MATHEU
Identification, Motivation and Support of Mathematical Talents in European Schools

MANUAL

Volume 1

Editor
Gregory Makrides, INTERCOLLEGE, Cyprus

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MATH.EU Project

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1.1 Partnership

The partnership of MATHEU consists of nine institutions from eight countries. These are

	Institution	
1.	CYPRUS (INTERCOLLEGE)	
2.	BULGARIA (ACADEMY OF SCIENCES)	
3.	CYPRUS (UNIVERSITY OF CYPRUS)	
4.	CZECH REPUBLIC (CHARLES UNIVERSITY)	
5.	GERMANY(UNIVERSITY DUISBURG-ESSEN)	
6.	GREECE (UNIVERSITY OF CRETE)	
7.	ITALY (UNIVERSITY OF PALERMO)	
8.	ROMANIA (NORTH UNIVERSITY)	
9.	HUNGARY (UNIVERSITY OF MISKOLC)	

The description of the partner institutions

Partner 1: (INTERCOLLEGE / School of Education- Coordinating Institution, CYPRUS):
Intercollege is a private tertiary education institution in Cyprus with some 200 full time faculty in a variety of undergraduate and post-graduate programmes. Intercollege's degree programmes are accredited by the Cyprus Government, the NCA of USA and some European Accrediting Bodies. The Department of Mathematics offers a degree programme in Applied Mathematics for Science and Technology. It aims to produce scientists who can contribute to the development of new science and technologies. The Department of Education offers a pre-primary, a primary education degree programme and a Master's in Education aiming in providing the future teachers with the necessary tools and methods in becoming effective teachers. In the light of the fact that Intercollege is striving to become the largest research university in Cyprus these methods are becoming all too important for the reassurance of quality services (education/research) in Cyprus and Europe. The project is in cooperation between the School of Education (Primary Education Department) and the School of Sciences and Engineering (mathematics Department). The college, through the coordinator of the project, actively contributes to the development of talented students in mathematics as it provides full support for the activities of the Cyprus Mathematical Society. The college also offers full-tuition scholarships to competent students who want to study mathematics. The School of Education and the School of Sciences and Engineering have previous experience in European projects under Comenius, Minerva, Leonardo and Lingua.

Partner 2: (Institute of Mathematics-Academy of Sciences, BULGARIA)
The Institute of Mathematics and Informatics (IMI), Bulgarian Academy of Sciences, is an independent legal entity. It organizes and carries out research in Pure and Applied Mathematics and Informatics. It is also engaged with the education and preparation of highly qualified specialists in virtually all important branches of Mathematics and Informatics. Currently IMI comprises about 180 persons. 130 of them have science degree (Doctor of Sciences and\or PhD). One third of its staff is temporarily abroad, working for other research and educational institutions worldwide. IMI promotes science and tries to attract young talents to a scientific occupational career.
For more than 25 years IMI is the main driving force behind the activities connected with the early identification and development of young Bulgarians with mathematical talent. There is a sustainable group of specialists from different departments at IMI (many of them former participants in national and international mathematical competitions) that supports scientifically all major national competitions and the preparation for international competitions. The group is tightly networked with groups and organizations performing similar activities in other countries. This includes the World Federation of National Mathematics Competitions (WFNMC, see http://www.amt.canberra.edu.au/wfnmc.html), European Kangaroo, the newly established Mathematical Society of South-East Europe (MASSEE), etc. Within the framework of the forthcoming Congress of MASSEE there will be a special mini-symposium on

attracting talent to science where higher ability secondary school and university students will deliver the results of their own research-like activities.
IMI is the birthplace of the first International Olympiad in Informatics (1989) and co-initiator of the Mathematical Balkaniad for secondary school students (early 80's).
The existing Bulgarian Know-How in the early identification, motivation and support of higher ability students will be critically estimated and compared with the similar knowledge in other countries. On this basis new materials will be prepared to facilitate the work of teachers and other educators with the specific group of mathematically talented youngsters.

Partner 3: (University of Cyprus, CYPRUS)

The University of Cyprus was established in 1989 and admitted its first students in 1992. Admission to the University is by national entrance examinations and the competition for places is intense. The ratio of candidates to available places is approximately 10 to 1. The main objectives of the University of Cyprus are twofold: the promotion of scholarship and education through teaching and research and the enhancement of the cultural, social and economic development of Cyprus. Research is promoted and funded in all departments for its contribution to scholarship in general and for its local and international applications. Original research is one of the primary activities of the academic staff at the University of Cyprus. The University's research programmes cover a wide range of fields that correspond to existing departments. The University is a member of a number of international university organisations and networks. It also cooperates, through inter-state and inter-university agreements, with universities and research centres in Europe and internationally, for the promotion of science, scholarly research and exchange of information. Moreover, the University cooperates with various institutions in Cyprus on research programmes that are specifically aimed at the needs of Cypriot economy and society in general.
Department of Education: The mission of the Department of Education is as follows:

- Producing and disseminating knowledge in the Pedagogical Sciences
- Identifying, researching and studying educational issues
- Educating primary and pre-primary teachers for Cyprus schools
- Providing graduate programmes with the aim of preparing research personnel and people who will assume leadership position within the educational system
- Providing in-service training and staff development courses for school personnel

The research interests of the faculty members cover a broad spectrum of areas ranging from issues related to educational assessment and evaluation, to educational management, to sociology and psychology and finally to didactic of the specific subject areas.

Partner 4: (Charles University, CHECZ REPUBLIC)

Charles University is located in Prague, CZ. It is a public institution.
Charles University's Faculty of Education mission is to prepare teachers for all types and levels of schools, prepare specialists and scientists in the area of pedagogy, educational psychology and didactics. Depending on the type of study, the Faculty of Education awards Bachelor, Master and Doctor degrees. In the area of international co-operation, the Faculty of Education focuses on various types of projects in the Socrates programme (Comenius, Lingua, Grundtvig, Minerva, Arion, Erasmus).

Partner 5: (University of Duisburg, GERMANY)

The university of Duisburg was established in 1972. Today, there are about15 000 students enrolled in five faculties. 210 professors and 650 other full-time academic staff are engaged in teaching and research. Students from 110 different countries are enrolled at the university which guarantees a lively international student community. Despite the relatively small size of the university, a wide range of courses are available on fields as various as the social sciences, economics and languages, as well as the natural sciences, mathematics and engineering. Alongside these traditional course programmes, the university of Duisburg offers a variety of unique degree courses and interdisciplinary courses of study aimed at international students. These courses are designed as Bachelor or Master degree programmes. The Institute of Mathematics has a strong emphasis to Applied Mathematics (optimisation, mathematical and numerical methods of image processing, wavelet analysis, approximation theory and probability theory). There are two regular professorships at our Institute for Education of Teachers (Didactics of Mathematics). Since January 2003 the university of Duisburg and the university of Essen have been merged to the new university of Duisburg and Essen.

Partner 6: (University of Crete, GREECE)

The University of Crete is a multi-disciplinary, research-oriented Institution, situated in the cities of Rethymnon and Heraklion on the island of Crete, Greece. It is a University with state-of-the art curricula and graduate programmes, considerable international cooperation and initiatives. The aim of the University of Crete is the promotion of science and knowledge through education and research, as well as the participation in the cultural, social and financial future of the region and of the country as a whole.

It started operating in 1977. To-day, 8723 students attend the University (7187 undergraduate and 1535 graduate). The University consists of 544 teaching and research staff members and more than 300 administrative employees.

The following Schools operate: a) The School of Letters consisting of the Departments of: Philology, History-Archaeology, Philosophy and Social Studies, b) The School of Social Sciences consisting of the Departments of: Sociology, Economics, Psychology and Political Science, c) The School of Education consisting of the Departments of: Primary Education and pre-School Education, d) The School of Science consisting of the Departments of: Mathematics, physics, Biology, Computer Science, Chemistry, Applied Mathematics and Material Science, e) The School of Health Sciences consisting of the Faculty of Medicine.

Partner 7: (University of Palermo, ITALY)

The Department of Mathematics and Applications was established in 1990. Before this date an Institute of Mathematics that has a very ancient history existed. The activity of the School of Mathematics in Palermo goes back to 1884 (“Circolo Matematico di Palermo”). Its activity has been manifold from studies of theoretical mathematics to studies of applied mathematics. Work has concentrated on the Fundamentals of Mathematics as well as the Didactics of Mathematics.

Numerous types of teachers of Mathematics both for the requirement of Sicily and for other zones of Italy were trained. The Department offers the followings types of degrees:

1. Mathematics
2. Mathematics Applied to industry and finance
3. Computer Science
4. Mathematics for Computer Science and scientific communication

Since 1979 the Department of Mathematics has developed a group for research in Mathematics Education: G.R.I.M. (Gruppo di Ricerca sull’Insegnamento delle Matemtaiche). The group mainly deals with research in didactics of Mathematics as well as the relationships with the schools in Sicily through the establishment of post university programmes for elementary school, middle school and upper school teachers.

The activity of research of this group is documented in the web site: http://dipmat.math.unipa.it/~grim /

In the site one can find:

1. the on-line magazine "Quaderni di Ricerca in Didattica": http://dipmat.math.unipa.it/~grim/menuquad.htm;
2. the proceedings of the international group http://dipmat.math.unipa.it/~grim/21project.htm;
3. the didactic materials elaborated by the teachers and the students of the university courses for the formation of the future elementary and upper secondary school teachers;
4. Articles and a list of Italian thesis for degrees and doctorates.

Partner 8: (North University of Baia Mare, ROMANIA)

Established in 1961 as a Pedagogical Higher Education Institute, nowadays North University of Baia Mare (NUBM) offers more than 35 graduate, postgraduate and PhD programmes covering a wide

range of fields. NUBM comprises four Faculties (Faculty of Engineering, Faculty of Mineral Resources and Environment Science, Faculty of Sciences, Faculty of Letters) and a University College with branches located at Satu Mare and Bistrita.
Department of Mathematics and Computer Science, the largest department of NUBM offers the following programmes: Mathematics and Physics, Mathematics and Computer Science, Computer Science (all graduate programmes), Computer Science (postgraduate programme) and Numerical Methods with Applications to CAD (Master degree programme).
There exists a long tradition at NUBM in training high quality teachers for the pre-university educational system as well as in organizing continuous education training programmes for teachers. The research work of the full staff is done in the frame of the Research Center “Pure and Applied Mathematics”, which has been ranked 8th out of 31 university research centres accredited by the National Council for University Research in 2001. A working group (Centre for Excellence in Mathematics) is involved in training talented high school and university students for mathematical competitions.

Partner 9: (University of Miskolc, Hungary)

The mission of the university is formed by a combination of major insights, commitments, values and efforts: the establishment and maintenance of an integrated HE institution that meets the standards of the age by producing well trained and highly qualified professionals, and by active participation in the scientific and social life of the nation and the world. The university is committed to the continuous adjustment in the contents and structure of its academic programmes to respond to the global and European developments in HE. The widespread international relations of the university have changed. Previous relationships (mainly with countries in Eastern Europe) have been transformed or, in some cases, ceased. On the other hand, broadening has taken place through the growing number of partner institutions in Western Europe in the form of joint international projects and bilateral agreements. The University of Miskolc did join the major European educational and research projects (TEMPUS, CEEPUS, SOCRATES, LEONARDO, NATO, 4th and 5th Framework Programmes, EUREKA, etc.), and the university has been participating in two other actions (LINGUA, MINERVA) of the SOCRATES programme.
In addition to the partners presented above we had contributions from a number of Math Societies. These were:

- Cyprus Mathematical Society
- Cyprus Union of Mathematicians
- Hellenic Mathematical Society
- Mathematical Society of South Eastern Europe
- European Mathematical Society
- Bulgarian Union of Mathematicians
- Romanian Mathematical Society
- Czech Mathematical Society
- Hungarian Mathematical Society

1.2 Reference/acknowledgement

Acknowledgement is given to members of Math Societies who voluntarily contributed to the development of material for the MATHEU project.
Particularly Prof.Mircea Becheanu from the Romanian Mathematical Society who collected material from all the partners and developed the identification tool.
Prof. Svetoslav Bilchev from the Bulgarian Mathematical Society who developed ladders
Dr Emilia Velikova from MASSEE who developed ladders.
Mr Andreas Philippou from the Cyprus Mathematical Society and Mr Marios Antoniades from the Union of Cypriot Mathematicians who put together the multiple choice part of the identification tool.

1.3 Mathematics as one of the major priority subjects in European Union

The decision of the European Union, COM/2001/678 says, «In a society of knowledge, Democracy requires the citizens to have scientific and technological knowledge as part of the basic competence». The future aims of the European Educational Systems, which was agreed on 12 February 2001 from the Education Council in Stockholm, identify Mathematics as one of the major priority subjects. The basic objective is the increase of interest in mathematics from early age and the impulsion of youth to follow careers in these subjects.

The types of students who will be able to contribute to the research of these fields are more likely to be students who are talented in these fields and more specifically in mathematics.
Certain activities towards this objective are already taking place in the partner countries. The idea of MATHEU was to bring together experts from the partner countries and to exchange ideas, background knowledge and experience and to develop together a system that will work for all member countries in the EU.

Talented students in Mathematics have to be discovered in early stages and in a systematic way. The usual method for identifying such students is through competitions but it is generally acceptable that many talented students in Mathematics are never discovered because they do not participate in competitions or because they were not among the top ten during the competition process or they are not able to work under strict time limits.
European countries have to find ways to keep their talents and brains in Europe. In order to accomplish this, mathematicians, academicians and educators have to work together in design a programme, which will change attitudes of governments, universities and foundations in favour of supporting the gain of mathematical talents in Europe and decrease the brain drain outside the European Community. Talented students need attention, affection, support, training, recognition and identification. The partners of MATHEU gave a promise to offer solutions to all these challenges for the development of European talented students through their teachers, educational administrators and other statutory corporations - institutions – government bodies, as well as through the direct links via the Internet.

1.4 - The aims of the project/manual:

In many European schools the mathematics curriculum is designed to serve the average and special needs students without identifying and supporting potentially talented/competent students in Mathematics. The aim of MATHEU was to develop methods and educational tools, which will help the educators to identify and motivate talented students in Mathematics as well as to support their development within the European Community without any discrimination. MATHEU merged forces and established a network through the Mathematical Societies and universities in the European area to support its aims as well as to use new technologies in the support, dissemination and sustainability of the developed structure of cooperation.

The main activities of MATHEU included:

- Analysis of the flexibility of existing mathematics curricula in European Schools with emphasis on the partner countries focusing on talented students
- Analysis of methods and tools used in European countries for the identification, motivation and support of talented students in Mathematics
- Design methods and tools for identifying potentially talented students in both primary and secondary education levels and for training teachers so that they can bring the students to express their 'talent' in Mathematics (talent as ability to face and solve problematic situation and to appreciate the role of theoretical thought)
- Design special pedagogical methods and subject material for the development and promotion of talented students in European schools
- Develop methods/solutions and a programme for changing attitudes within government , universities and foundations in providing fellowships and support in order to keep mathematical brains in Europe
- Design a special Web-site devoted to this project which will enable the sustainability of the project aims

Main outcomes were:

- A European Manual with methods and tools for identifying, motivating and supporting talented students in Mathematics
- An information programme for government, universities and foundations
- A training course for both primary and secondary school educators for identifying and developing talented students in Mathematics

The project contributes to a “brain gain-effect” for the European Community and helps in the aims of the Education Council of the European Union as agreed on 12 February 2001 in Stockholm to set Mathematics as one of the major priority subjects.

1.5 - The experiment

The overall purpose of this evaluation was to prove in an experiment the idea of the ladders and the selection of students by a special tool. For this reason we invited from each partner 2 students to run a special procedure, and focussed the outcomes in individual portfolios.

1.6 Organization of the Material

A “ladder” in this case is a self-contained mathematical text, focused on a specific topic, which could be used by teachers or by students in their work in and beyond the classroom. In essence the ladder is a sequence of mathematical problems, explanations and questions for self-testing ordered in slowly increasing degree of difficulty. By working on the text the student could elevate his/her mathematical knowledge to essentially higher levels. This is where the name “ladder” comes from: a device for climbing to a higher level, an instrument facilitating the process of overcoming different difficulties. Using the ladder the students (but also their teachers) could enrich, deepen and test their knowledge on a specific mathematical topic. The lower part of the ladder is rooted in the normal curriculum material studied in the class. As “steps” one has the mathematical problems, definitions and explanations, pieces of information and other challenges that the learner has to master in order to acquire the higher level of understanding the material. Depending on their individual abilities the students will advance i.e. “climb” to different heights on the ladder. The degree of advancement will single out higher ability students. Therefore the ladders will help identify talented students too.

If the ladder is well designed and consists of interesting and challenging problems, it will attract and motivate the students to apply more time and energy in studying mathematics.

It is important to design the ladders in such a way that the level of difficulty increases slowly (a small distance between two consecutive steps) and the students are capable of climbing the steps even without the help of the teacher. The definitions and the explanations should help this happen. The presence of questions and problems the solution of which is commented later will allow the student to check whether or not he/she understands what is going on.

Last but not least: offering ladders to students and teachers will not require restructuring of the whole educational process in school. It is close to some traditional practices which were abandoned in the last decades. The perturbation (if any) of the normal educational process will be small and, correspondingly, the level of resistance on the side of teachers and school authorities minimal.

1.7 How to be used by teachers

This book aims to help you with identifying and developing mathematical talents in their classrooms. Students may show their special talent in mathematics in various ways. There exist several lists of characteristics of talented students – see Chapter xxx. Let us mention here those which are common in most of the lists. Students talented in mathematics are likely to learn and understand mathematical ideas quickly, work systematically and accurately, see mathematical relationships, make connections between the concepts and procedures they have learned, apply their knowledge to new or unfamiliar situations, communicate their reasoning and justify their methods, take a creative approach to solving mathematical problems, persist in completing tasks, construct and handle high levels of abstraction, have strong critical thinking skills and are self-critical, can produce original and imaginative work.

This variety in characteristics induces the difficulties in the identification of students talented in mathematics. No student demonstrates all characteristics, but he/she shows a significant number of them. The identification is not, and cannot be, perfect.

In this book, the identification through problem solving is used. There are … topics covered at two age levels, 9 to 14 and 15 to 18. Each topic is tackled by gradually more complex problems. Some of them are furnished with the main ideas of the related theory, some of them not.

The chapters are not meant to be used as a non-separable whole, to start at the beginning and go through to the end. You know your students; you know their previous knowledge, their abilities and skills, their independence in finding relevant information etc. You are the main person for deciding how to use the ladders with your students, which problems to use, which resources to recommend them etc.

The identification tool can be used during the whole year but identification at the beginning of the school year is recommended. This can give you an indication about students who may exhibit a high ability and talent in mathematics. Students who do not do well on the identification tool does not mean that they do not necessarily have a talent in mathematics. All potential students should be given a ladder to work on. The progress they make on the ladder will indicate advanced abilities and the level they reach within a ladder will indicate a talent in mathematics.

1.8 How to be used by students

This book aims to help you to recognize and develop your mathematical talent. Some of you have already been identified as mathematical talents during various opportunities – competitions, application of your mathematical knowledge in other of your fields of interest etc, and some have not.
Through sets of gradually more complex problems in several topics you can either develop your talent or identify it. You are not supposed to start with the first problem in the chapter. If you find it too easy, you can continue further and work on problems that are challenging for you. For some of them you will need to find additional mathematical information. There are rich information resources on Internet, in books as well as support from your teachers and other people around you.
Even very talented students may have difficulties in solving all the problems in a ladder. They may be problems that are too difficult for you at the moment. But working systematically and broadening your repertoire of knowledge and skills will help you to solve the problems that you find unsolvable at the moment.
All authors wish you good luck in developing your talent in mathematics and much enjoyment with the problems of the manual.

