones, and especially in Arabs, students of his work, when measuring devices require exact correlations between angles and lengths, for which they construct tables.

(3)



Thales used, perhaps for the first time, an early T device, an artifact to measure height or the distance of a remote object, i.e. a ship (Barbin 1996). Thales in effect transferred the distance of the ship to a point of known distance, and it is not perhaps incidental that he had a conceptual tool, the properties of isosceles triangles.

During the Middle Ages and in the Renaissance, the early instrument used by Thales was developed into a group of tools known as Quadrants.

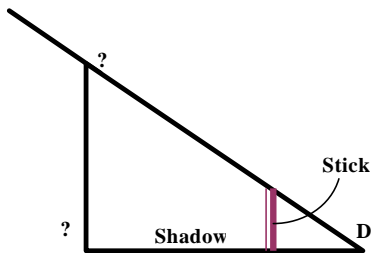


The figure on the left dating from the 16th century shows the mode of calculating the height of a distant object with the help of Quadrants (Oxford Science History Museum).

It is worth noting that this device does not have a goniometer, so its function relies on its displacement and on the calculation of the ratio of similarity every time, thereby in essence on two different values of the tangent to the angle that is formed by the pointer with the

horizontal level.

(4) If we had to choose the key-statement concerning the development of measuring tools



via the similarity concept, this would have to be statement 18 in Euclid’s Optics: "Measure a given height, by the ratio, while sun is in the sky".

Here Euclid uses the sunray AD and the shadow of a remote object (a stick) in order to create similar triangles and then he calculates the given height using the proportion of the sides of the triangles. One might say that Euclid has established here a prototype

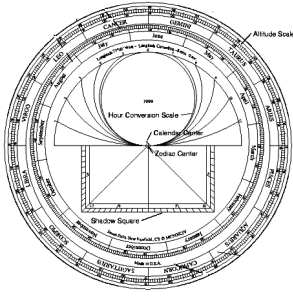
of the similarity of triangles, which can be considered the intergraded mathematical idea of some of the measuring tools of the Middle Ages.

Here, we might also suppose that Euclid had a conception of linearity in the correlation of the lengths of sides when the angles do not vary.

(5) It would not be inaccurate to claim that the difficulty created in measurements by the non-linear correlation of angle and length leads mathematicians to the construction of tables of measurements.

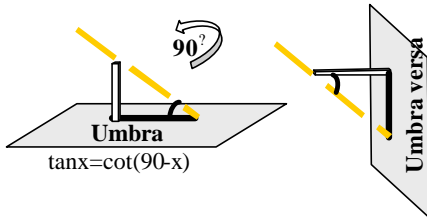
In our case, two are the measurement tools that required the creation of tables, the sun-clock and the astrolabe.

The development of the astrolabe by the Arabs is the outcome of their preference for measuring time not only by the shadow (sun clock), but also by the sun’s height (Sarton 1975); this results in the creation of the first tabulation of tangents.



The construction as well as the use of an astrolabe requires a table of given measurements that correlate the angle or the arc with the height; specifically, the shadow square indicated the ratio of the opposite perpendicular to the adjoining one--what we now specify as the tangent of an angle in a right triangle.

The Arabs also made a distinction between the two modes of positioning a sun clock. Sun clocks were positioned either on a horizontal plane or on the vertical wall of a building (Smith 1958 p. 620). The Arabs appear to have attributed to these different positions a simple, on the face of it, but significant spatial correlation of the angle of the sunrays to the shadow.

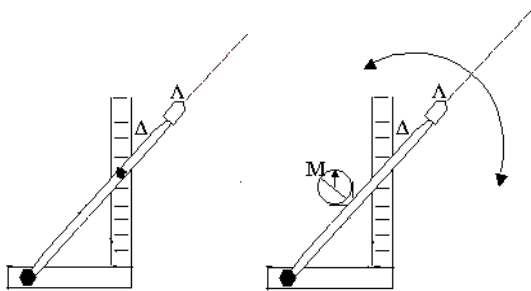


In the horizontal sun clock, the slant and the shadow were linked through the concept of the cotangent, while in the vertical ones through the tangent. In Latin authors of the late Middle Ages, this distinction also appeared in the way the two shadows were named: Umbra recta (Horizontal shadow) and Umbra versa (Vertical shadow). This distinction must be associated also with the fact of the different tabulations for tangents and cotangents.