1.9 Comparative study

Part I: Identification-Motivation

Introduction

In this part of the comparative study we will try to give a clear picture of what is happening in eight European countries (Bulgaria, Cyprus, the Czech Republic, Greece, Germany, Italy, Romania and the United Kingdom) in order to Identify, Motivate and Support Mathematical Talents. We start by giving as many details as we can about the activities that are organized by the Math Societies, and the Math Unions, the Universities and the governments as they have been given by the member countries participating in the project.

- **Cyprus**

Mathematics education in Cyprus is rising. There is great interest amongst primary school students and the Cyprus Maths Society identifies and motivates students through competitions. Students are trying hard to enter the national team. The Cyprus Math Society (CMS) supports students through publications, a preparation programme and by organizing a Math Summer School.

The Cyprus Math Society, which was established in 1983, aims to promote mathematical education and science. The CMS is a non-profit organization supported by the voluntary work of its members. The CMS (Cyprus Math Society) counts over 600 members. In order to promote its aims, CMS organizes Mathematical competitions among students all over Cyprus, and takes part in international math competitions. The CMS organizes annually mathematical conferences and seminars.

In Cyprus in order to identify talented students they use 1.City competitions e.g. Nicosia, Limassol, Larnaca and Ammohostos, Paphos, 2.National Competitions either for The Gymnasium or the Lyceum and Selection examinations, 3.National Mathematical Olympiad for the 4^{th} to the 12^{th} grades.

As far as motivation is concerned Cyprus organizers Awards ceremonies for National Teams for 1.BMO, 2.JBMO, 3.IMO, 4. International Contest for Primary School pupils. Cyprus supports talented students with 1.Preparation programme, 2.Summer Math School and 3.Publications.

- **Bulgaria**

The Mathematics and Informatics (IMI) institution in Bulgaria organizes and carries out research in Pure and Applied Mathematics and Informatics. It is also engaged with the education and preparation of highly qualified specialists in virtually all important branches of Mathematics and Informatics. Currently the Institute of Mathematics and Informatics (IMI) comprises about 180 persons. 130 of them have scientific degree (Doctor of Sciences). One third of its staff is temporarily abroad, working for other research and educational institutions worldwide. The Institute of Mathematics and Informatics (IMI) promotes science and tries to attract young talents to scientific occupational careers. Moreover the Institute has a clear vision of identifying talented students. The mathematicians are not willing to accept that identifying talented students is important.

The Union of Bulgarian Mathematicians played an important role in training students for taking part in Math Olympiads (both regional and international). The Union is preparing students by giving lectures and organizing seminars in cities all over Bulgaria. They are trying to identify and train talented students by individual work with promising students during the time of ordinary classes in mathematics and with out of school mathematical activities like excursions, mathematical evenings, days of famous mathematicians and mathematical competitions. In Bulgaria the best students know that they can study in good universities in the US. This is a great motivation and most of them do very well in competitions. The school, the teachers and the parents think highly of the students that take part in such events. Unfortunately, Europe is unfriendly to talented students and the US absorbs all the talent of the world. e.g. If a student goes to France to study he will need to cover his living expenses, on the other hand in the US everything is paid.

As far as Identification of talented students is concerned, in Bulgaria there are 1. TV, Internet and journal competitions, 2.School mathematics competitions (City, inter city), 3.National Competitions (Winter, Spring, 'Atanas Radev", 'Sly Peter", 'Ivan Salabashev', 'Akad Kiril Popov', 'Peter Beron', 'Chernorizec Hrabar', Language Schools', Christmas, Easter, Mathematics Tournaments (Sofia, Pazardjic, Kardiali), 4.National Olympiad (School Round, City Round, Regional Round, National Round, 5. Selection For IMO (National Competitions and Olympiads, International Competitions and Olympiads, Spring Conference of the Union of Bulgarian Mathematicians, One Month Summer School, Special Control Papers During the Summer School, Final Evaluation), 6.International Competitions, 7.Balkan Olympiad of Mathematics (Junior level, Senior level), 8.Tournament of the Towns, 9.Kangaroo Competitions (French Kangaroo, Australian Kangaroo), 10.International Mathematical Olympiad (Junior level, Senior level).

Bulgaria in order to motivate students allows them to participate in 1. National Teams for: i. Balkan Mathematical Olympiads – Junior, ii. IMO – Junior and Senior levels, 2. KANGAROO Contests – European, 3. Tournament of the Towns, 4. Studies in Europe, USA, Canada etc, 5. Awards Ceremony. There is also Society support environment and organization of a training process in three stages: 1) force of the students' interest; 2) formatting high level of knowledge and skills; 3) developing of students' abilities. The organization of a training process which depends on the students' interests and abilities. The personality of the training (leading) teachers is a strong motive for the students. The strongest motive for the students is the possibility to be university students without entry exams. Furthermore Bulgarian talented students have a high prestige. Finally a strong motive for Bulgarian students is parents' support.

Bulgaria in order to support students arranges 1. Individual work with promising students during the time of ordinary classes in mathematics, 2.extracurricular events with mathematical activities such as excursions, mathematical evenings, days of famous mathematicians and mathematical contests etc, 3. School mathematical circles, 4. City mathematical circles such as Ordinary and Special groups 5.Short mathematical circles such as Green, Summer, Winter and Sea, 6. Extramural mathematical circles that supply students with problems, solutions, evaluation and commentaries, 7. Correspondence Preparation with Problems and Literature, 8. Lectures on special subjects of mathematics by the invited university professors and leading teachers, 9. Spring Conference of the Union of Bulgarian Mathematicians, 10. School Institute of Mathematics and Informatics with the aim: Preparation of Mathematical Essays by students that includes School Round, City Round, Regional Round, National Round.
Moreover another aim of the School Institute of Mathematics and Informatics is the Selection for the Centre of Excellence in Education in Boston in the USA.

- **Czech Republic**

The Czech Republic organizes many math competitions and takes part in math Olympiads. Particularly for the 1st stage (from the 4th class of BS (9 years) and experimentally from the 3rd class) there is the Mathematical Olympiad and Kangaroo. For the 2nd stage (from the 6th class of BS (11 years) there is the Mathematical Olympiad, Kangaroo, other competitions (Dejte Ulavydohromody, Pythagoridda, Prazsled strela, Dopplerova vlua) and many local competitions (at schools, in towns). For the 3rd stage (from the 1st class of SS (15 years) the Mathematical Olympiad and Kangaroo as well as for the 4th stage (from the 1st year of studies in university (19 years)). The Categories for Mathematical Olympiad are: 1.Z4, Z5, Z6, Z7, Z8, Z9 – for BS, 2.C- for the 1st class of SS, 3.B- for the 2nd class of SS, 4.A- for the 3rd and 4th classes of SS, 5. All together about 20-30 pupils, 6.Several rounds. The Categories for Kangaroo are 1.Klobaluek- the 4th and 5th classes of BS, 2.Benjamin- the 6th and 7th classes of BS, 3.Kadet- the 8th and 9th classes of BS, 4.Junior- the 1st and 2nd classes of SS, 5.Student- the 3rd and 4th classes of SS, 6.All together about 300,000 pupils participate. All the activities are supported by the state.

The most current support forms in practice in the Czech Republic for the 1st stage (from the 4th class of BS (9 years) (experimentally from the 3rd class) are special classes with extended teaching of foreign languages or mathematics after the special exam and Correspondence seminars (competitions) are organized. For the 2nd stage (from the 6th class of BS (11 years) classes with extended teaching, especially of foreign language or mathematics (or physics) (after the special exam), Correspondence seminars, Holiday camps with teaching math and physics, foreign languages, sports, Clil are organized. The Czech Republic for 3rd stage (from the 1st class of SS (15 years)) organizes special classes with extended teaching of foreign languages or mathematics or other subjects (physics, chemistry, informatics, physical training, special sports, and art), Correspondence seminars, SOC in many branches as Olympiads, Preparing competitions for pupils of BS, and Clil. For the 4th stage (from the 1st year of studies in university (19 years) the staff organizes special faculties, especially faculty of mathematics and physics, SVOC in many branches (as OU SS), Preparing competitions for pupils of BS & SS. Finally, for the 5th stage (from PhD studies (usually 24 years) the Czech Republic organizes PhD studies and prepares competitions for pupils of BS & SS.

- **Germany**

In the university of Duisburg and Essen they are trying to attract students by organizing workshops and other activities but the number of mathematics students is dropping since there is no professional organization for promoting talented students.

Germany uses Mathematical Competitions in order to identify talented students in mathematics. The first one is “Bundeswettsewers Mathematir” that has 3 Rounds.
The first Round includes 4 homework (1-March→1-June), the second Round includes 4 homework (1-June→1-Sept) and the third 3.Round involves Colloquium. In 2003 there were 1146 participants, 90% usually in classes 9-13.
“Bundeswettsewers Mathematir” is organized by the “Vertin Bildung and Begabung e.v.” and it is supported by the Ministry for Education and Research.
In Germany the Mathematic Olympiads (Univ. Rostoor) exist since 1994. There are also other competitions such as Arbeitsgemeisdaft des “bundeswite” Sdilerwettscwerse and Volrswapastiltang. Since 2003 there are 14 projects in the programme Verbeslerng des Mathematir-unlimcsrles. Germany with the view to motivate students gives prices, these prices are: Price I, II, III, A, No.

As far as support of talented students at universities is concerned Studienstiftang des Deutschen Volres gives stipends to excellent students (all areas). There are further private foundations that support talented students in mathematics.

- **Greece**

There are no schools for the talented students in Greece. Such schools closed 20 years ago and the European Union does not encourage these practices anymore. The Hellenic Mathematical Society offers some help to the talented students. The Hellenic Mathematical Society organizes many activities like seminars, competitions and publications in order to prepare students for international Olympiads. All activities of the society are supported by the Ministry of Education.

Still, a major drawback is the university entry examinations. Everybody studies hard to get into university and there is no time for extra effort in mathematics. The Math Society issues one journal but this journal does not really apply for the “Olympiad brains”. In Greece there are many outstanding students that never took part in any math competition because they believe that it is a waste of time.

Most of the students that get the medals finally become Mathematicians. In Greece the Hellenic Mathematical Society organizes competitions with the aim of identifying the best students in mathematics. These competitions are the following ones: 1.Thales. This competition takes place by the end of October, at a local level. It is open to anybody willing to participate, from 2nd Gymnasium (age 12) to final school year (age 17); there are separate questions for each class. The Syllabus includes whatever the students have learnt up to the previous class. Approximately 15000 students took part in 2002/3. 2. Euclid. This competition takes place by Mid-December in Athens. The participation depends on the grade in Thales competition. This grade is set by the Organizers, according to the results of the participants. The participants come from 2nd Gymnasium (age 12) to final school year (age 17): there are separate questions for each class. The Syllabus includes whatever the students have learnt up to the past three months of their current class. Approximately 1600 students took part in 2002/3. 3. Archimedes. This competition takes place by the beginning of February, in Athens. The participation is by invitation according to success in Euclid. This competition is open to young students from the Gymnasium and students from the Lyceum. The Syllabus includes the same one for IMO. 300 students took part in 2002/3. 4) Other Competitions such as Informatics (http://www.epy.gr) and two competitions using the Internet and then final competition. We can add to all these participation in BOI and IOI.

In Greece the Hellenic Mathematical Society in order to motivate students gives prizes in each competition. Especially, more than 300 students per class are given a prize, organized locally for Thales competition. About 50 or 60 students per class are given a prize for Euclid competition. 25 Gymnasium students and 25 Lyceum students receive prizes for Archimedes. Moreover about 25 Young 25 Old participate in further internal competitions and training. The BMO and IMO teams are selected from these groups.

The Hellenic Mathematical Society tries to inform, encourage and support students through schools and Local educational Authorities to participate in Thales competitions. The participation in Euclid competition is made by invitation according to the grade obtained in Thales. The participation in Archimedes is made by invitation according to success in Euclid. Unfortunately students in Greece are largely self-taught.

- **Italy**

The University of Palermo, in cooperation with the University of Pisa, organizes regional math competitions in order to identify talented students in mathematics. These competitions are: 1.Mathematical competitions to provincial level held in Palermo and concern students of the age of 11-13. 2. Mathematical competitions to national level organized by in University of Milan (Lettera Pristem: Palermo-Milan) and University of Pisa. 3. The role of the processes of socialization of the knowledge in situations of teaching/learning. 4. The construction of particular didactic situations that can allow this socialization. 5. The possibility to be able to socialize. 5. The possibility to be able to socialize procedures, schemes of reasoning, decisive strategies of situations/problem. Italy in order to support students supplies them with the following: The winners of the mathematical competitions for the ‘Scuole Superiori’ (16-18) as Tutor in the mathematical competitions have been used for the middle school (11-13). Further more it made Site wed with problems and solutions and many Publications.

- **Romania**

In Romania there exists a long tradition in training high quality teachers for the pre-university educational system, as well as in organizing continuous education training programmes for teachers. A working group (Centre for Excellence in Mathematics) is involved in training talented high school and university students for mathematical competitions (both regional and national).
The aim of the Romanian Math Society, that was founded in 1910 and has more than 10,000 members, is to support the existence of the math community in Romania. Since 1895 the RMS issues on a monthly basis a journal of mathematical culture for the youth (Gazette Mathematica). This journal, for more than 100 years, has contributed substantially to the development of the Romanian mathematics education and later to the survival of the Romanian School of Mathematics. The journal is addressed to students and teachers that are interested in Mathematics and publishes mathematical

papers, subjects from the Olympic exams and competitions. The Society at the moment is in disagreement with the Ministry of Education regarding the regulations of the math Olympiads. The Ministry is more interested in the number of prizes given instead of encouraging the spirit of getting students involved with mathematics. The Society tries to attract and motivate students to study mathematics. In Romania students are leaving the country and are going to the US (brain drain). 10 years ago students used to go to Germany but the legislation for work permit was a major drawback. Students might go to Europe for study. After completing their PhD they return to Romania where they cannot find a job. The US ranked the schools in Romania and the best students from the best schools are accepted immediately to US Universities. The MOE has decided to decrease the prizes in the national Olympiads and this led to a decrease of motivation not only in participating in such Olympiads, but in studying mathematics as well. Romania has a long tradition in selecting and training talented people. For the record the first competition for primary school students took place in 1885. In 1897 the first attempt was made to organize a national mathematics contest. In 1902 the Annual Contest “Gazeta Matematica” was made by mail. Seven years later Contestants (selected among the best solvers) gave a written and oral examination. In 1949: National Mathematics Olympiad was created and organized by RMS and the Ministry of Education. There are now many competitions in Romania. There are: Mathematics Olympiad for all the school levels (primary, secondary, and high school), School Round, City Round, District (County) Round, Final (National) Round. 550-600 students (Forms 7-12) qualify each year for the Final Round. More over there is the annual competition "Gazeta Matematica" organized each summer for the best solvers of the journal. There are also International Competitions such as JBMO, BMO, IMO. A Team selection takes place during the Final Round of the National Mathematical Olympiad.

Furthermore there are Special training stages, Kangaroo, Intercounty Mathematics Competitions (organized during the academic year, most of them between the County Round and Final Round of the National Mathematical Olympiad). The most important of them are: "Gh. Titeica" , "Gr. C. Moisil", "Tr. Lalescu" and "L. Duican".

The most important motive for students is the prospect of Studies in the US. There also other motives for talented students in mathematics such as prizes. In the first competition for primary school students in 1885 when 70 participants were examined, 9 boys, 2 girls were awarded prizes. Nowadays, there is an award ceremony at the Final Round of NMO. The prizes are awarded by the Ministry of Education, the County Authorities where NMO is organized, the Romanian Mathematical Society, sponsors etc. Selection in the JBMO, BMO and IMO extended teams is another very strong motive for the Romanian students. At county level there are prizes awarded by RMS, and local sponsors. Medalists at JBMO, BMO, IMO are awarded special prizes in a special ceremony by the Minister of Education, the Prime Minister or even the President of Romania. Also winners are offered excursions abroad.

Romania has been trying to support talented students in mathematics for many years. For instance it published the First issue of Scientific Recreations: “mathematics; physics; chemistry etc.” with problems, notes, and articles in 1883.

The first issue of Gazeta Matematica, the most respectable Romanian journal, devoted to elementary mathematics was published in 1895; it is mainly responsible for creating, improving, and keeping up a high interest in attracting talented students. It has 12 issues/ a year and it has been published without interruption so far. The first International Mathematics Olympiad (IMO) was held in Bucharest in 1959, in 1999 the 40^{th} Olympic was also held in Bucharest. During the years 1971-1973 Classes of excellence: Mathematics, Physics, Chemistry and Biology were organised by the Minister of Education, Mircea Malita. The National Net of Centres of Excellence (main cities) was established in 2001.The Centre of Excellence in Maramures country was established in October 2002. Moreover the journal named ‘LUCRARILE SEMINATULUI DE CREATIVITATE MATEMATICA’ which means “Seminar on Creative Mathematics”, provides students with articles, notes written by students, secondary school or high school teachers, and university teachers. In this journal there are articles dedicated to heuristic of solving competitions problems and most of them are designed to develop inventive skills by problem solving; this opens the way to research work.

The Centre for Excellence in Mathematics (NUBM) was founded in October 1991 and since then it has been giving seminars on Creative Mathematics. In 2000 “The Centre for Training Gifted Students” was founded and in 2001 the Centre for Excellence in Mathematics was founded. Its main aims and scopes are: 1.To work with early university level students. These problem solvers can make early research in mathematics, 2.To identify and train talented high school students for mathematical competitions such as National Mathematics Olympiad, Inter-county Mathematics Contests, American Math Competitions, 3.To publish the last materials such as the Journal “Seminarul de Creativitate Matematica”, problems and other maths books, 4.To organize the Inter county Mathematics Competition“Gr. C. Moisoil”: problems, proposals and coordination, 5.To help Winter Mathematics Camps organizers: lecturers and training students. 6.To train university students for Mathematics Competitions (“Tr. Lalescu” math.comp’ International Maths Competition: IMC – 2001 (Prague); 2002 (Warsaw)).

- **United Kingdom**

The United Kingdom organizes math competitions in order to identify talented students in mathematics, such as 1.Junior Challenge Years 7 and 8, (Ages 12 and 13). The students are given a 60 minute paper with 25 multiple-choice questions. In this competition 240.000 entrants from over 3.200 schools take part. 2. The follow-on round to the Junior Olympiad is the Junior Mathematical Olympiad. This event is normally held on the first or second Tuesday of June. In 2003 this event was held on Tuesday 10th June. 1.204 of the best performing students, in the 2003 Junior Challenge, were invited to take part in the Junior Mathematical Olympiad. The Junior Mathematical Olympiad is a two-hour paper which has two sections: Section A has ten questions and pupils are required to give the answer only. Section B has six questions for which full written answers are required. 3. Intermediate Challenge, (Years 10, 11 and 12 Ages 14, 15 and 16). The students are given a 60 minute paper with 25 multiple-choice questions. Over 207,000 entrants take Part from over 2.700 schools. 4. The follow-on competitions to Intermediate Challenge are the European Kangaroo and the IMOK Olympiad (Intermediate Mathematical Olympiad and Kangaroo). 1.000 pupils in each of Y9, Y10 and Y11 (E&W), S2, S3 and S4 (Scot.), and Y10, Y11 and Y12 (N.I.) are invited to sit the European Kangaroo. This is a one-hour multiple-choice paper with 25 questions taken by students across Europe and beyond. The lowest year group takes the Kangaroo 'Grey' paper and the top two year groups both take the Kangaroo 'Pink' paper. In 2003 the European Kangaroo took place on Thursday 20th March. All those taking part receive a Certificate of Participation or a Certificates of Merit. 5. (Senior Challenge, Years 13 and 14, Ages 17 and 18). The students are given a 90 minute paper with 25 multiple-choice questions. Over 60.000 entrants take part from over 1.500 schools. 6. British Mathematical (Round 1). Up to 1.000 high scorers will be invited to participate in the British Mathematical (Round 1). This can lead to BMO2. Six participants will make it to the International Mathematical Olympiad. 7. BMO2. 8. International Mathematical Olympiad, (only six participants).
As far as the motivation in mathematics education in the United Kingdom is concerned it is connected with the participation in the national team. As far as Junior Challenge Years 7 and 8, (Ages 12 and 13) is concerned, high scorers are invited to participate in the Junior Olympiad (UK JMC).
All those who take part in the Junior Mathematical Olympiad receive a certificate: the top 25% receive Certificates of Distinction and the rest receive Certificates of Participation. Medals are awarded to very good candidates and the top 50 also receive a prize. High scorers in Intermediate Challenge, (Years 10, 11 and 12 Ages 14, 15 and 16) are invited to participate in the follow on rounds: Intermediate Mathematical Olympiad and Kangaroo (IMOK). All those taking part in the European Kangaroo receive a Certificate of Participation or a Certificate of Merit. Up to 1,000 high scorers in Senior Challenge, (Years 13 and 14, Ages 17 and 18) are invited to participate in the British Mathematical (Round 1). High scorers in the British Mathematical (Round 1) can participate in BMO2, and from those, six participants will make it to the International Mathematical Olympiad.

In table 5 we present all the Identification, Motivation and Support activities that each country provides so far to its pupils as far as mathematics education is concerned. We give the names of the competitions and Olympiads, times of the competitors, grades of the students, the prize names, and names of Certificates of Participation etc.

Results-Comments

Results speak for themselves and allow us to observe that other countries have as a major goal to identify, motivate and support talented students in mathematics and use as many competitions as they can; they prepare their students for the competitions and create the best conditions they can to promote them.

Particularly, the Identification process is done in Cyprus by using 1.City competitions, 2.National Competitions, 2.National Mathematical Olympiads. In Bulgaria there are many competitions, 1.TV, Internet and journal competitions, 2.School mathematics competitions, 3.National Competitions, 4.National Olympiad, 5.Selection for IMO, 6.International Competitions, 7.Balkan Olympiad of Mathematics, 8.Tournament of the Towns, 9.Kangaroo Competitions,10.International Mathematical Olympiad. The Czech Republic has 1, Mathematical Olympiad, 2.Kangaroo,3. Local competitions (at schools, in towns). In Germany the Identification of the best students in mathematics is done through 1. Mathematical Competitions, 2. Mathematical Olympiads. In Greece and Italy there are mathematical competitions and in Romania identification of the best students in mathematics is done by 1. Mathematical Competitions, 2. Mathematical Olympiads. Finally, in the United Kingdom identification of the best students in mathematics is done by 1. Mathematical Competitions, 2. Mathematical Olympiad. The motivation of pupils in mathematics education is connected with the participation in the national team such as 1..BMO, 2.JBMO, 3.IMO, the opportunity to study in other countries through a scholarship (especially in the USA, Canada, Europe), the Awards ceremonies, the Certificate of Participation or a Certificate of Merit. There is also the prestige gained by a talented pupil which is taken into account both by him/her and by the family.

The above countries try to support their talented students in many ways. For instance Bulgaria does individual work with promising students during the time of ordinary classes in mathematics. It has a special school where there are mathematical activities. Furthermore it has 3School mathematical circles, City mathematical circles, Short mathematical circles, and Extramural mathematical circles. It also provides students through Correspondence Preparation and lectures on special subjects of mathematics by the invited university professors and leading teachers. It organizes the Spring Conference of the Union of Bulgarian Mathematicians and it has the School Institute of Mathematics and Informatics with the aim: Preparation of Mathematical Essays by students. Cyprus has Preparation programmes, and a Summer Math School. It also provides to students many publications. The Czech Republic has Special classes with extended teaching of foreign languages or mathematics (after the special exam), correspondence seminars (competitions). There are *Classes with extended teaching of foreign language or mathematics or physics after the special exam and Holiday camps with teaching math and physics, foreign languages, and sports. In addition it prepares competitions for pupils of BS & SS.

In Greece there are math competitions but students are largely self-taught. Italy provides web sites and publications. Finally Romania provides many publications and journals devoted to elementary mathematics; it is mainly responsible for the creation, improvement, and keeping up of a high interest in attracting talented students. Moreover it provides Seminars on Creative Mathematics and articles dedicated to heuristic of solving competition problems and designed to develop inventive skills by problem solving and open the way to research work. In Romania there is a Centre for Training Gifted Students"2001; a Centre for Excellence in Mathematics whose aims and scope are: 1.To work with early university level students, 2.To identify and train talented high school students for mathematical competitions such as National Mathematics Olympiad, Inter county Mathematics Contests, American Math Competitions, 3.To publish the latest materials, Journals, problems, books etc, 4.To organize the Inter county Mathematics Competitions, 5.To help Winter Mathematics Camp organizers: lecturers and training students and finally 6.To train university students for Mathematics Competitions.

But the important question is who is going to provide financial support to talented students in order to keep those intelligent people in Europe?

Part II: Support of talented pupils

In this paragraph we present the Support (by 1.Government, Ministry, 2. Institutions, Universities, Foundations, 3. Societies, 4. Individual support, 5. Publications/journals, 6. Local authorities, 7. Through teacher training of talented pupils for mathematics competitions that can be given by the countries (partners) that participate in this project.

- **Bulgaria**

The support to the talented students is provided by:
1. Government - The Ministry of Education and Science pays only the salaries and the social insurance of the staff in the "Centre for Students Technical and Scientific Creativity" (CSTSC).
2. Institutions (Universities, Foundations, etc) - The Universities assist with lecturers, rooms and facilities for extra-curricular work with talented students. The Foundations assist with some small amounts of money for individual and joint projects connected with the work with talented students ("Bistra and Galina" in Rousse, for example). The Foundations like "St Ciril and St Methodius" and "Eureca" give awards to the best teachers involved in the work with talented students. Some private companies (as "Mobiltel", for example) provide the necessary money for airplane tickets for the Bulgarian command for IMO.
3. Societies – The Union of Bulgarian Mathematicians (UBM) focuses on all kinds of support for the work with talented students - money, lecturers, rooms, facilities. Union of Scientists (for example - in Rousse) assists in publishing scientific papers of some leading teachers involved in work with talented students.
4. Individual support - from the parents of talented students and from some "former talented students" living abroad or in Bulgaria.
5. Publications/journals - organize courses with awards (money, books, taking part in summer camps, etc.) to the best students.
6. Local authorities - The local governments pay the necessary money for electricity, water and heating for the "CSTSC". The Schools Boards of Trustees (SBT) aid some seminars with leading teachers about extra-curricular work with talented students and some summer, winter, autumn preparation camps with talented students.
7. Through teacher training - UBM and some SBT aid seminars with leading teachers about extra-curricular work with talented students.

- **Cyprus**

The support to talented students is provided by:
1. The Government (Ministry of Education and Culture) through the pre-service teachers training of mathematicians. The pre-service training programme helps teachers to identify and provide basic support to talented students in their classroom.
2. The Cyprus Mathematical Society (CMS):
i) With collaboration with the Ministry of Education and Culture (financial support) CMS provides to the most talented students extracurricular lessons on a regular basis.
ii) Organizes mathematics summer school. This school provides to talented students learning opportunities and activities based on students' special interests for developing their potential.

- **Czech Republic**

The support to the talented students is provided by:
1. Material and financial support
i) Ministry of Education, Youth and Sports

ii) Regional school authorities
iii)Union of Czech Mathematicians and physicists
iv) Sponsors (minimally)
2) Organizational support
i) Schools
ii) Municipalities
iii) Individuals
Types of organizational support:
Preparation of problems and their solutions
Correction of solutions
Organization of competitions at various levels
Organization of courses for talented students
Organization of correspondence seminars

- **Germany**

The support to the talented students is provided by:
1. Government
The Ministry for Education and Research (Bundesminsterium fuer Bildung und Forschung) supports financially the following two main events (national competitions)
i) Bundeswettbewerb Mathematik
ii) Jugend forscht.
The "Studienstiftung des Deutschen Volkes" supports mainly students at Universities but exceptional talented students at high schools are also welcome.
2. Institutions
Usually foundations offer support to students at universities. The "Volkswagenstiftung" has started a programme "Improvement of mathematical education" in the year 2001 which includes 14 projects in Mathematical Didactics.
Many universities have started special programmes and initiatives to make mathematics more attractive to students at schools. Usually they are of regional character; as an example we mention the following internet addresses
http://www.ma.tum.de (Technical University Muenchen)
http://www.uni-duisburg.de/FB11/SMS (Univeristy of Duisburg--Essen)
3. Societies
The German Mathematical Society (Deutsche Mathematiker Vereinigung, DMV) provides a documentation "Begabtenfoerderung im Fach Mathematik" which surveys current activities in supporting talented students, see
www.mathematik.de
4. Individuals.

Based on initiatives of mathematics teachers and interested students local clubs have been established at many schools. A list of addresses can be found under www.mathematik.de or at the end of this Section

Mint-EC
Verein mathematisch-naturwissenschaftlicher Excellence-Centre an Schulen e.V.
http://www.mint-ec.de
Arbeitsgemeinschaft der bundesweiten Schuelerwettbewerbe
Mathematik-Olympiaden e. V. (Univ. Rostock)

Nach Bundesländern gruppieren Übersichtskarte

06122 Halle	Landesweite Korrespondenzzirkel der mathematisch-naturwissenschaftlichen Spezialschulen

07743 Jena	Arbeitsgruppe zur Förderung mathematischer Begabungen im Grundschulalter - Schülerzirkel "Die Matheasse"	website
07743 Jena	Carl-Zeiss-Gymnasium mit math.-naturw.-techn. Spezialklassen	website
07443 Jena	Mathematik-Olympiade	website
09120 Chemnitz	Bezirkskomitee Chemnitz zur Förderung mathematisch-naturwissenschaftlich begabter und interessierter Schüler	website
10099 Berlin	Mathematische Spezialklasse an der Andreas-Oberschule in Zusammenarbeit mit der Humboldt-Universität zu Berlin, Institut für Mathematik	website
10117 Berlin	Spezialklasse an der Andreas-Oberschule (Senatsverwaltung für Schule, Jugend und Sport)	
10117 Berlin	Kooperation der Humboldt-Universität mit Oberschulen (Senatsverwaltung für Schule, Jugend und Sport)	
14469 Potsdam	Mathematikklub des Treffpunkts Freizeit Potsdam	
14974 Potsdam	Pädagogisches Landesinstitut Brandenburg (PLIB)	
15711 Königs Wusterhausen	Friedrich-Wilhelm-Gymnasium	
15711 Königs Wusterhausen	Schülerfreizeitzentrum der JUH e.V.	
17235 Neustrelitz	Gymnasium Carolinum	
17491 Greifswald	Alexander-von-Humboldt-Gymnasium	
18051 Rostock	Korrespondenzzirkel Mathematik an der Universität	
18051 Rostock	Kreativität und Beharrlichkeit-Zauberworte für die Mathematik	website
18528 Bergen	Ernst-Moritz-Arndt-Gymnasium	
20146 Hamburg	Schülerzirkel Mathematik	
20146 Hamburg	Förderkurse für mathematisch besonders befähigte Schüler (Hamburger Modell	website
20146 Hamburg	Mathezirkel zur Förderung mathematisch interessierter Grundschulkinder	
20146 Hamburg	Besondere mathematische Begabung im Grundschulalter - ein Forschungs- und Förderprojekt	
21629 Neu Wulmstorf	Talentförderung Mathematik am Gymnasium Neu Wulmstorf	website
27570 Bremerhaven	Schülerzirkel Mathematik	
28215 Bremen	Schülerzirkel Mathematik	
34281 Gudensberg	Synapse	
34369 Hofgeismar	primatha	
35578 Wetzlar	Zentrum für Mathematik e.V.	website
37073 Göttingen	Mathematischer Korrespondenzzirkel	website
38104 Braunschweig	CJD Jugenddorf - Christopherusschule BS	website
39114 Magdeburg	Korrespondenzzirkel des Olympiadekomitees für die Mathematik-Olympiade in Sachsen-Anhalt	
39114 Magdeburg	Spezialistenlager (Wochenlehrgänge mit Vorträgen, Seminaren und	

	Übungen)	
39126 Magdeburg	Landesweite Korrespondenzzirkel der mathematisch-naturwissenschaftlichen Spezialschulen	
41334 Nettetal	PIN Privates (Psychologisches Institut am Niederrhein)	
64625 Bensheim	Samstagsakademie	website
64625 Bensheim	Zentrum für Mathematik e.V.	website
64625 Bensheim	MatheTreff 3456	website
66119 Saarbrücken	Gymnasium am Schloss	
67663 Kaiserslautern	Fachbereich Mathematik, Universität Kaiserslautern, Arbeitsgruppe Technomathematik	
66763 Dillingen	Zentrum für Begabtenförderung - Technisch-Wissenschaftliches Gymnasium Dillingen	
70029 Stuttgart	Problem des Monats (Ministerium für Kultus, Jugend und Sport BW)	website
79102 Freiburg	Freiburg-Seminar für Mathematik und Naturwissenschaften	website
85579 Neubiberg	Begabtenförderung Mathematik e.V.	
89017 Ulm	MINT	
97070 Würzburg	Landeswettbewerb Mathematik Bayern	
99867 Gotha	Mathe-Club	

5. Publications/Journals: see www.mathematik.de
6. Local Authorities: No information available
7. Through teacher training: No information available

- **Italy**

The support to the talented students is provided by:
1. Material and financial support

i)U.M.I. (Unione Matematica Italiana)
ii)Groups of Research in Mathematics Education (in Departments of Mathematics in Italian Universities: Torino, Genova, Trieste, Udine, Padova, Modena, Pisa, Siena, Roma, Bari, Catania, Palermo.
iii)P.RI.ST.EM., Progetto Ricerche Storiche E Metodologiche (University “Bocconi” of Milano)
iv)G.R.I.M. (Gruppo di Ricerca sull’Insegnamento delle Matematiche), Department of Mathematics, University of Palermo

- MATHESIS (National Association of Teachers of Mathematics with branches in many Italian cities)
- A.I.C.M. (Associazione Insegnanti e Cultori di Matematiche, Sicilian Association of Teachers, Students and lovers of mathematics
- Ministry of Education
- Regional school authorities
- Sponsors (minimally)

2. Organizational support

i) Schools
ii) Municipalities
iii) Individuals
iv) And all groups mentioned in 1 above

Types of organizational support:
i)Preparation of problems and their solutions

ii)Correction of solutions
iii)Organization of competitions at various levels
iv)Organization of correspondence seminars
v)Organization of Conferences for students and teachers implicated in competitions
vi)Web sites with problems of competitions and solutions:

1. http://www.collegiopiox.com/contents/home/gm_2004.htm (page of PRISTEM, University Bocconi Milano.
2. http://olimpiadi.ing.unipi.it/ (page of UMI for Olimpiadi di Matematica, there are also the on-line journals for students with problems and many solutions for Italian students)
3. http://dipmat.math.unipa.it/~grim/aicm/index.htm (page of AICM in the web site of GRIM, Palermo)

Games and Recreational Mathematics

1. **Project of the Math Olympiads**

All the news concerning the Olympiads of mathematics: forum, children's paper, news, history of the Olympiads. Italian language
http://olimpiadi.ing.unipi.it/

2. **Puzzles and mathematical games**

About ten puzzles and more known games, easily accessible and well illustrated.
http://digilander.iol.it/enigmiegiochi/index.htm

3. **"Base cinque"**

The amusing side of mathematics, edited by Gianfranco B. A rich collection of mathematical questions, of humorous wisecracks and of sympathetic nervous system test. Italian language.
http://digilander.iol.it/basecinque/index.htm

4. **Amusing mathematics**

Edited by prof Giovanni Pontani, teacher of mathematics and physics.
Among the index books: questions and games, historical notes, curiosity, things that make a mathematician to recognize a mathematician. Italian language
http://space.tin.it/clubnet/hlhmpo/
http://www.matematicadivertente.com/

5. **Math Magic**

A collection of mathematical games to which anyone can contribute in order to improve the given solutions or to propose new.
http://www.stetson.edu/~efriedma/mathmagic/archive.html

6. **The page of Tangram**

http://www.worldtel.it/varie/giochi/tangram/tangram.html

7. What the challenge has beginning

http://digilander.iol.it/atlantide75/giochi.htm
Other information for web sites in mathematics education, history, competitions, at address: http://math.unipa.it/~grim/SITI.htm

- **<u>Romania</u>**

The support to the talented students is provided by:

1. Government.
Financial support for organizing the main mathematical competition for talented students - National Mathematical Olympiad (local, county and final rounds). The Ministry of Education has a special yearly budget to support the participation of all qualified students to the final round.
The Ministry of Education also supports all expenses needed by the Romanian IMO team. All IMOs medallists are usually awarded significant prizes (in money, excursions abroad etc.) in the framework of a special ceremony organized by the Romanian presidency.

2. Institutions.
Many universities in Romania have their own special activities (mathematics contests, training programmes etc.) for talented high school students. Most of the universities accept (without passing the entrance exam) students who were awarded prizes at certain math competitions.
3. Societies.
The Romanian Mathematical Society was and still is deeply involved in supporting talented students in several ways. It is one of the main organizers of math competitions and has its own system of awarding talented students.
4. Individual support.
Parents themselves offer their support to talented students.
5. Publications / journals.
There are several journals, even newspapers that give support or contribute to the organization of math contests or make publicity regarding the best students. They also provide various awards.
6. Local authorities.
Local authorities support the organization of local and county rounds of maths competitions, offer support for attending the final round of NMO and also provide several kinds of awards (money, books, certificates of distinction, excursions etc.)
7. Through teacher training.

The teachers involved in training talented students are rewarded by their school authorities, by local authorities and, for IMO medallists, even by the Ministry of Education (from time to time, depending on the Minister’s view on training talented students).

Discussion and Observation

It is evident as we mentioned before that European countries have as a major goal to identify, motivate and support talented students in mathematics and use as many competitions as they can. They prepare their students for the competitions and create the best conditions they can to promote them. It is also evident that all those countries have been convinced that it is of major importance to keep the talented students in mathematics in Europe, thus they are willing to offer significant support to the talented students. Governments, Ministries of Education and Science Institutions, Universities, Foundations, Societies, individuals, local authorities, publications and journals, do offer their support but not so much financially. What is lacking is a systematic support system by the Educational Authorities, which could provide a sustainable development for talented students in mathematics.