It appears, therefore, that with the Arabs a conceptual shift occurs from the mainly qualitative renderings of correlations by the Greeks (triangle similarity, and ratio of linear segments) to quantitative-numerical correlations, which are the initial steps in the process that leads to the concept of the trigonometric tangent as a function.

It appears, therefore, that with the Arabs a conceptual shift occurs from the mainly qualitative renderings of correlations by the Greeks (triangle similarity, and ratio of linear segments) to quantitative-numerical correlations, which are the initial steps in the process that leads to the concept of the trigonometric tangent as a function.

(6) The course of our pilot research project until today has indicated two significant modifications to be made both to the device and to the software.



To begin with, the measuring device was simplified as much as possible in order to become more user-friendly, without however losing its basic characteristics. The two new versions of the device are shown in the following figures.

Furthermore the geometrical software was replaced by another application that allows for a dynamic handling of symbols through the use of the Logo programming language. This replacement was deemed necessary since the geometrical software did not facilitate students to arrive at a clear formula for the function, especially in the second phase.

References

- 1) Artigue, M. (2001): Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. Paper presented at CAME 2001, Freudenthal Institute, University of Utrecht.

- 2) Barbin, E. (1996): The role of problems in the history and teaching of mathematics, in Ronald Calinger (Ed) ‘Vita Mathematica’ The Mathematical Association of America (publish)
- 3) Cobb Paul (2002): Reasoning With Tools and Inscriptions *The Journal of the Learning Sciences*, 11(2&3), 187–215, Lawrence Erlbaum Associates, Inc.
- 4) Freudenthal H(1983).: Didactical Phenomenology of Mathematical Structures, *D.:* Reidel Publishing Company, Utrecht.
- 5). Freudenthal, H. (1991): Revisiting Mathematics Education, *China Lectures. Dordrecht: Kluwer Academic Publishers*, Utrecht.
- 6) Hershkowitz., R . Kieran, P. Schwarz, B. (2003): Potential and pitfalls of joining together computer artifacts (representatives) towards a construction of meaning in secondary school algebra, *Proceedings of the 2nd conference on ‘Mathematics at the secondary school’ Athens*
- 7) Kaput, J. (1996): The role of physical and cybernetic phenomena in building intimacy with mathematical representations. In P. Clarkson (Ed.), ‘*Technology in Mathematics Education: Proceedings of the 19th Annual Conference of the Mathematics Education Research Group of Australia*’ (MERGA) (pp. 20–29) Melbourne, Australia: Deakin University Press.
- 8) Maor Eli (1998): Trigonometric delights, *Princeton University Press*.
- 9) Mariotti M.A., et al.: 1997, Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of the 21st PME Conference*, Lathi.
- 10) Mariotti, M.A. Cerulli, M (2001), Semiotic mediation for algebra teaching and learning. *Proceedings of the 25th PME Conference*, Utrecht.
- 11) McLuhan, Marshall (1964). *Understanding Media: The Extensions of Man*. New York: McGraw-Hill.
- 12) Sarton George *Introduction to the History of Science*, Krieger Publishing Company (June 1975)
- 13) Sfard, A (2002) : Analyzing Tools: Perspectives on the Role of Designed Artifacts in Mathematics Learning, *The Journal of the Learning Sciences*, 11 (2&3), pp. 153–161
- 14) Smith D. E (1958): *History of Mathematics*, Dover Publications Inc.
- 15) Spagnolo F. (1999): A theoretical-experimental model for research of epistemological obstacles, Cairo (Egitto) pp. 14-18 November, *International Conference on Mathematics Education into the 21st Century*.
- 16) Treffers, A. (1987): Three dimensions: A model of goal and theory description in mathematics education. *The Wiskobas project. Dordrecht: Kluwer Academic Publishers*.
- 17) Vygotsky, L. S. (1978): *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.