Table 5: Identification, Motivation, Support activities

COUNTRY	IDENTIFICATION	MOTIVATION	SUPPORT
CYPRUS	1.City competition *Nicosia *Limassol *Larnaca and Ammohostos *Paphos 2.National Competitions *Gymnasium *10^{th} Grade *11th + 12th grade *Selection examinations 3.National Mathematical Olympiad 4^{th} grade to 12^{th} grade	Awards ceremony National Teams for 1.BMO 2.JBMO 3.IMO 4.International Contest for Primary School pupils	1.Preparation programme 2.Summer Math School 3.Publications
BULGARIA	1.TV, Internet and journal competitions 2.Schools mathematics competitions *city *inter city 3.National Competitions *Winter *Spring * ‘Atanas Radev” * ‘Sly Peter” * ‘Ivan Salabashev’ * ‘Akad Kiril Popov’ * ‘Peter Beron’ * ‘Chernorizec Hrabar’ *Language Schools’ *Christmas *Easter *Mathematics Tournaments (Sofia, Pazardjic,	1. National Teams for: i. Balkan Mathematical Olympiads – Junior ii. IMO – Junior and Senior levels 2. KANGAROO Contests – European 3. Tournament of the Towns 4. Studies in Europe, USA, Canada etc 5. Awards Ceremony - Society support environment, - organization of a training process in three stages: 1) force of the students' interest; 2) formatting high level of knowledge and skills; 3) developing of students' abilities. - organization of a training process which depends on the students' interest and abilities - personality of the training (leading) teachers; - possibility to be university students without entry exams;	1.Idividual work with promising students during the time of ordinary classes in mathematics 2.extra school mathematical activities *excursions *mathematical evenings *days of famous mathematicians & mathematical contests etc 3.School mathematical circles 4.City mathematical circles *Ordinary groups *Special groups

	Kardiali). 4.National Olympiad *School Round *City Round *Regional Round *National Round 5.Selection For IMO *National Competitions and Olympiads *International Competitions and Olympiads *Spring Conference of the Union of Bulgarian Mathematicians *One Month Summer School *Special Control Papers During the Summer School *Final Evaluation 6.International Competitions 7.Balkan Olympiad of Mathematics *Junior level *Senior level 8.Tournament of the Towns 9.Kangaroo Competitions *French Kangaroo *Australian Kangaroo 10.International Mathematical Olympiad *Junior level *Senior level	- prestige; - parents' support;	5.Short mathematical circles *Green *Summer *Winter *Sea 6.Extramural mathematical circles *problems *solutions *evaluation *commentaries 7.Correspondental Preparation *Problems *Literature 8.Lectures on special subjects of mathematics by the invited university professors and leading teachers 9.Spring Conference of the Union of Bulgarian Mathematicians 10.School Institute of Mathematics and

			Informatics with the aim: Preparation of Mathematical Essays BY students *School Round *City Round *Regional Round *National Round *Selection for the Center of Excellence in Education, Boston, USA
CZECH REPUBLIC	1st **stage**: from the 4th class of BS (9 years) experimentally from the 3rd class *Mathematical Olympiad *Kangaroo 2nd **stage**: from the 6th class of BS (11 years) *Mathematical Olympiad *Kangaroo *Other competitions (Dejte Ulavydohromody, Pythagoridda, Prazsled strela, Dopplerova vlua, …) *Many local competitions (at schools, in towns) 3rd **stage**: from the 1st class of SS (15 years) *Mathematical Olympiad		<u>The most current forms in practice</u> 1st **stage**: from the 4th class of BS (9 years) experimentally from the 3rd class *Special classes with extended teaching of foreign languages or mathematics after the special exam) *Correspondence seminars (competitions) 2nd **stage**: from the 6th class of BS (11 years) *Classes with extended teaching (especially) of foreign language or mathematics (or physics) (after the special exam) *Correspondence seminars *Holidays camps with teaching math and physics, foreign languages, sports, *Clil

	*Kangaroo 4th **stage**: from the 1st year of studies in university (19 years) <u>Mathematical Olympiad</u> Categories: *Z4, Z5, Z6, Z7, Z8, Z9 – for BS *C- for the 1st class of SS *B- for the 2nd class of SS *A- for the 3rd and 4th classes of SS *Together about 20-30 pupils *Several rounds <u>Kangaroo</u> Categories: *Klobaluek- the 4th and 5th classes of BS *Benjamin- the 6th and 7th classes of BS *Kadet- the 8th and 9th classes of BS *Junior- the 1st and 2nd classes of SS *Student- the 3rd and 4th classes of SS *Together about 300,000 pupils		3rd **stage**: from the 1st class of SS (15 years) *Special classes with extended teaching of foreign languages or mathematics or other branches (physics, chemistry, informatics, physical training, special sports, art, …) *Correspondence seminars *SOC in many branches as Olympiads *Preparing competitions for pupils of BS *Clil 4th **stage**: from the 1st year of studies in university (19 years) *Special faculties, especially faculty of mathematics and physics *SVOC in many branches (as OU SS) *Preparing competitions for pupils of BS & SS 5th **stage**: from PhD studies (usually 24 years) *PhD studies *Preparing competitions for pupils of BS & SS

GERMANY	Mathematical Competitions "Bundeswettsewers Mathematir" 1.Round: 4 homework1-March→1-June 2.Round: 4 homework 1-June→1-Sept. 3.Round: Colloquium *In 2003: 1146 participants, 90% usually in classes 9-13. Price I, II, III, A, No *Organized by the "Vertin Bildung and Begabung e.v." (Supported by Ministry for Education and Research) *Mathematics Olympiads (Univ. Rostoor) since 1994 *Arbeitsgemeisdaft des "bundeswite" Sdilerwettscwerse *Volrswapastiltang Since 2/03 14 projects in program Verbeslerng des Mathematir-unlimcsrles.	Price I,II,III,A,No	Support of talented students at universities *Studienstiftang des Deutschen Volres gives stipends to excellent students (all areas) *Further private foundations

GREECE	1)Hellenic Mathematical Society competitions: 1.Thales Time: End of October,Place: Locally Participation: Open to anybody willing to participate, Approximately 15000 students took part in 2002/3. Level: From 2nd Gymnasium (age 12) to final school year (age 17), Separate questions for each class. Syllabus: Whatever the students have learnt up to the previous class. 2.Euclid Time : Mid December Place: Athens Participation: By invitation according to grade in Thales, This grade is set by the Organisers, according to the results of participants, Approximately 1600 students took part in 2002/3. Level: As in Thales. Syllabus: Whatever the students have learnt up to the past three months of their current class. 3.Archimedes Time: beginning of February, Place: Athens Participation: By invitation according to success	1.Thales Prizes: More than 300 students *per class* are given a prize, organised locally 2.Euclid Prizes: About 50 or 60 students *per class.* 3.Archimedes Prizes: 25 Young 25 Olds ones About 25 Young 25 Old participate in further internal competitions and training. From them, BMO and IMO teams are selected.	3. Thales Information and encouragement through schools and Local Education Authority. 2.Euclid Participation: By invitation according to grade in Thales, 3.Archimedes Participation: By invitation according to success in Euclid, Students are largely self-

	in Euclid, 300 students took part in 2002/3. Level: Young (Gymnasium) and Old (Lyceum). Syllabus: As for IMO. 4) Other Competitions a)Informatics (http://www.epy.gr) Two competitions using the Internet and then final competition, Participation in BOI and IOI,		taught.

ITALY	1.Mathematical competitions to provincial level (Palermo age 11-13) 2.Mathematical competitions to national level University of Milan (Lettera Pristem: Palermo-Milan) and University in Pisa 3.The role of the processes of socialization of the knowledge in situations of teaching/learning 4.The construction of particular didactic situations that can allow this socialization 5.The possibility to be able to socialize 5.The possibility to be able to socialize procedures, schemes of reasoning, decisive strategies of situations/problem		1. The winners of the mathematical competitions for the 'Scuole Superiori' (16-18) as Tutor in the mathematical competitions have been used for the middle school (11-13). 2.Website with problems and solutions 3.You introduces the website 4.Publications
ROMANIA	Romania: a long tradition in selecting and training talented people *1885: First competition for primary school students *1897: First attempt to organize a <u>national</u> mathematics contest *1902: Annual Contest "Gazeta Matematica" (by mail) *1909: Contestants (selected among the best solvers) give a written and oral examination *1949: National Mathematics Olympiad is created (organized by RMS and Ministry of Education) *Mathematics Olympiad in Romania (levels: primary, secondary, high school) School Round City Round District (County) Round	Studies in US *1885: First competition for primary school students 70 participants; 11 prizes awarded (9 boys, 2 girls) - Awards ceremony at the Final Round of NMO: prizes awarded by the Ministry of Education, the County Authorities where NMO is organized, Romanian Mathematical Society, sponsors - Selection in the JBMO, BMO and IMO extended teams - At county level: prizes awarded by RMS, local sponsors - Medalists at JBMO, BMO, IMO are awarded special prizes (ceremony, Minister of Education, Prime Minister or even President of Romania) - Also awarded excursions abroad for the medalists.	*1883: First issue of Scientific Recreations: "mathematics; physics; chemistry etc." by problems, notes, articles (→1889) *1895 (Sept. 15): First issue of Gazeta Matematica the most respectable Romanian journal devoted to elementary mathematics; it is mainly responsible for the creating, improvement, and keeping up of a high interest in attracting talented students 12 issues/year; published continuously so far *1959: First edition of International Mathematics

	Final (National Round) 550-600 students (Forms 7-12) qualify each year for the Final Round The annual competition "Gazeta Matematica" organized each summer for the best solvers of the journal International Competitions -JBMO -BMO -IMO Team selection during the Final Round of National Mathematical Olympiad Special training stages Kangaroo Intercounty Mathematics Competitions (organized during the academic year, most of them between the County Round and Final Round of the National Mathematical Olympiad) The most important of them: "Gh. Titeica" "Gr. C. Moisil" "Tr. Lalescu" "L. Duican"		Olympiad (IMO): Bucharest 1960, 1978 (20^{th} ed.), 1999 (40^{th} ed.): Bucharest *1971-1973: Classes of excellence: Mathematics, Physics, Chemistry and Biology (Minister of Education: Mircea Malita) *2001: National Net of Centers of Excellence (main cities) *2002 (Oct. 1): Center of Excellence in Maramures country *The journal: LUCRARILE SEMINATULUI DE CREATIVITATE MATEMATICA (Seminar on Creative Mathematics) Articles, notes written by students, secondary school or high school teachers, university teachers: • There are articles dedicated to heuristic of solving competitions problems • Most of them are designed to develop inventive skills by problem solving and open the way to research work Center for Excellence in Mathematics (NUBM)

			Founded: October 1991: Seminar on Creative Mathematics 2000: “Center for Training Gifted Students”2001: Center for Excellence in Mathematics Aims and Scope: 1.To work with early university level students. Problem solvers → early research in mathematics 2.To identify and train talented high school students for mathematical competitions: *National Mathematics Olympiad *Inter county Mathematics Contests *American Math Competitions 3.To publish the latest materials * Journal “Seminarul de Creativitate Matematica” *problems books *books 4.To organize the Inter county Mathematics Competition“Gr. C. Moisoil”: problems proposals & coordination 5.To help Winter Mathematics Camps organizers: lecturers and training students (At 2), 4) and 5) in collaboration with RMS-

			Maramures Br.) 6.To train university students for Mathematics Competitions ("Tr. Lalescu" math.comp' International Maths Competition: IMC – 2001 (Prague); 2002 (Warsaw) At the national level (NMO): Maramures County: 3-5 place in 2002 (there are 41 counties in Romania) Usually: 15-20 place
UNITED KINGDOM	1.Junior Challenge Years 7 and 8, Ages 12 and 13 Format: a 60 minute paper with 25 multiple-choice questions Number Taking Part: Over 240,000 entrants from over 3,200 schools. 2. Junior Mathematical Olympiad The follow-on round to the Junior Olympiad is the Junior Mathematical Olympiad. This was held on Tuesday 10th June 2003, and it is normally held on the first or second Tuesday of the month. 1,204 of the best performing students in the 2003 Junior Challenge were invited to take part in the Junior Mathematical Olympiad.	1.Junior Challenge Years 7 and 8, Ages 12 and 13 Follow-on round: High scorers will be invited to participate in the Junior Olympiad (UK JMC). 2. Junior Mathematical Olympiad All those who take part receive a certificate: the top 25% receive Certificates of Distinction and the rest receive Certificates of Participation. Medals are awarded to very good candidates and the top 50 also receive a prize.	

	The Junior Mathematical Olympiad is a two hour paper which has two sections: Section A has ten questions and pupils are required to give the answer only. Section B has six questions for which full written answers are required. 3. Intermediate Challenge, Years 10, 11 and 12 Ages 14, 15 and 16 Format: a 60 minute paper with 25 multiple-choice questions Number Taking Part: Over 207,000 entrants from over 2,700 schools. 4. Intermediate Mathematical Olympiad and Kangaroo (IMOK) The follow-on competitions to Intermediate Challenge are the European Kangaroo and the IMOK Olympiad. European Kangaroo 1,000 pupils in each of Y9, Y10 and Y11 (E&W), S2, S3 and S4 (Scot.), and Y10, Y11 and Y12 (N.I.) are invited to sit the European Kangaroo. This is a one-hour multiple-choice paper with 25 questions taken by students across Europe and beyond. The lowest year group takes the Kangaroo 'Grey' paper and the top two year groups both take the Kangaroo 'Pink' paper. The	3.Intermediate Challenge, Years 10, 11 and 12 Ages 14, 15 and 16 Follow-on rounds: High scorers will be invited to participate in the follow on rounds: Intermediate Mathematical Olympiad and Kangaroo (IMOK)	

	2003 European Kangaroo took place on Thursday 20th March. All those taking part receive a Certificate of Participation or a Certificates of Merit. 5.Senior Challenge, Years 13 and 14, Ages 17 and 18 Format: a 90 minute paper with 25 multiple-choice questions Number Taking Part: Over 60,000 entrants from over 1,500 schools. Follow-on rounds: 6. British Mathematical (Round 1) Up to 1,000 high scorers will be invited to participate in the: British Mathematical (Round 1). This can lead to BMO2. Six participants will make it to the International Mathematical Olympiad. 7.BMO2 8.International Mathematical Olympiad. Six participants	4.European Kangaroo All those taking part receive a Certificate of Participation or a Certificate of Merit. 5.Senior Challenge, Years 13 and 14, Ages 17 and 18 Follow-on rounds: Up to 1,000 high scorers will be invited to participate in the: British Mathematical (Round 1).	

		6. British Mathematical (Round 1) This can lead to BMO2. 7.BMO2 Six participants will make it to the International Mathematical Olympiad.	

CURRICULUM In European Schools
Part I:

Summary

In this part of the comparative study we will try to give a clear picture of what is happening in eight European countries (Bulgaria, Cyprus, Czech Republic, Greece, Germany, Italy, Romania and the United Kingdom) in order to realize whether the concepts that we decided to include in the project are being taught and at what age exactly the pupils are taught those notions. It is understood that it is not possible that all the countries will teach all those concepts at the same grade and to the same extent.

Introduction

The project participants decided how the concept elements should be separated: they decided to have two levels, the first one concerns the ages 9-14, Level 1 (Primary) and Secondary, Level 2 concerns the ages 15-18. We present the elements of the first level in ten categories. Then follows a comparison study that concerns the teaching of those categories in both levels and in eight European countries (Bulgaria, Cyprus, Czech Republic, Greece, Germany, Italy, Romania and the United Kingdom).

Elements of Level 1 (Primary), Ages 9-14:

Combinatorics (Pigeon hole principle (Dirichlet's Principle), Counting Finite Sets, Inclusion and Exclusion Principle),

Number Theory (Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm), Prime numbers (including decomposition of numbers), Properties of Numbers, Base representations of numbers),

Euclidean Plane Geometry (Dirichlet's, Principle in Geometry, Combinatorial Geometry, Cuttings and Coverings, Areas of Figures, Geometry of the Triangle, Geometry of the Circle),

Inequalities (Algebraic inequalities, Geometrical inequalities),

Polynomials (Factorization of polynomials, Linear and quadratic equations),

Simple Mathematical Modeling (instead of "Word Problems"), Word Problems, Processes Problems, Story Problems,

Functions (Dependences and Correspondences, Linear Functions),

Discrete Mathematics (Elementary Probability, Elements of Graph theory, Sequences),

Invariants (discovering invariants (beginning with divisibility), Game strategies based on invariants),

Transformations (Translations, Reflections, Rotations, Inversions (all defined geometrically), Properties of Figures, Composition of transformations (of the same type)) (Table 1).

Thus after a close examination of the curricula of each of the eight countries we are in a position to make a comparative study that shows which topic concept at what age and in which country is being taught. We also give a table (Table 1) that shows which topic concept at what age and in which country it is being taught as far as the first level is concerned.

Results

Combinatorics
As far as the compilation category is concerned which is Combinatorics, the Pigeon hole principle (Dirichlet's Principle) is not taught in any of the eight participant countries. The concept of Counting Finite Sets is being taught in Cyprus to 12-13 year-old students and in Romania to 10-14 year old students. This notion is not included in the Bulgarian, Czech, German, Greek, Italian and the United Kingdom school curricula. The Inclusion and Exclusion Principle is included only in Romania's curriculum and it concerns the 13-14 year-old students. In conclusion we can mention that only Cyprus and Romania deal with some elements of the Combinatorics category.

Number Theory
Number theory is a well known topic in all countries. The only difference is the age at which the students become familiar with number theory as can be seen in table 1. The students start to meet some of the concepts at the age of 9. Particularly, in Cyprus the students are taught the concept of Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) during the ages of 9 to 13; the Prime numbers (including decomposition of numbers) during the ages of 9 to 14 and Properties of Numbers during the ages of 11 to 13. Finally they are taught Base representations of numbers during the age of 12 to 13.

In Bulgaria Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) is taught to 10 to 11 year-old students and Base representations of numbers is taught to 11 to 12 year old students. 9-14 year old Bulgarian students are not taught Prime numbers (including decomposition of numbers), and Properties of Numbers.

In the Czech Republic, 11-12 year old students become familiar with Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) and Prime numbers (including decomposition of numbers). 10-11 year old Czech Republic students are taught Properties of Numbers. The Base representations of numbers are not taught in the Czech Republic as far as the 9-14 year-old students are concerned.

In Germany the concept of Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) is taught to 9-13 year-old students, the concepts of Prime numbers (including decomposition of numbers). Properties of Numbers and Base representations of numbers are taught to 11 year-old students.

In Greece the concept of Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) is taught to 9-13 year-old students. Prime numbers (including decomposition of numbers) are taught to 11-12 year-old students. Properties of Numbers are taught to 10-11 year old students. Base representations of numbers are not taught at all to 9-14 year old students.
In Italy 11-14 year old students are taught all the elements of the Number Theory category.

In Romania the concept of Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm) is taught to 9-13 year-old students, Prime numbers (including decomposition of numbers) is taught to 11-12 and 13-14 year-old students, Properties of Numbers is taught to 10-13 year-old students; Base representations of numbers is taught to 9-13 year-old students.
Euclidean plane geometry
As far as the third category is concerned, Euclidean plane geometry, the Dirichlet's, the Principle in Geometry, Combinatorial Geometry and Cuttings and Coverings, are not covered in the school curriculum of any of the eight participant countries.

The areas of figures, the geometry of the triangle and the circle are very well known to students of all the eight countries. The students are taught those notions, starting from the age of 9 until the age of 14. In each of the corresponding grades the notions have a different degree of extent and difficulty. Particularly, elements of Areas of Figures, Geometry of the Triangle, and Geometry of the Circle in Cyprus are taught to 9-13 year-old students.

In Bulgaria the areas of figures are taught to 11-14 year-old students and the geometry of the triangle is taught to 12-14 year-old students. The geometry of the circle does not appear in the Bulgarian curriculum for the 9-14 age groups.

In the Czech Republic the areas of figures are taught to 9-13 year-old students, geometry of the triangle is taught to 9-12 year-old students and the geometry of the circle is taught to 9-14 year-old students.

In Germany the geometry of the triangle is taught to 11-14 year-old students and the geometry of the circle is taught to 11-13 year-old students. Notions connected with areas of figures do not appear in the Bulgarian curriculum regarding the age of 9-14.

In Greece the areas of figures are taught to 9-14 year-old students, geometry of the triangle is taught to 10-13 year-old students and the geometry of the circle is taught to 9-14 year-old students.

In Italy 11-14 year-old students are taught elements of the areas of figures and geometry of the triangle and the circle.
The areas of figures are taught to 9-13 year-old Romanian students. Moreover geometry of the triangle and the circle are taught to 11-13 year-old students in Romania. In the United Kingdom 11-14 year-old students are taught elements of the areas of figures and geometry of the triangle and the circle.

Iinequalities
The category of inequalities, algebraic, is taught to students by teachers of mathematics in all the countries except the Czech Republic. Especially, the teaching of algebraic inequalities in Cyprus concerns the 9-14 years students, in Bulgaria the 12-14 year-old students, in Germany the 11-13 year-old students, in Greece the
13-14 year-old students, in Italy the 11-14 year-old students in Romania the 10-14 year-old students and finally in the United Kingdom the 11-14 year-old students. The geometric inequalities do not appear in Cyprus, Czech, Germany, Greece, Romania and the United Kingdom. The Bulgarian and Italian curricula include these notions in their curricula, in grade 7 for Bulgaria and grades 6 to 8 for Italy.

Polynomials
The fifth category of level 1, Polynomials, does not appear at all in the Bulgarian and Italian curricula. The linear and quadratic equations are taught mostly in the 8th grade in the rest of the countries. Moreover the students from Romania become familiar with the procedure of factorization of Polynomial.

Simple mathematical modeling
Students in Cyprus, Bulgaria, Germany, Greece, Romania and the United Kingdom are asked to get familiar with the category that concerns simple mathematical modeling. The teaching of Simple mathematical modeling in Cyprus concerns the 9-14 year students, in

Bulgaria the 10-14 year-old students, in Germany the 10-11 year-old students, in Greece the 9-14 year-old students, in Romania the 10-13 year-old students and finally in the United Kingdom the 9-14 year-old students. Italy and the Czech Republic do not include in their school curriculum regarding the 9-14 year-old students, the category of Simple mathematical modeling. It is not also included in the curriculum for 9-14 year-old students in Italy and the Czech Republic.

Functions

The following category (table 1) named 'functions' appears in the school curricula in Bulgaria, Germany, Greece, Romania and the United Kingdom. In the United Kingdom the students meet functions from the beginning of the 4th grade and continue to the 8th grade, in Bulgaria both in 5th to 8th grade, and the students from the rest of the above countries in the 8th grade. Only in Germany are the students in the 8th grade taught linear functions.

Cyprus, the Czech Republic and Italy do not include the category of functions in their school curriculum aimed at 9-14 year-old students.

Discrete Mathematics

The notions of elementary probability and Graph theory are common to all the participant countries, even in grade 4 as in Cyprus, Germany, Romania and the United Kingdom, and other grades for the rest of the countries. Especially the teaching of elementary Probability in Cyprus concerns the 9-12 year-old students, in Bulgaria the 10-13 year-old students, in the Czech Republic the 11-12 year-old students, in Italy the 11-14 year-old students, in Romania the 9-13 year-old students and finally in the United Kingdom the 9-14 year-old students. The teaching of Elements of Graph theory in Germany concerns the 9 year-old students, in Greece the 11-12 year-old students, in Italy the 11-14 year-old students and in Romania the 11-14 year-old students. The sequences do not appear in any country.

Invariants

The category of Invariants only appears in the Italian curriculum and it concerns the 11-14 year-old students.

Transformations

As far as the last category is concerned, (Transformations), the only country that does not teach it at all is Bulgaria. The teaching of Translations, Reflections, Rotations, Inversions (all defined geometrically) is taught in Cyprus in the 5th grade, in the Czech Republic to 9-13 year-old students, in Germany to 9-14 year-old students, in Italy to 11-14 year-old students, in Romania to 9-14 year-old students and finally in the United Kingdom to 9-14 year-old students. The elements of this category named Properties of Figures and Compositions of transformations (of the same type) do not appear in any country.

We now present the elements of the second level (Level 2) that counted nineteen categories.

Elements of Level 2 (Secondary), Ages 15-18:

Combinatorics (Pigeonhole principle (Algebra, Geometry), Counting, Logic combinatorial problems, Probability, Inclusion, Exclusion principle,
Number Theory Properties of integers (divisibility), Prime numbers, Irrationality, Diophantine equations, Congruencies, Applications (e.g. in cryptography),
Geometry in the plane Triangles, Polygons, Circles, Optimization problems, Loci, Construction problems, Transformations (Similarity, Inversion), Special Theorems (Euler, Ceva, Simpson, etc.), Metric properties (area, etc.), Geometric inequalities,
Solid Geometry (Properties of lines and planes, Solids, convex bodies, Metric properties, External problems, Proofs with an exit to space, Geometric inequalities in the space),
Inequalities (Algebraic, Trigonometric)
Polynomials (Properties, Euclidean division, roots, factorization, Solutions, Symmetric functions of roots),
Mathematical modeling, word problems, logic,
Functions (Graphs, Equations, properties and relations of functions, Special unctions (trigonometric, exponential, etc.)),
Complex Numbers (Properties, Applications to geometry),
Discrete Mathematics (Recurrence, Algorithms, Propositional logic, Graph theory),
Mathematical induction (Variants of induction methods, Applications in algebra, geometry, combinatorics, etc.),
Trigonometry (Properties, Applications in geometry),
Sequences (Properties, limits, Series (Finite, infinite, convergence))
Statistics (Properties, basic concepts, Applications),
Invariants (Applications),
Linear Algebra (Vectors, Determinants, Matrices),
Analytical Geometry (Lines, Conic sections, Applications),
Transformation Methods (Philosophy of the transformation theory, Applications),
Mathematical Games
Thus after a close examination of each of the eight countries' curricula we are in a position to make a comparative study that shows which topic concept, at what age and in which country is being taught and to give also a table (Table 2) that shows which topic concept, at what age and in which country is being taught as far as the second level is concerned.

Results
The first examined category is Combinatorics
As far as the compilation category is concerned which is Compinatorics, the Pigeonhole principle only applies to the Czech Republic where Logic combinatorial problems are also taught.
The concepts of Counting, Logic combinatorial problems, Probability are taught in Cyprus in the 12th grade, and in Bulgaria in the 10th grade. From this category (table 2) only the Probability concept is taught in Germany and in Greece. Mainly the teaching of Probability in Germany concerns the 19 year-old students, in Greece the 14-15 and 17-18 year-old students.
Italy does not include Combinatorics in the school programme. In Romania Logic combinatorial problems and Probability exist in the school programme and concerns the students of the 10th grade. Finally, as far as Combinatorics is concerned the concepts of Counting and

Probability appear in mathematics curriculum of the United Kingdom. The concepts of Counting are taught to 17-18 year-old students and Probability to 14-18 year-old students.

The second examined category is Number Theory

Elements of Number Theory are taught in all the participant countries except Italy. Italy does not include the Numbers Theory in its programme. The numbers Theory that refers to Properties of integers (divisibility), Prime numbers, Irrationality, Diophantine equations appear in the 11th grade in Cyprus. The concepts of Congruencies, Applications (e.g. in cryptography) are not taught in Cyprus.

The concepts of Irrationality and Congruencies, Applications (e.g. in cryptography) are the only concepts of the Number Theory category which are taught to 14-15 year-old Greek students.

In the Czech Republic Properties of integers (divisibility) and Prime numbers are taught to students of the 10th grade.

In Germany Irrationality is taught to the students of the 10th grade.

Properties of integers (divisibility), Prime numbers, Irrationality appear in the 11th grade in Greece. In Romania Irrationality is taught to 14-15 year-old Students.

Properties of integers (divisibility), Prime numbers, Irrationality appear in the United Kingdom curriculum of 9th to 12th grade.

The third examined category is Geometry in the plane

Geometry in the plane seems to be very popular in European countries. All the concepts which belong to this category start to be taught in the 9th grade and go through to the 12th grade in almost every country.

Particularly, in Cyprus the students are taught the Triangle (remarkable points of the triangle) during the ages of 14 to 16, the Polygons during the age of 14 to 17, the Circles during the ages of 15 to 17, the Loci during the ages of 15 to 18, Construction problems during the ages of 14 to 18, Transformations (Similarity, Inversion) during the ages of 15 to 17, Special Theorems (Euler, Ceva, Simpson, etc.) during the ages of 14 to 15, Metric properties (area, etc.) during the ages of 17 to 18, Geometric inequalities during the age of 14 to 16. In Cyprus the students of this level are not taught Optimization problems.

In Bulgaria students are taught the Triangle (remarkable points of the triangle) during the ages of 14 to 16, the Polygons during the age of 14 to 16, Construction problems during the ages of 14 to 16, Transformations (Similarity, Inversion) during the ages of 14 to 15, Special Theorems (Euler, Ceva, Simpson, etc.) during the ages of 15 to 16, and Geometric inequalities during the ages of 15 to 16. In Bulgaria the students of this level are not taught Circles, Optimization problems, Loci and Metric properties (area, etc).

In the Czech Republic the students are taught the Polygons during the ages of 15 to 16, the Circles during the ages of 15 to 16, the Loci during the ages of 15 to 16, Construction problems during the ages of 15 to 16, Transformations (Similarity, Inversion) during the ages of 14 to 16, Special Theorems (Euler, Ceva, Simpson, etc.) during the ages of 15 to 16, Metric properties (area, etc.) during the ages of 17 to 18.

In the Czech Republic the students are not taught the Triangle (remarkable points of the triangle), Optimization problems and Geometric Inequalities.

In Germany the students are only taught the Loci at the age of 15 and 19, Transformations (Similarity, Inversion) at the age of 15, Special Theorems (Euler, Ceva, Simpson, etc.) and Geometric Inequalities from the elements of the third category at the age of 15.

In Greece the students are taught the Triangle (remarkable points of the triangle) in the 10th grade, the Polygons and the Loci during the10th and the 11th grades, Construction problems in the 11th grade, Transformations (Similarity, Inversion) during the 9th and 10th grades. Metric properties (area, etc.) are taught to Greek students during the ages of 15 to 17, Geometric inequalities during the ages of 14 to 15. In Greece the students of level 2 are not taught Circles, Optimization problems and Special Theorems (Euler, Ceva, Simpson, etc.)

In Italy students are taught the Triangle (remarkable points of the triangle) and the Polygons between the ages of 15 to 16, the Circles between the ages of 14 to 17, Transformations (Similarity, Inversion) during the ages of 15 to 17, Special Theorems (Euler, Ceva, Simpson, etc.) during the ages of 14 to 18, Metric properties (area, etc.) during the ages of 15 to 16, Geometric Inequalities during the age of 14 to 15. In Italy the students of level 2 are not taught Circles, Optimization problems, Loci and Construction problems during the ages of 14 to 18.
In Romania students are taught the Loci and Geometric Inequalities in the 11th grade and Special Theorems (Euler, Ceva, Simpson, etc.) in the 12th grade.
Finally, Polygons, Circles and Loci are the only elements of this category, named Geometry in the Plane, that are taught to English students from the 9th to 12th grades.

The fourth category is Solid Geometry
As far as Solid Geometry is concerned the concepts of Proofs with an exit to space, and Geometric inequalities in the space are not covered in the school curricula of the eight participant countries.
Properties of lines and planes and Solids convex bodies are taught in Cyprus in the 9th and 10th grades. In Cyprus the students of level 2 are not taught Metric properties, and Optimization problems.
In Bulgaria Properties of lines and planes, Solids convex bodies and Optimization Problems are taught in the 12th grade. In Bulgaria the students of level 2 are not taught with Metric properties.
In the Czech Republic Properties of lines and planes are taught to students in grades 11 and 12, Solids convex bodies and Optimization Problems are taught in grades 9 to 12. In the Czech Republic the students of level 2 are not taught Metric properties and Optimization Problems.
German students of the 10th grade have to study only Solids convex bodies and 9th grade students of Greece have to study Solids convex bodies too.
In Italy 10 to 17 year-old students have to understand the concepts of Properties of lines and planes and Solids convex bodies and the Metric problems by the time they reach the age of 16. The students of Italy are not taught Optimization Problems.
In Romania Properties of lines and planes are taught to students in grade 11 and Solids convex bodies are taught to students in grade 10. The Romanian students are not taught Metric and Optimization Problems.
Solids convex bodies are also compulsory for the United Kingdom students during 9th to 12th grades. It is the only element of the forth category that is taught to English students regarding level 2.

The fifth category is Inequalities
The concepts of Algebraic and Trigonometric inequalities do not appear in Germany and Italy for level 2.
Trigonometric inequalities are taught in the Czech Republic in the11th grade and in Romania in the 10th grade. The students of the rest of the countries are only taught Algebraic Inequalities starting from the ages of 14. Especially in Cyprus the students are taught Inequalities during the ages of 15 to 18, in Bulgaria and the Czech Republic during the ages of 15 to 16, in Greece during the ages of 14 to 16, in Romania during the ages of 15 to 17 and in the United Kingdom during the ages of 14 to 18.

The sixth category is Polynomials

Polynomials are a compulsory category for students after the 9^{th} grade in all eight participant countries (Table 2). In particular, in Cyprus students are taught the Properties, Euclidean division, roots, factorization of polynomials during the ages of 14 to 18 and the Solutions of polynomials during the ages of 14 to 15. In Cyprus students are not taught Symmetric functions of roots of polynomials.
Bulgarian students are taught Properties, Euclidean division, roots, factorization of polynomials in the 9^{th} grade. The Bulgarian students are not taught Solutions of polynomials and Symmetric functions of roots of polynomials.
In the Czech Republic the students are taught the Properties, Euclidean division, roots, factorization of polynomials during the ages of 15 to 17 and Solutions of polynomials during the ages of 15 to 16. Symmetric functions of roots of polynomials are excluded from the curricula of this country.
In Greece the students are taught the Properties, Euclidean division, roots, factorization of polynomials during the ages of 16 to 17, Solutions of polynomials during the ages of 14 to 15, Symmetric functions of roots of polynomials during the ages of 16 to 17.
In Italy Properties, Euclidean division, roots, factorization of polynomials are taught in the 9^{th} grade.
In the United Kingdom Properties, Euclidean division, roots, factorization of polynomials, Solutions of polynomials, Symmetric functions of roots of polynomials are taught during the 9^{th} to 12^{th} grades.

The seventh category is Mathematical modeling

The seventh topic category of level 2, named Mathematical modeling, word problems, logic, appear only in the Bulgarian and the United Kingdom curricula, the category is taught during the 9^{th} to 12^{th} grades.

The eighth category is Functions

Functions are also a very popular topic in the mathematics curricula and it is taught in every country (Table 2). Graphs, Equations, properties and relations of functions, Special functions (trigonometric, exponential, etc.) are some of the notions of this specific category that are greatly valued by everybody in all participant countries.
Particularly, in Cyprus students are taught Graphs during the ages of 14 to 18, the properties and relations of functions during the ages of 15 to 18 and Special functions (trigonometric, exponential, etc.) during the ages of 16 to 18.
In Bulgaria students are taught Graphs during the ages of 15 to 16, the equations, properties and relations of functions during the ages of 14 to 18 and Special functions (trigonometric, exponential, etc.) during the ages of 14 to 18.
In the Czech Republic students are taught Graphs during the ages of 14 to 17, the equations, properties and relations of functions during the ages of 16 to 17 and Special functions (trigonometric, exponential, etc.) during the ages of 14 to 17.
In Germany students are taught equations, properties and relations of functions and Special functions (trigonometric, exponential, etc.) at the age of 16. In Germany students are not taught Graphs at the level 2.
In Greece students are taught Graphs during the ages of 14 to 18, equations, properties and relations of functions during the ages of 15 to 16 and Special functions (trigonometric, exponential, etc.) during the ages of 14 to 17.
In Italy students are also taught special functions (trigonometric, exponential, etc.) in the 11^{th} grade. Italian students are not taught equations, properties and relations of functions.
In Romania students are taught Graphs and equations, properties and relations of functions during the ages of 14 to 17 and Special functions (trigonometric, exponential, etc.) during the ages of 15 to 17.
In the United Kingdom students are taught Graphs and equations, properties and relations of functions and Special functions (trigonometric, exponential, etc.) during the ages of 14 to 18.

The ninth category is Complex Numbers
The ninth category concerning Complex Numbers is not taught in Germany and Italy. In Cyprus students are taught Properties and application of Complex Numbers during the ages of 16 to 18. In Bulgaria students are taught only the Properties of Complex Numbers in the 12th grade. In the Czech Republic the Properties of the Complex Numbers is taught in the 12th grade. In Greece students are taught Properties and application of Complex Numbers during the ages of 17 to 18. In Romania and the United Kingdom Complex Numbers Properties and their Applications to geometry are taught in the 10th and 12th grades respectively.
The tenth category is Discrete Mathematics
The topic category named Discrete Mathematics is taught exclusively in the United Kingdom in the last grade.

The eleventh category is Mathematical induction
Variants of induction methods are included in the Cyprus, the Czech Republic, Greek, Romanian and British math curricula. Particularly, in Cyprus and the Czech Republic students are taught Variants of induction methods during the ages of 16 to 18, in Greece during the ages of 16 to 17, in Romania during the ages of 14 to 15 and the United Kingdom during the ages of 14 to 18.

The twelfth category is Trigonometry
The topic of Trigonometry (Properties, Applications in geometry) seems be of high importance in all the countries except Bulgaria.
Mainly, in Cyprus students are taught Trigonometric properties during the ages of 14 to 18 and applications to geometry during the ages of 17 to 18. In the Czech Republic students are only taught Trigonometric properties during the ages of 16 to 17. In Germany students are taught Trigonometric properties and applications to geometry at the age of 16. In Greece students are taught Trigonometric properties during the ages of 14 to 17, and applications to geometry during the ages of 16 to 17. In Italy students are taught Trigonometric properties during the ages of 17 to 18, and applications to geometry during the ages of 15 to 18. In Romania students are taught Trigonometric properties and applications to geometry during the ages of 16 to 17. In the United Kingdom students are taught Trigonometric properties during the ages of 14 to 18.

The thirteenth category is Sequences
The next topic is of importance in a lot of the curricula. It is the topic of Sequences that is taught in the majority of the countries in the 10 or 11th grades. In particular in Cyprus and Greece students are taught properties and limits of Sequences during the ages of 16 to 18. In Bulgaria, Italy and Romania students are taught properties and limits of Sequences in the 11th grade. In the Czech Republic students are taught properties and limits of Sequences during the ages of 17 to 19. In Germany students learn about properties and limits of Sequences at the age of 17. In the United Kingdom students are taught properties and limits of Sequences during the ages of 17 to 18
Furthermore the concept of Sequences is taught only in Cyprus and the United Kingdom in the last grade.

The fourteenth category is Statistics
The basic concepts, Properties, basic concepts, Applications of Statistics are taught in the last grades in Cyprus, Germany and Greece. In Bulgaria students are taught Properties, basic concepts and applications in Statistics during the ages of 16 to 18, in Romania during the ages of 15 to 16 and in the United Kingdom during the ages of 14 to 18.

The fifteenth category is Invariants
Invariants and its Applications are taught only in Germany to the fifteen year-old students.

The sixteenth category is Linear Algebra

The country that does no include vectors in its education system is Italy. On the whole in Cyprus and Greece students are taught vectors during the ages of 16 to 17, in Bulgaria during the ages of 17 to 18, in the Czech Republic during the ages of 16 to 18, in Germany at the age of 15, in Greece during the ages of 14 to 17, in Romania during the ages of 15 to 18 and finally, in the United Kingdom during the ages of 14 to 18.

Determinals are taught in Germany and to 19 year-old students and in the United Kingdom to students of the 12th grade.

Matrices are also taught in Germany to 19 year-old students, in the United Kingdom in the last grade and in Romania in the 11th grade.

The seventeenth category is Analytical Geometry

The seventeenth topic of Analytical Geometry is taught in three countries in the 12th grade. These countries are the Czech Republic, Greece and the United Kingdom.

The eighteenth category is Transformation Methods

The next topic concerns Transformation Methods. Only two countries deal with the Philosophy of the transformation theory and its applications; these are Romania where Transformation Methods are taught in the10th grade and the United Kingdom where Transformation Methods are taught in grades 9 to12.

The nineteenth category is Mathematical Games

Finally, the last topic in the list of level 2 is the Mathematical Games which is taught exclusively by the English mathematics teachers in the last grade, grade 12th, (Table 2).

CURRICULUM In European Schools
Part II:

Introduction

In this part of the comparative study we will give two tables (Table 3, Table 4) with the curricula in eight European countries (Bulgaria, Cyprus, the Czech Republic, Greece, Germany, Italy, Romania and the United Kingdom) in order to have a clear picture of what is being taught in these countries and to what extent.

The national curricula of each country set the legal requirements of the teaching and learning of mathematics, and provide information to help teachers implement mathematics in their school. The national curriculum lies at the heart of our policies to raise standards. It sets out a clear, full and statutory entitlement to learning for all pupils. It determines the context of what will be taught, and sets attainment targets for learning. It also determines how performance will be assessed and reported. An effective national curriculum therefore gives teachers, pupils, parents, employers and the wider community a clear and shared understanding of the skills and knowledge that young people will gain at school. It allows schools to meet the individual learning needs of pupils and to develop a distinctive character and ethos rooted in their local communities. It provides a framework within which all partners in education can support young people on the road of further learning. Getting the National Curriculum right presents difficult choices and balances.

It must be robust enough to define and defend the core of knowledge and cultural experience which is the entitlement of every pupil and at the same time flexible enough to give teachers the scope to build their teaching around it in ways which will enhance its delivery to their pupils. The focus of this National Curriculum, together with the wider school curriculum, is therefore to ensure that pupils develop from an early age the essential literacy and numeric skills they need to learn; to provide them with a guaranteed, full and rounded entitlement to learning; to foster their creativity; and to give teachers discretion to find the best ways to inspire in their pupils a joy and commitment to learning that will last a lifetime.

An entitlement to learning must be an entitlement for all pupils. This National Curriculum includes for the first time a detailed, overarching statement on inclusion which makes clear the principles schools must follow in their teaching right across the curriculum, to ensure they have the chance to succeed, whatever their individual needs and the potential barriers to their learning may be.

Each of the European countries has developed its own curriculum based more or less on the principles of the National Curriculum. Below we try to give as many details as we can about the mathematics topics as they are given by the participating countries in the project. The teaching material in grades 4 to 12 is presented in tables 3 and 4.

Table 1: Level 1

AGES 9-14	CYPRUS	BULGARIA	CZECH REPUBLIC	GERMANY	GREECE	ITALY	ROMANIA	UNITED KINGDOM
• **COMBINATORICS**								
Pigeon hole principle (Dirichlet's Principle)								
Counting Finite Sets	12-13						10-14	
Inclusion and Exclusion Principle							13-14	
• **NUMBER THEORY**								
*Divisibility of numbers (criteria, Euclidean division, Euclidean Algorithm)	9-13	10-11	11-12	9-13	9-13	11-14	9-13	
*Prime numbers (including	10-13		11-12	11	11-12	11-14	11-12	9-14
decomposition of numbers)	9-14					11-14	13-14	
*Properties of Numbers	11-13	11-12	10-11	11	10-11	11-14	10-13	9-14
*Base representations of numbers	12-13			11		11-14	9--13	9-14
• **EUCLIDEAN PLANE GEOMETRY**								
*Dirichlet's Principle in Geometry								
*Combinatorial Geometry								
*Cuttings and Coverings								
*Areas of Figures	9-13	11-14	9-13		9-14	11-14	9-13	9-14
*Geometry of the Triangle	9-13	12-14	9-12	11-14	10-13	11-14	11-13	9-14
*Geometry of the Circle	9-13		9-14	11-13	9-14	11-14	11-13	9-14
• **INEQUALITIES**								
*Algebraic inequalities	9-14	12-14		11-13	13-14	11-14	10-14	9-14
*Geometrical inequalities		12-13				11-14		
• **POLYNOMIALS**								
*Factorization of polynomials							13-14	
*Linear and quadratic							10-14	9-14
equations	13-14		13-14	14	13-14			
• **SIMPLE MATHEMATICAL MODELLING** (instead of "Word Problems")	9-14	10-14		10-11	9-14		10-13	9-14
*Word Problems								
*Processes Problems								
*Story Problems								
• **FUNCTIONS**		11-14		14	13-14		13-14	9-14
*Dependences and								

Correspondences *Linear Functions				14				
• **DISCRETE MATHEMATICS** *Elementary Probability *Elements of Graph theory *Sequences	[9-12]	10-13	11-12	 9	 11-12	11-14 11-14	9-13 11-14	9-14
• **INVARIANTS** *Discovering invariants (beginning with divisibility) *Game strategies based on invariants						11-14		
• **TRANSFORMATIONS** *Translations, Reflections, Rotations, Inversions (all defined geometrically) *Properties of Figures *Composition of transformations (of the same type)	10-11		9-13	9-14	9-14	11-14	10-14	9-14

Table 2: Level 2

AGES 15+	CYPRUS	BULGARIA	CZECH REPUBLIC	GERMANY	GREECE	ITALY	ROMANIA	UK
<u>Combinatorics</u>								
• Pigeonhole principle			18-19					
➢ Algebra								
➢ Geometry								
• Counting	17-18	15-16						17-18
• Logic combinatorial problems	17-18	15-16	18-19				15-16	
• Probability	17-18	15-16		19	14-15, 17-18		15-16	14-18
• Inclusion, Exclusion principle								
<u>Number Theory</u>								
• Properties of integers (divisibility)	16-17		15-16		16-17			14-18
• Prime numbers	15-17		15-16		16-17			14-18
• Irrationality	16-17	14-15		15-16	16-17		14-15	14-18
• Diophantine equations	16-17							
• Congruencies								
• Applications (e.g. in cryptography)		14-15						
<u>Geometry in the plane</u>								
• Triangle (remarkable points of the triangle)	14-16	14-16			15-16	14-16		
• Polygons	14-17	14-16	15-16		15-17	14-16		14-18
• Circles	15-17		15-16			14-17		14-18
• Optimization problems								
• Loci	15-18		15-16	15,19	15-17		16-17	14-18
• Construction problems	14-18	14-16	15-16		16-17			
• Transformations (Similarity, Inversion)	15-17	14-15	14-16	15	14-16 15-16	15-16 15-16		
• Special Theorems (Euler, Ceva, Simpson, etc.)	14-15 17-18	15-16	15-16 15-16 17-18	15		14-18	17-18	
• Metric properties (area, etc.)	15-16	15-16		15	15-17	15-16	16-17	
• Geometric inequalities	14-15				14-15	14-15		

AGES 15+	CYPRUS	BULGARIA	CZECH REPUBLIC	GERMANY	GREECE	ITALY	ROMANIA	UK
Solid Geometry								
• Properties of lines and planes	14-16	17-18	16-18			16-17	16-17	
• Solids, convex bodies	14-16	17-18	14-18	15,16	14-15	16-17	15-16	14-18
• Metric properties						16-18		
• Optimization problems		17-18						
• Proofs with an exit to space								
• Geometric inequalities in the space								
Inequalities								
• Algebraic	15-18	15-16	15-16		14-16		15-17	14-18
• Trigonometric			16-17				15-16	
Polynomials								
• Properties, Euclidean division, roots, factorization	14-18	14-15	15-17	17	16-17	14-15	5-16	14-18
• Solutions	14-15		15-16		14-15			14-18
• Symmetric functions of roots				16	16-17		15-16	14-18
Mathematical modeling, word problems, logic		14-17						14-18
Functions			14-19					
• Graphs	14-18	15-16	14-17		14-18	17-18	14-17	14-18
• Equations, properties and relations of functions	15-18	14-18	16-17	16	15-16		14-17	14-18
• Special functions (trigonometric, exponential, etc.)	16-18	14-18	14-17	16	14-17	17-18	15-17	14-18
Complex Numbers								
• Properties	16-18	17-18	17-18		17-18		15-16	17-18
• Applications to geometry	16-18				17-18		15-16	17-18
Discrete Mathematics								
• Recurrence								17-18
• Algorithms								
• Propositional logic								17-18
• Graph theory								

AGES 15+	CYPRUS	BULGARIA	CZECH REPUBLIC	GERMANY	GREECE	ITALY	ROMANIA	UK
Mathematical induction • Variants of induction methods • Applic. in algebra, geometry, combinatorics, etc.	16-18		16-18		16-17		14-15	14-18
Trigonometry • Properties • Applications in geometry	14-18 17-18		16-17	16 16	14-17 16-17	17-18 15-18	16-17 16-17	14-18
Sequences • Properties, limits Series (Finite, infinite, convergence)	16-18 17-18	16-17 16-17	17-19 17-19	15 17	16-18	16-17	15-16 16-17	14-18 17-18 17-18
Statistics • Properties, basic concepts • Applications	17-18	16-18		19 19	17-18		15-16	14-18 14-18
Invariants Applications				15				
Linear Algebra • Vectors • Determinants • Matrices	16-17	17-18	16-18	17 19 19	14-17		15-18 16-17	14-18 17-18 17-18
Analytical Geometry • Lines • Conic sections • Applications			17-18		17-18			17-18
Transformation Methods • Philosophy of the transformation theory • Applications							15-16	14-18
Mathematical Games								17-18

Table 3: curriculum of level 1

	4th grade 9-10	5th grade 10-11	6th grade 11-12	7th grade 12-13	8th grade 13-14
CYPRUS	**Integral numbers: 0-10000, 10000-1000000** Order, writing, comparison, ordering Operations with integral numbers **Addition, subtraction, multiplication, division.** Factoring of numbers Fractions Unit, part Similar and dissimilar fractions operations with fractions, Addition, subtraction with Similar fractions Addition, subtraction with dissimilar fractions The concept of decimal comparisons place value; one tenth, one hundredth **Operations with decimals numbers** Addition, subtraction Rounding **Measuring** Concepts Perimeter, area, susceptibility, weight, time Measure units (money, length, time) **Geometry** **3D shapes** cubes, cuboids, **Analogie** constructions recognition of facet, edge, top **Polygons** (triangle, square, rectangle, parallelograms, pentagon, hexagon) **Kinds of triangles** Names, classification, properties, constructions,	**integral numbers up to 1000000000** place value of a digital (millions, milliard) prime numbers and composite numbers maximum common factor least common multiple writing numbers, comparison **operations with integrals** multiplication with three figure numbers two figures division estimation of addition, subtraction, multiplication, division properties of operations total factorisation of numbers fraction **The concept of fractions** Unit, part of group of things Similar and dissimilar fractions **Sequence placing** **Mixed numbers** Reduction Transformation of fraction into decimals numbers **Operations with fractions** Addition, subtraction, multiplication, division **The concept of Decimal numbers** place value; (tens, hundreds, thousands) colleration with fractions **operations with decimals**	**integral numbers up to milliards** place value of a digital (millions, milliard) writing numbers comparison, Sequence placing prime numbers and composite numbers maximum common factor least common multiple total factorisation quadratic numbers **Powers** (concept, recognition, writing) **operations with integrals** multiplication with three figure numbers two figures division estimation of addition, subtraction, multiplication, division properties of operations total factorisation of numbers fraction **The concept of fractions** Unit, part of group of things Comparison and equality (Similar and dissimilar fractions) Sequence placing Mixed numbers Reduction Transformation of fraction into decimals	**Sets** Subsets Equal sets, unequal sets, Properties of equal sets Union and section of two or more sets Graphical representation of union and section of two or more sets Complement of a set The use of sets in order for problems to be solved **Natural numbers** Definition of natural numbers Comparison of natural numbers Equal or unequal natural numbers Properties of equality or inequality Use of the four operations n the set of natural numbers Priority of operations Equations Solution of problems connected with real numbers **Basic geometric concepts** Point, line, plane shapes, constructions, line segment, middle of a line segment, units of length, addition or subtraction of line segment, definition of an angle, kinds of angles, construction of	**Rational numbers** Recognition of positive and negative numbers, opposite numbers Comparison of rational numbers, definition of absolute value, elimination of parenthesis, estimation of arithmetic value of algebraic expressions **Powers of rational numbers that have as an exponent an integral number** Properties of powers Powers of rational numbers that have as an exponent an positive or negative integral number Transformation of a powers of rational numbers that have as an exponent positive to a power with an exponent a negative integral number **Inequalities** Inequalities of the form $ax+b>c$ or $ax+b<c$ and graphic representation of its solution **uations** Types of equations *with a solution or without a solution, or indefinite equations) Verification of the solution of an equation **s** The concept of ration, and analogy Properties of analogies, problems of merits finding the unknown value I an analogy **Areas of plane shapes** **Stereographic** (Polyhedrons, prism, parallelepiped, cuboids, Pyramid, scroll, cone, sphere, Problems of

	Angles Right angle Construction, comparison **Line** Line, line segment, Parallel and intersecting lines **Circle** Construction, ray, diameter Symmetry -axis of symmetry, complementation of missing parts of symmetrical figures **Statistics** Graphs Explanation Collection and recording data Construction Ordered pair Probabilities **Problem solving** Strategies of problem solving	Addition, subtraction, Multiplication with integral and decimal number Division with decimal Estimation of addition and subtraction Rounding **Ratio, Proportions, Percentages** The concept of ratio, Equality of two ratios Proportional amounts **Measure units** Perimeter, area, susceptibility, weight, time Area of rectangle square, parallelogram, triangle Volume of cubes, cuboids, Metric units (cm, m, km, l,g, cm^2, m^2, cm^3 Measuring time (min, sec, month, day, year etc,) **Geometry** **3D shapes** cubes, cuboids, pyramids recognition of facet, edge, top constructions Polygons (triangle, square, rectangle, parallelograms, pentagon, hexagon) **Kinds of triangles** Names, classification, properties, constructions, **Angles** Kinds of angles (Right, obtuse, acute) Construction, comparison Measuring angles **Line** Line, line segment, Parallel , perpendicular and intersecting lines **Circle** Construction, ray, diameter Perimeter, area	numbers **Operations with fractions** Addition, subtraction, multiplication, division **The concept of Decimal numbers** place value; (tens, hundreds, thousands) correlation with fractions **operations with decimals** Addition, subtraction, Multiplication with integral number Division with decimal Estimation of addition and subtraction Rounding of decimal numbers **Ratio, Proportions, Percentages** The concept of ratio, Equality of two ratios Finding of the missing term Use of analogies in problem solving Simple analogy method The concepts of percentages correlation with fractions and decimals finding of percentages finding of the whole amount if the percentage is known **Measure units** Concepts: Perimeter, area, susceptibility, weight, time Area of rectangle square, parallelogram,	an angle, comparison of angles, addition and subtraction of angles, construction of lines and perpendicular lines, definition of distance of a point into a line, property of midperpendicular , construction of a cross lines, definitions of adjacent, convex, reflex, vertical angles, complementary, supplementary, adjacent and supplementary angles, bisector of an angle, construction of a bisector of an angle, definition and construction of a circle and elements of a circle, Difference of a circle and secular sector, arc, chord , definition of central angle, relation between an arc and a central angle, positions of a line and a circle, distance the central of the circle into a line in comparison with the ray **Parallel lines** **Triangles** **Tetragon** **Powers of natural numbers** **Square Root** **Perimeter** **Areas** **Pythagorean theorem** **Divisibility of natural numbers** **Fractions** **Decimal numbers**	area and volume measuring)

		Symmetry -axis of symmetry, complementation of missing parts of symmetrical figures **Statistics** Graphs (bar chats, etc) Explanation Collection and recording data Construction Ordered pair Mean value Probabilities Writing results Conclusion predictions **Problem solving** Stage of problem solving Strategies of problem solving (construction of a design, table) choice of the right of the four operations use of logic reasoning estimate and control write the question in order to complete a problem **Problems of** Exploring Integral, fractions, decimals geometry measuring proportions	triangle Area of the external area of cubes, cuboids, pyramids Volume of cubes, cuboids, Metric units (cm, m, km l,g, cm^2, m^2, cm^3 Measuring time (min, sec, month, day, year etc) Measuring of angles **Geometry** **3D shapes** cubes, cuboids, pyramids recognition of facet, edge, top area of external areas constructions **Polygons** (triangle, square, rectangle, parallelograms, pentagon, hexagon) **Kinds of triangles** Names, classification, properties, constructions, **Angles** Kinds of angles (Right, obtuse, acute) Complement and supplementary angles Construction, comparison Measuring angles **Line** Line, line segment, Parallel , perpendicular and intersecting lines Circle Construction, ray, diameter Perimeter, area Symmetry -axis of symmetry, complementation of		

			missing parts of symmetrical figures Statistics Graphs (bar charts, etc) Explanation Collection and recording data Construction Ordered pair Mean value Probabilities Writing results Conclusion predictions **Problem solving** **Stage of problem solving** Strategies of problem solving (construction of a design, table) choice of the right of the four operations use of logic reasoning estimate and control write the question in order to complete a problem **Problems of** Exploring Integral, fractions, decimals geometry measuring proportions, percentages, partitioning, interest		

BULGARIA		**Numbers. Algebra**. **Fractions. Decimal Numbers. Application in:** Figures and Solids; Logical Knowledge; Modelling. **Figures and Solids. Geometrical Figures and Solids. Application in:** Functions. Measuring; Logical Knowledge; Modelling. **Numbers. Algebra. Dividing. Application in**: Logical Knowledge. Numbers. Algebra. **Common Fractions. Application in:** Logical Knowledge; Elements of Probability and Statistics; Modelling.	**Numbers. Algebra. Powers. Application in:** Logical Knowledge; Elements of Probability and Statistics. **Numbers. Algebra. Rational Numbers. Application in:** Functions. Measuring; Logical Knowledge; Elements of Probability and Statistics. **Figures and Solids. Geometric Figures and Solids. Application in:** **Functions.** Measuring; Logical Knowledge; Modelling; Elements of Probability and Statistics. Numbers. Algebra. **Whole Expressions. Application in:** Logical knowledge.	**Numbers. Algebra. Whole Expressions. Application in:** Logical Knowledge; Modelling. **Figures and Solids. General Geometrical Figures.A pplication in:** Functions. Measuring; Logical Knowledge; Elements of Probability and Statistics. **Numbers. Algebra Equation. Application in:** Logical Knowledge; Modelling. **Figures and Solids. Equal Triangles. Application in:** Logical Knowledge. **Numbers. Algebra. Inequalities. Application in:** Figures and Solids; Logical Knowledge; Modelling. Figures and Solids. **Parallelogram. Trapezium. Application in:** Logical Knowledge.	**Numbers. Algebra** **A Square root. Application in:** Logical Knowledge; Modelling. **Numbers. Algebra. Quadratic Equation. Application in**: Logical Knowledge. **Figures and Solids. Vectors. Middle segment. Application in**: Logical Knowledge; Modelling. **Functions. Measuring. Functions. Applications in:** Elements of Probability and Statistics; Modelling. **Functions. Measuring. Equalities. Applications in**: Numbers. Algebra. Logical Knowledge; Modelling. Numbers. Algebra. **Systems of Linear Equations with two Unknown Quantities. Application in:** Logical Knowledge. Numbers. Algebra. **Systems of Linear Inequalities with one Unknown Quantity.** Application in: Logical Knowledge. **Figures and Solids.** **A Circle and a Polygon. Application in:** Functions. Measuring; Logical Knowledge.
CZECH REPUBLIC	**Numbers up to 1 000 000** (order, number line, rounding, operations, estimations) **Fractions** (unit, part, fraction; numerator, denominator, half, quarter, third, fifth, tenth) **Parallel and intersecting lines,** perpendicular, circle (mutual position of two lines in	**Natural numbers** (natural numbers up to one million and over a million, the sequence of natural numbers, number line, recording numbers in the decimal number system, arithmetical operations with natural numbers and	**Consolidation of the** knowledge and skills that children bring with them from Primary school **Decimal numbers** (decimal numbers, comparison of decimal numbers,	**Fractions** (proper fraction, equal fractions, reduction of fractions, operations with fractions, common denominator, reciprocal fraction, mixed numbers) **Integers** (positive and	**Square and square root.** Pythagoras theorem **Powers with natural exponents** (operations, decimal notation) **Expressions** (number expressions, variable, polynomial) **Linear equations** (equality,

	the plane, centre and radius) **Symmetry** (axis of symmetry, symmetrical figures, isosceles triangle, equilateral triangle) Area of a square, rectangle, net of a cube and cuboids	their properties **Decimal numbers** (fractions with denominators of 10 and 100 and recording these as decimals, place value; one tenth, one hundredth) **2D and 3D shapes** (construction of a rectangle - including a square, further units of area (are, hectare, km^2, mm^2), surface area of a cuboid and a cube; using different sets of bricks to introduce the concept of volume **Tables, graphs, diagrams** (variable, independent variable, dependent variable; graphs, system of coordinates	addition and subtraction of decimal numbers, multiplication and division of a decimal number by a natural number, multiplication and division of decimal numbers, written algorithms, properties of arithmetical operations with decimal numbers) **Divisibility of natural numbers** (multiple, factor; divisibility tests for 2, 3, 5, 10; primes, decomposition of a number into prime factors, common factor, highest common factor, common multiple, least common multiple) **Angle and its size** (angle as a measure of turn, size of an angle, degree, minute, protractor; acute, right, obtuse and reflex Axial symmetry (congruence of geometrical figures, mirror or reflective symmetry, line of symmetry; symmetrical shapes angles; supplementary, vertically opposite angles; addition and subtraction of angles, multiplication and division of angles and	negative numbers, opposite number, absolute value, operations, negative decimals; rational numbers, their order, operations with rational numbers) **Ratio. Direct and inverse proportion** (scale, coordinate system; graph of a direct and inverse proportion, rule of three) **Percentages.** Interests **Congruence**, rotation (congruence of plane figures; symmetry of triangles, theorems, symmetrical figures) **Quadrilaterals, prism** (parallelogram and its properties, rectangle, rhombus, square; perimeter and area of a parallelogram, area of a triangle; trapezium, perimeter and area of a trapezium; prism, volume and surface area of a prism)	linear equations and their roots, equivalent linear equations) **Circle, cylinder** (circle, circumference of a circle, mutual position of a circle and a line – tangent, secant,; mutual position of two circles; number π; cylinder, its net, bases, curved face, volume and surface area) **Constructions** (basic constructions of triangles and quadrilaterals; sets of points of a given property)

			their sizes by two) **Triangle** (external and internal angles of a triangle, isosceles and equilateral triangle, altitudes of a triangle, medians and centroid of a triangle, circumscribed and inscribed circles, triangular inequality) **Volume and surface area of a cuboid** (volume of a cuboid, units of volume: cm^3, m^3, dm^3, mm^3, hl, dl, cl, ml, cuboid - including cube – volumes, nets for cuboids, surface area of cuboids, face and body diagonals, 2D isometric representation of cuboids)		
GERMANY	**Calculating with numbers between 0 to 100** (repetition and intensifying) **Numbers between 0 and 1000** **Addition with numbers between 0 and 1000** **Simple tasks for multiplication and division** **Measure units** (money, length, time) **Geometry** (measurements, symmetry)	**Numbers between 0 to 1 000000** **Addition and subtraction with numbers between 0 and 1 000000** **Multiplication with numbers between 0 and 1 000000** **Division** (writing method) **Measure units (length, time, weight and volume)** **Special tasks** (tables, graphics, texts) **Geometry** (angle of 90 degree, parallels and applications)	**Calculating in contexts:** Measure units and exercises dealing with money/currencies, length, time, and weight. Calculating from one measure unit to another Rounding up and down Modelling, interpretation and answering context tasks **Calculating with natural numbers:** The set of natural numbers The sequence of natural numbers Even and odd numbers, arithmetic progressions	**Calculating in contexts** Tasks for computations with fractions and decimals Computation of areas and volumes Connection between different scaling units **Calculus for fractions** Basic operations with fractions Order fractions according to size Addition and subtraction of fractions Multiplication with fractions (Permanence model, operator model, inverse fractions) Division of fractions by natural numbers Division of fractions by	**Elementary calculus with percents** Description via fractions, 1/4 =25 % Fractions, decimal numbers, percents Visualization with diagrams Determination of percents of a given object Determination of the value of the percentage **Calculus with rational numbers** Introduction of negative numbers (permanence principle) Order relations for rational numbers Addition, subtraction, multiplication and division Mixed exercises (laws and training with fractions) **Algebra** (Introduction) Letters as names for a variable

			Representation of natural numbers on a line Definitions: sum, product, difference and quotient Ranking of operations: (multiplication goes before addition) Operations + -- * : Commutativity, Associativity and Distributives laws Relations < , > , = (Reflexivity, symmetry, transitivity) Algorithms in written form **Divisibility:** Divisibility in N divisors and sets of multiples Rules for divisibility (last number is 2, sum of the decimal cipher coefficients is divisible by 3) Prime factor decomposition Greatest common divisor and the smallest common multiple **Geometry:** fundamental definitions in the plane and space, nets, Line characterized by: each point has two neighbours Half line characterized by: subset of line where	fractions **Decimals and its calculus.** Tables for the decimal system Decimals by fraction methods Fractions in decimals and the converse Order fractions and decimal fractions according to the size Addition, subtraction, multiplication and division with decimals Notation with periods (e.g. 3,753 period) **Geometry** (angles, triangle, circle) Angles (different sorts, right angles) Drawing of triangles, four-angels and classification Polygon Triangles ABC (with same sides etc.) Four-angles: ABCD Trapeze, parallelogram, quadrate, Identification of circles and drawings Half circle, sector, radius Volume of cubes.	Substitution of a value for a variable in an equation Inequalities with a variable Set of solutions (depending on the ground set) Equivalent calculations Algorithms: 1. simplification of terms 2. addition rule 3. multiplication rule **Relations -- (Anti) proportionality** definition of "mapping to", different kind of representations tables, axes, Proportionality and its properties; y = ax Applications tasks Computational and graphical solutions **Geometry** (fundamental definitions) Basic definitions, reflection at a line constructive geometric method of reflections Triangle, circle, line, line segment, half line Line segment and length, Distance between two parallels Length of a line segmentAB Distance of a point and a line Parallelism and orthogonality of lines Special kind of angles Reflection at lines at its properties/ symmetry at axes Basic constructions with lineal and circle (construction of the midpoint of a line segment, construction of the orthogonal line) construction of parallels, finding the half of an angle] **Geometry** (reflection, triangles, circle)

			one point has only one neighbour Subset of the line between two points: line segments Arrows (vectors) and directed lines Relations: orthogonal, parallel, congruent Operations: subset, intersection, union of sets Reflection at a line and symmetry at axes Cube, quader, pyramid, cylinder, cone, ball Nets: volume of rectangles Models, maps, plans, drawings and lattices Instruments in geometry: circle, instruments for drawing lines and angles. **Fractions and representations of fractions** Fractions and representations of fractions Representations of fractions in the environment Decomposition of measure units in fractions Different representation of fractions (equivalence classes)		triangular forms sum of the angles in a triangle congruence and area of triangles basic constructions at triangles Secants, tangents at circles Theorem of Thales and its converse

GREECE	The course is more or less descriptive, enabling students to learn and familiarize themselves with concepts and figures. It includes: **Solid Geometry** (parallelepipeds, cubes), concept of length, breadth, height. Points, straight lines, angles. Parallel lines, lines in general, circle. Polygons. **Symmetry.** Perimeter and area of figures. Volume, weight. Calculation with coins. **Decimal system for integers, order of numbers.** Addition, subtraction, multiplication, division. Problems involving the four operations. **Fractions** (operations, problems). **Decimal numbers** (notation, operations, problems).	**Integers in decimal** notation (operations, divisibility criteria, problems). **Fractions** (operations, comparison, rounding, conversion to decimals). **Geometric solids** (parallelepipeds, angles, parallel lines, perpendicular lines). Measurement of magnitudes (length, area, angle, weight, time). **Operations and problems** involving decimal numbers. **Operations and problems** involving fractions. **Polygons** (triangles, quadrilaterals, parallelograms). Drawing in scale. **Area.** **Circle** (circumference, area), inscription of regular polygons. **Statistics** (averages, manipulation of numerical data, charts).	**Revision of properties of numbers.** Introduction to the use of a letter in place of numbers. **Problems** (numerical or geometrical) leading to equations with one unknown. Expressions (such as area of a triangle) involving the use of two variables. **Prime and composite numbers.** Representation of a number as product of primes. Least common multiple. **Powers of ten.** Properties of three dimensional figures and their development. Measurement of various magnitudes. Scale. **Proportion**. Proportional and inversely proportional quantities. Percentage. Interest. **Idea of a graph.**	**Natural and decimal numbers** (operations, Euclidean algorithm, divisibility, powers). **Measurement of magnitudes** (areas and volumes of various figures). **Fractions (**operations, conversion to decimal). **Proportion and scales** (percentage, reading a map). **Basic geometric figures** (distance, perpendicularity, parallel lines, circle). Angles (up to sum of angles in a triangle). **Plane figures** (equality of triangles, parallelograms and their areas). **Rational numbers** (including negative numbers and order).	**Rational numbers** (properties of powers, long division). **Equations and inequalities** (solution of). **Real numbers,** square roots, irrational numbers, Cartesian Plane, Pythagoras Theorem. **Basic Trigonometry (**sine, cosine, tangent. Case of 30^0, 45^0, 60^0). **Functions and their graphs** (y = ax + b, y = a/x). Statistics (diagrams, frequency, mean, median). **Symmetry** (with respect to point or line). **Measurement of circle** (regular polygons, circumference, area). **Measurement of 3D figures** (prism, pyramid, cylinder, cone, sphere).
ROMANIA	**Natural numbers:** how they appear, classes (units, thousands, millions, billions); order, writing, comparison, ordering, rounding Features of the numeration system we are using: decimal and positional writing with roman digits **Operations with natural**	**Natural numbers** Writing and reading natural numbers; sequence of natural numbers Representation of natural numbers on the axis. Comparing and ordering natural numbers Addition of natural	**Algebra** **Natural numbers** **The set of natural numbers** Divisor, multiple Criteria of divisibility by 10, 2, 3 Properties of the divisibility relation in	**Algebra** **Set of integer numbers** **Sets** The notion of set; relations (membership, equality, inclusion): operations	**Algebra** **Real numbers** □ ⊂ □ ⊂ □ ⊂ □. Various forms of writing a real number. Representation on the axis. Approximations. Absolute value of a real number. Intervals Intersection and union of intervals. Operations with real

	numbers: -addition and subtraction; specific terminology:term, sum etc. -multiplication by 1,10, 100; -multiplication by using distribution with respect to addition (without using this terminology) -multiplication by more factors; different cases; specific terminology -division with remainder; specific terminology -division by 10, 100, 1000; other cases **Problems** which are solved by at most three operations Problems which are solved by the figurative method Problems of estimating; which are solved by trials Problems to organize data in tables Problems which involve more than three operations; of logic and probabilities **Fractions** -the concept of fraction; equal fractions; representations by pictures equi-unity fractions, sub-unity and upper-unity fractions comparison of fractions sum and difference of fractions with the same denominator finding the fraction from an integer **Finding an unknown number from a relation of the form** ?+a=b; ?-a=b; ?+a <b etc. **Intuitive elements of geometry** geometric figures: point, segment, polygons, angle, parallel lines **perpendicular lines**	numbers Subtraction of natural numbers Multiplication of natural numbers; the order of doing arithmetic operations Division with remainder of natural numbers The order of doing arithmetic operations. Common factor Divisor, multiple, divisibility with 10, 2, 5. Even numbers and odd numbers Solving and forming equations; inequations and problems that lead to the arithmetic operations studied (including elements of grouping data) Power with natural exponent of a natural number; **perfect squares* Square and cubic power of a natural number Comparing and ordering of powers; rules to compare powers Order of doing the operations. **Rules of computation with powers* Decimal system of numeration **History of the evolution of the systems of writing numbers* **Bases of numeration* True propositions and false propositions "And", "Or", "Not"; "if-then" Sets (description and writing); element, relation of membership **The sets □ and □*** Negative integer numbers.	□ Prime numbers and composite numbers Decomposition of natural numbers as a product of powers of prime numbers Common divisors of two or more natural numbers; least common divisor; Prime numbers each others Common multiples of two or more natural numbers; greatest common multiple **2. Operations with positive irrational numbers** Different forms of representing a rational number. Representation by drawing or on the number axis Summation of positive rational numbers Subtraction of positive rational numbers Multiplication of positive rational numbers Division of positive rational numbers The ordering of processing arithmetic operations. Equations **Ratios and proportions** Ratios Proportions; the fundamental property	(intersections, union, difference, Cartesian product) **Integer numbers** The set of integer numbers; representation on the axis; operations; order of doing operations divisibility in □ : definition, divisor, multiple; equations; in equations. **Set of rational numbers** Set of rational numbers (□); representation on the axis of rational numbers, the opposite of a rational number; absolute value of a rational number (modulus). The inclusions □ ⊂□ ⊂□ . Writing the rational numbers in decimal or fractional form Sum of rational numbers, properties Subtraction of rational numbers Comparison of rational numbers Multiplication of rational numbers, properties, order of doing operations Division of rational numbers, properties, order of doing operations Power of a rational number with integer exponent. Rules of	numbers of the form a $\sqrt{b},\ b>0$ (sum, subtraction, multiplication, division, powering) Formulae of brief computations $(a\pm b)^2=a^2\pm 2ab+b^2$ $(a+b)(a-b)=a^2-b^2$ $(a+b+c)^2=a^2+b^2+c^2+2$ * $(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm$ * $(a\pm b)(a^2\mp ab+b^2)=a^3\pm b$ Decomposition into factors Ratios of real numbers represented by symbols (letters) Operations $(+,-,*,/,\hat{})$ **Functions** The concept of function Functions defined on finite sets, expressed by means of diagrams, tables, formulae; graph representation Functions of the form $f:□\to□,\ f(x)=ax+b(a,$ Graph Functions of the form $f:A\to□,\ f(x)=ax+b(a,$, where A is an interval or a finite set; Graph representation **Equations and inequations** Equations of the form $ax+b=0,\ a,b\in□$ Equations of the form $ax+by+c=0,\ a,b,c\in□$ Systems of equations of the form

	special quadrilaterals: rectangle, rhombus, square, **parallelogram, trapezoid** perimeter (rectangle and square) area; area of rectangle and square **exercises for observation of objects with forms of:** cube, sphere, prism, pyramid, cylinder, cone; unfolding the cube, (parallelepiped) cuboid and assembling some components **Measurements using nonconventional standards** Units for measurement of -length: metre, multiples, submultiples, transformations -capacity: litre, multiples, submultiples, transformations -**surface* (area): square metres -time: hour, minute, day, week, month, year, decade, century, millennium -coins and banknotes	The set of integer numbers. Representation of an integer number on the axis Relations between sets, subset Operations with sets (intersections, union, difference) Examples of finite sets; the set of divisors of a natural number Examples of infinite sets; the set of multiples of a natural number. **Rational numbers** Fraction; representation of fractions by means of drawings Equiunitary fractions; subunitary fractions; upper-unity fractions Equal fractions. Equivalent representations of fractions; sequence of equal fractions; positive rational numbers The common denominator of fractions Summation and subtraction of positive rational fractions (only for fractions whose common denominator can be calculated by observation). Comparing fractions Finding the fraction from a number Writing the fractions with denominators powers 10 in decimal form Comparing, ordering, representation on the axis of a rational number written in decimal form. Rounding	of proportions, finding an unknown term Percentages. Solving problems that involve percentages Quantities directly proportional. Graphic representation of the direct proportionality The rule of three Quantities inversely proportional. Graphic representation of inverse proportionality. The compound rule of three Graphic representation of data (graphics using bars); Elements of grouping data and of probability **Integer numbers** Integer number; representation on the numbers axis; opposite number; absolute value Comparing and ordering integer numbers Representation of a point of integer coordinates in an orthogonal system of axes Addition of integer numbers Subtraction of integer numbers Multiplication of integer numbers. Multiples of an integer number Division of integer numbers when the	computations with powers Order of doing operations and the use of brackets Solving in $\square$ the equations of the form $ax+b=0,\ a\in\square^*,\ b\in\square$ Problems that can be solved using equations Ratios; proportions; derived proportions; percentages; percentage ratio; sequence of equal ratios Arithmetic mean and weighted arithmetic mean **Real numbers** Square root of a natural number (perfect square) Square root of a positive rational number: The algorithm for computing the square root. Approximations Examples of irrational numbers; irrationality of $\sqrt{2}$ (proof-not compulsory); set or real numbers; absolute value; ordering, representation on the axis by approximations Rules of computations $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$; $\sqrt{a}/\sqrt{b}=\sqrt{a/b},\ a>0,\ b$. Inserting factors under square root. Extracting factors from the square root	$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2+b_2y+c_2=0 \end{cases}$ ' a_i, b_i, c_i Solution by reduction and substitution method; geometric interpretation Solving problems by means of equations and system of equations Solving in $\square$ the equation $ax^2+bx+c=0,\ \ a,b,c\in\square$, by decomposing in factors or squares. Formula. Inequations of the form $ax+b>0(\leq,>,\geq),\ a,b\in\square$ *Systems of two equations of this form **Summarizing themes for preparing the graduation exam** **Geometry** **Relations between points, lines and planes** Known geometric bodies: cube, rectangle parallelepiped, pyramid, cylinder, cone, sphere (description, representation in the plane, unfolding, presentation of round bodies as rotation bodies) Points, lines, planes: conventions of drawing and notation Determination of a line, of the plane Tetrahedron. Pyramid Relative positions of two lines in space (using the bodies already studied); parallels axiom; parallelism in the space Angles with sides respectively parallel (without proof), angle of two lines in the space; perpendicular lines

		Summation and subtraction of numbers with a finite number of non-zero decimal digits Multiplication of numbers with a finite number of non-zero digits (multiplication by $10^2, n \in$ □ ; multiplication by a natural number; multiplication of two numbers written with decimal point) Powers with natural exponent of a number with a finite number of non-zero decimal digits Divisions of natural numbers giving as result a decimal number Periodicity Division of natural numbers by $10^2, (n \in$ □ $)$ or by a natural number or by a decimal number Order of doing operations with decimal numbers. Decimal approximations Solving and forming equations, inequations and problems which involve the operations studied Arithmetic mean of two or more numbers; applications Ratio; percent. The set of rational numbers □ **Elements of geometry and measurement units** Geometric figures: straight line, curbs, polygons; angles, triangles, quadrangles (presentation by description and	dividend is a multiple of the divisor Divisors of an integer number The power of an integer number with natural exponent Computing rules with powers The order of doing arithmetic operations and the use of brackets Solving some equations in □ Solving some inequalities in □ . **Geometry** **Geometric figures and bodies** Geometric instruments (marked rule, unmarked rule; compass, set square) Using them in drawing several configurations Geometric figures: triangles, quadrangles, circles, segments, zigzag lines, lines, curbs (presented by description and drawing): the intersection of two circles (intuitive presentation) Geometric bodies: cube, parallelepiped, pyramid, sphere, cylinder, cone (description; distinguishing their elements: vertices, edges, faces;	Geometric mean **Algebraic Computation** Computation with (real) numbers by symbols: sum, subtraction, multiplication, power with integer exponent; rules of computation for powers Formulae $(a \pm b)^2 = a^2 \pm 2ab + b^2$; $(a-b)(a+b) = a^2 - b^2$; * $(a+b+c)^2 = a^2 + b^2 + c^2$. Solving the equation of the form $x^2 = a$, with $a \in$ □ Numerical applications **Elements of organizing data** Orthogonal system of axes; representation of points in the plane using the orthogonal system; the distance between two points in the plane; solving simple geometric problems using the representation of points in an orthogonal system; representation of some real numbers on axis, using ruler and compass. Representation by tables, diagrams and graphics of the functional dependence. Computation of the	Relative positions of a line with respect to a plane *theorem related to lines parallel to a plane Line perpendicular on a plane; distance from a point to a plane; Altitude of pyramid Symmetry axes of parallelepiped Relative positions of two planes Parallel planes; distance between two parallel planes Prism: altitude, right prism *oblique prism Sections parallel to the base in bodies already studied; trunk of pyramid **Orthogonal projections on a plane** Projections of points, lines, segments The angle of a line with a plane; length of the projection of a segment. The theorem of the three perpendiculars. Computing the distance from a point to a line. *Reciprocally of 3 perpendiculars theorem Dihedral angle; plane angle corresponding to a dihedral angle of two planes; perpendicular planes Computation of some distances and measures of angles on the faces or interior of bodies already studied. **Computation of areas and volumes** Area and volume of a geometric body Lateral area, total area and volume of right prism having the base an equilateral triangle, a square of a regular hexagon. Lateral area, total area and volume of regular triangular pyramid, trunk of pyramid

		drawing; observation of their elements: sides, vertices, angles) Geometric instruments. Drawing geometric figures and measurement of length and angles Perpendicular lines. Parallel lines. *Localization in the plane of a point with integer coordinates Construction of figures using the symmetry and translation Geometric bodies (description; identifying their elements: vertices, edges, faces) Measurement and estimation of some length, perimeters and areas, using different standards Measure units for length; transformations; area of square and rectangle equivalent surfaces Measure units for volume; transformations; volume of cube and rectangular parallelepeped Measure for capacity; transformations Measure for mass; transformations Measure for time; transformations	unfolding the cube and rectangle parallepiped) Identification of some plane geometric figures on the faces of known geometric bodies **The line** Point, line, plane, half-plane, half-line, segment (description, representation) Relative positions of a point with respect to a line; collinear points; “through two distinct points passes one line only” Relative positions of two lines: concurrent lines, parallel lines; lines not situated in the same plane Distance between two points; length of a segment; congruent figures Congruent segments; midpoint of a segment; construction of a segment congruent to a given segment **Angles** Definition, notations, its elements; interior, exterior, null angle; angle with sides in continuation Measurement of angles with protactor; construction using the protactor of an angle of given measure. Congruent angles	probability of some events. **Equations and systems of equations** Properties of the relation “=” in the set of real numbers Equations of the form $ax+b=0,\ a,b\in\mathbb{R}$; the set of solutions; equivalent equations; solving equations *Using formulae in solving equations reducible to the equations of this form Inequations of the form $ax+b>0\ (<,\geq,\leq),\ a,b\in$ Writing the set of solutions Equations of the form $ax+by+c=0,\ a,b,c\in\mathbb{R}$ Systems of equations with two unknowns of the form $\begin{cases} a_1x+b_1=c_1 \\ a_2x+b_2=c_2 \end{cases}$; solution by reduction method and substitution method. Solving simple practical problems, using equation, inequations and systems of equations. **Geometry Summarizing and completion** Vertical angles Congruency of triangles. Important lines in triangle Necessary and sufficient conditions for	quadrilateral regular and trunk of pyramid hexagonal regular Cylinder circular right: description, unfolding, sections parallel to the base and axial sections, total area, volume Cone circular right: description, unfolding, sections parallel to the base and axial sections, total area, volume. Sphere: description, area, volume **Summarizing and synthesis themes for preparing the graduation exam**.

			Right angle; acute angle; obtuse angle Calculations with measures of angles expressed in hexadecimal degrees, minutes, seconds Adjacent angles; bisector of an angle; supplementary angles; complementary angles Vertical angles; their congruence; angles around a point their sum **Congruency of triangles** Triangle: definition, its elements; different kind of triangles: scalar, isosceles, equilateral, rectangle, obtuseness angle, acute angle (definition, drawing); perimeter of a triangle, exterior angle of a triangle (definition) Construction of triangles: the cases SAS, ASA, SSS. Congruency of arbitrary triangles, congruency criteria of triangles SAS, ASA, SSS (obtained from the cases of construction) Elements of geometric reasoning (hypothesis, conclusion, proof, axiom, theorem) Method of congruent	a parallelogram to be a rectangle, a rhombus, a square Center of symmetry and axis of symmetry for the polygons studied. Trapezoid, classification; isosceles trapezoid, properties Areas (triangles, quadrangles); computing the area of some surfaces using nets, cuttings, pavements, or using formulae Computation of lateral and total area of some polyhedrons, using given unfolding **Similarity of triangles** Ratios and proportions formed with length of segments; proportional segments, Theorem of …. Equidistant parallels. Thales theorem. Reciprocal of Thales theorem. Dividing a segment into parts proportional to given numbers (segments) Middle line in a triangle; properties. Center of gravity of a triangle Middle line in a trapezoid; properties Similar triangles. Fundamental theorem of similarity Similarity criteria for triangles **Metric relations in a rectangle triangle** Orthogonal projections	

			triangles. **Perpendicularity (orthogonality)** Perpendicular lines (definition, notation, construction with set square); oblique; distance from a point to a line Cases of construction and congruence criteria for rectangle triangle Midperpendicular of a segment; the property of points situated on the midperpendicular of a segment (proof); construction of the midperpendicular using ruler and compass; the concurrency of the midperpendiculars of a triangle; **the circle circumscribed to a triangle* The property of points situated on the bisector of an angle (proof); construction of bisector using ruler and compass; the concurrency of the bisectors in a triangle; *circle inscribed in a triangle **Parallelism** Parallel lines (definition, notation); construction of parallel lines (by translation); axiom of parallels “Two distinct lines, parallel to a third line,	on a line Altitude’s theorem; cathetus’ theorem Pythagoras theorem; reciprocals of Pythagoras theorem Constant ratios in a rectangle triangle: sin, cos, tg, ctg; use trigonometric tables; writing the table for the angles $30^0, 45^0$ and 60^0. Solving the rectangle triangle. Areas of polygons studied. **Circle** Circle: definition; elements of a circle: center, radius, chord, diameter, arc, interior, external; disc Angle at center; measuring arcs; congruent arcs Chords and arcs in circle Angle inscribed in a circle; triangle inscribed in a circle; quadrangle inscribed in a circle; regular polygon: construction, elements *inscribable quadrangle Relative positions of a line with respect to a circle; tangent to a circle from an external point; circumscribed triangle to circle Relative positions of two circles Computation of elements in regular polygons: equilateral triangle, square, regular	

			are parallel to each other” Criteria of parallelism (theorems regarding the angles obtained by two parallel lines and a secant) **Properties of triangles** The sum of measures of the angles of a triangle (proof); theorem of external angle The altitude in a triangle (definition, drawing in the cases: acute triangle, rectangle triangle, obtuse triangle); the area of a triangle (intuitively, on a net of squares); the median (definition; property of the median to divide the triangle into two triangles of equal areas) Concurrency of altitudes and medians (without proof) The symmetry with respect to a line; properties of isosceles triangle (angles, important lines, symmetry); properties of equilateral triangle (angles; important lines, symmetry). **Quadrangles** Convex quadrangle (definition, drawing); particular quadrangles	hexagon (side, hypothenuse, area, perimeter) Length of a circle and area of the disc; length of an arc of circle; area of sector of circle.	

			(parallelogram, rectangle, rhombus, square, trapezoid - definition, drawing) The sum of angles of a convex quadrangle Parallelogram: properties (of sides, angles, diagonals); symmetry with respect to a point Particular parallelograms: properties		
United Kingom	**Key stage 2** **Using and applying number** Problem solving Reasoning **Numbers and the number system** Counting Numbers patterns and sequences Integers Fractions, percentages and ratio Decimals **Calculations** **Number operations and the** relationship between them Mental methods written methods calculator methods **Solving numerical problems** **Using and applying shape, space and measures** Problem solving Communicating Reasoning **Understanding properties of shape** **Understanding properties of position and movement** **Understanding measures** **Using and applying handling data** Problem solving	**Key stage 3** **Number and Algebra** **Using and applying number and algebra** Problem solving Communicating Reasoning **Number and the number system** Integers Powers and roots Fractions Decimals Percentages **Calculations** **Number operations and the relationships between them** Written methods **Solving numerical problems** **Equations, formulae and identities** Use of symbols Index notation Equations Linear equations Formulae Direct proportion Simultaneous linear equations Inequalities	**Key stage 4** **Number and algebra** Problem solving Communicating Reasoning **Numbers and the number system** Integers Powers and roots Fractions Decimals Percentages Ratio **Calculations** Number operations and the relationships between them Written methods Calculator methods **Solving numerical problems** **Equations, formulae and identities** Use of symbols Index notation Inequalities Linear equations Formulae **Sequences, functions and graphs** Sequences Graphs of linear	**Key stage 4** **HIGH MATHEMATICS** **Number and algebra** Problem solving Communicating Reasoning **Numbers and the number system** Integers Powers and roots Fractions Decimals Percentages Ratio **Calculations** Number operations and the relationships between them Written methods Calculator methods **Solving numerical problems** **Equations, formulae and identities** Use of symbols Index notation Inequalities Linear equations Formulae **Sequences, functions and graphs** Sequences Graphs of linear	

	Communicating Processing, representing and interpreting data	Numerical methods **Sequences, functions and graphs** Sequences Functions Gradients **Shape, space and measures** Problem solving Communication Reasoning **Geometrical reasoning** Angles Properties of triangles and other rectilinear shapes Properties of circles 3-D shapes **Transformations and coordinates** Specifying transformations Properties of transformations Coordinates **Measures and construction** Measures Construction Mensuration Loci **Handling data** **Using and applying handling data** Problem solving Communicating Reasoning **Specifying the problem and planning** **Collection data** **Processing and representing data** **Interpreting and discussing results** **Breadth of study (activities)**	functions Gradients Interpret graphical information **Shape, space and measures** **Using and applying shape, space and measures** Problem solving Communicating Reasoning **Geometric reasoning** Angles Properties of circles 3-D shapes **Transformations and coordinates** Specifying transformations Properties of transformations Coordinates **Measures and construction** Measures Construction Mensuration Loci **Handling data** **Using and applying handling data** Problem solving Communicating Reasoning **Specifying the problem and planning** **Collecting data** **Processing and representing data** **Interpreting and discussing results** **Breadth of study**	functions Gradients Interpret graphical information Quadric functions Other functions transformation of functions **Shape, space and measures** **Using and applying shape, space and measures** Problem solving Communicating Reasoning **Geometric reasoning** Angles Properties of circles 30D shapes **Transformations and coordinates** Specifying transformations Properties of transformations Coordinates vectors **Measures and construction** Measures Construction Mensuration Loci **Handling data** **Using and applying handling data** Problem solving Communicating Reasoning **Specifying the problem and planning** **Collecting data** **Processing and representing data** **Interpreting and discussing results**	

				Breadth of study	
ITALY 14-18 anni					**The number** General resumption of the integers and the arithmetic of the Primary School: the multiples and the divisors of a number; the numbers primes, the composed numbers; minimum common multiple, maximum common divisor; power elevation, operations with the powers; **Mastering and ampliamento of concept of number; rational numbers:** The Fraction like an operator and like a quotient; Decimal writing of rational numbers; The comparison between rational number relatives. The Irrational numbers: sense of the square root and of the extraction of root; the square root as inverse operation of the elevation to the square. square root of a product and a quotient. **Geometry** *General resumption of* Solid and plain geometry of the Primary School. Mastering of analysis of plain figures. Meaningful elements and characteristic property of triangles and quadrilaterals. Concave and convex polygons.

					Regular polygons, circle and circumference. The geometric transformations: the concept of "equal in comparison to" and of invariant. Intuitive notion of geometric transformation. The isometries: translations, rotations, symmetries. Analysis in concrete contexts of transformations not isometric. Relationships among geometric quantity Concept of contour and surface. Calculation of perimeters and areas of some plain figures. The similarity Theorems of Pythagoras and Euclide. Introduction to the concept of system of reference: the Cartesian coordinates, the Cartesian plan. **The Measure** The geometric quantity. The international system of measure. **Data and previsions** Phases of a statistic investigation. Concept of champion of a population. Examples of representative and not representative champion. Probability of an event: evaluation of probability in simple cases **Connected historical aspects to the mathematics** The method of Eratostene for the measure of the ray of the Earth. **The measure to distance in the medieval geometry**

Table 4: curriculum of the level 2

	9th grade 14-15	10^{10h} grade 15-16	11^{h} grade 16-17	12th grade 17-18	18-19
CYPRUS	**Algebra** Algebraic representations Polynomials Factorisation of Polynomials and Algebraic fractions, **Geometry** Basic geometric notions Equality of triangles Trigonometry Geometry of parallelograms and trapeziums **Graphs** Graphs of the form y=ax, y=ax+b, y=k, x=l **Systems** of linear equations with two unknowns.	**Algebra** Equations and Systems of equations First degree systems Systems of linear equations with two unknowns. (Solution of the equations both algebraic and graphically) Problem solving with the help of an equation and two equations **Roots** Properties of roots Powers with fractional exponent **Functions** The concept of correspondence Ways of representation of correspondence The concept of function Domain, codomain Graphs of functions Line, y=kx+b The function y=a/x The function $y=ax^2+bx+c$ Equations and inequalities of second degree Sum and product of roots of second degree equation Sigh of the expression ax^2+bx+c The trinomial Second degree inequalities Geometry Repetition of basic concepts **Equality of triangles** Definition of triangles equality Criterions of equality of triangles Properties and criteria of isosceles triangles **Parallelograms -trapezium**	**Graphical representation of line** Systems of first degree equations Solution of second degree equation Graph of the y=ax2+bx+c Pythagorean theorem Arithmetic progression Geometric progression Problem solving in progressions Logarithmic and exponential equations Properties of logarithms Logarithmic equations Exponential equations **Trigonometry** Trigonometric numbers of any angle Sine low Cosine low Area of triangle Solving of triangle Trigonometric equations **Geometry** Areas Similar shapes Regular polygons circle (only for students in "direction") **Algebra** Absolute value of a real number **Functions** Domain, codomain , equality, operations, composition, inverse function, **Limits** of functions, properties of limits **Complex numbers**	**Statistics** Basic notions Presentation of statistical data Eigenvalue of a distribution **Combinatorics** The n! Counting low Commutation of n objects Allocation of n objects sur k Combinations of n objects sur k **Probabilities** **Polyhedron** **Stereometry** Area and volume of scroll, cone, colure, sphere (only for students in "direction") **Functions** Functions that are parametric defined **Function derivatives** Derivatives of functions that are parametric defined Applications **Graphs** The theorem of the middle value of differential inference Local extremums Concave and convex functions, inflexion point Asymptotes Graphs of functions Problems with maximum and minimum **Inverse trigonometric functions** **Anti-derivative** Definition Basic derivatives types Properties Methods of derivative	

		Definition of a parallelogram Properties of a parallelogram Applications of the Properties of parallelograms Special parallelograms Rectangle Properties of the rectangle in to orthogonal triangle Rhomb Square Trapezium **Circle** The concept of circle and elements of circle Relations of central angles with the correspondence arcs and chord Position of a point and a circle Position of a line and a circle Position of two circles Inscribed angle Angle under chord and tangent **Similar triangles- similar Polygons** The concept of Similarity **Metric relation in to orthogonal triangle** Useful concepts and capacities Metric relation into orthogonal triangles **Trigonometry** Introduction in Trigonometry Trigometric numbers of acute angle Trigometric numbers of any angle Trigometric angle –trigometric circle Trigometric numbers of angle into trigometric circle Sign of trigometric numbers of an angle Relations of trigometric numbers of two supplementary angles	Application in the second degree equation. Representation of a complex number in the plane **Induction** **Sequence** **Progressions** **Exponential and logarithmic function** **Continuity of functions** **Derivative** Derivative of functions, derivative of composition of functions, derivative of exponential and logarithmic function Applications of derivative **Polynomials** Equality of polynomials, arithmetic value, operations with polynomials, root of polynomials. Fractions analysis into simple fractions **Analytic Geometry** Vectors Line equations **Geometry** Inscribed, circumscribed tetragons in a circle Normal polygons, calculation of circle Locus, analytical composition methods, constructions **Geometry of space** Position of two lines in space**,** skew lines Polyhedron, counting polyhedron Stereometry, scroll, cone, colure, sphere, spherical zone, spherical sector **Trigonometry** Sine low, cosine low, low of area, trigometric	**Definite derivative** Properties Applications **The symbol of sum S** Examples of very well known sums Properties Sums of n primary terms of a sequences **Compinatorics** Counting low Properties of combinations n sur k **Tables** **Statistics** **Conic sectors** Circle Parabola ellipse hyperbola Enrich lesson **Linear programming** Solution of equations systems and inequalities of first degree Applications of linear programming systems **Complex numbers** The set C of Complex numbers Geometric representations of complex numbers Polar coordinate of a complex number Trigonometric representation of a complex number Roots of a complex number **Applications of derivatives** Differential and its applications Problems with maximum and minimum Rate of change **Integral applications** Shell methods First theorem of Pappos **Analytic Geometry**	

			numbers of addition and subtraction angles of double arc Transposition of sum of trigonometric numbers into product Trigonometric equations Enrich lesson **Numbers theory** **Complex numbers** **Geometry** Analytical-synthetic methods Applications in constructions Applications in locus **Analytic structure** Binary relations Group annulus	Applications of analytic geometry into locus Transformations in the plane The general second degree equation	
BULGARIA	**Numbers. Algebra.** **1. Rational expressions. Rational Equations.** Applications in: Algorithms. Quadrate Equation. Biquadrate Equation. **Logical Knowledge.** **2. Logical Terminology and Conjunctions.** Contradiction.Contra-example. Rationality. Applications in: Necessary Condition. Sufficient Condition. Theorem and Conversely Theorem. **Modelling.** **Models with Linear and Quadratic Equations.** Evaluating of the Solution. Estimating and Control	**Numbers. Algebra.** **1. Real Number**. Application in: Real Number. Real Axis. One-to-one Correspondence. **Functions. Measuring.** **2. Quadratic Function.** Application in: Parabola.Vertex. Axis. Increasing and Decreasing Function. The Best Upper and Low Values of the Function. **Logical Knowledge.** **3. Evaluation of Trueness and Rationality of the Choice.** Application in: Logical Knowledge; Using of Graphic Method. **Numbers. Algebra.** **4. Rational Inequalities.** Application in: Quadrate Inequality. Biquadrate Inequality. Fractional Inequality.	**Functions. Measuring.** **1. Number Sequences.** Application in: Sequences. Monotone Number Sequences. Arithmetic and Geometric Progressions. **Logical Knowledge.`** **2. Logical Conjunctions.** Contradiction. Application in: Necessary and Sufficient Conditions. Formulating of Assertions. **Modelling.** **3. Modelling with Systems of Equations of Second Degree with Two Unknown Quantities.** Application in: Progressions. Simple and Compound Interest. **Probability and Statistics.** **4. Statistical Data.** Application in:	**Functions. Measuring.** **1. Elements of Mathematical Analysis.** Application in: Operations with Functions. Limit of Function. Composite Function. Continuous and Interrupted Function. Points of Interruption. Derivative of the Function. Local Extremum. Convex and Concave Function. Inflexed Point. Asymptote. **Logical Knowledge.** **2. Properties of Relations And Operations.** Application in: Using of Composite Function for Solving Problems. **Functions. Measuring.** **3. Rotation Solids.** Application in: Area of Surface. Volume. Inscribe and Circumscribe	

	of the Final Result. Application in: Real Situations. Solving Economic, Financial and etc. Problems. Numbers. **Algebra.** **4. Systems of Equations of Second Degree With Two Unknown** Quantities. Theorems of Equivalence. Ordered Pair of Numbers. Applications in: Equation of Second Degree with Two Unknown Quantities. Systems of Second Degree With Two Unknown Quantities. Solving Geometrical problems With Algebraic Knowledge. **Numbers. Algebra.** **5. Irrational Expressions.** **Applications in:** Standard Equivalence Transformations. Equivalence of Expressions. Rationalization of The Divider of The Fractions. **Logical Knowledge.** **6. Conjunctions for Existence and Contradiction.** Application in: Finding The Possible Values and The Numerical Value of An Irrational Expression. Rational Transformation of Irrational Expressions.	Method of Intervals. **Logical Knowledge.** **5. Logical Conjunctions: “And”, “Or” and “Equivalence”.** Application in: Choice of Right Algorithm For Solving Rational Inequalities and Evaluating of the Final Result. **Numbers. Algebra.** **6. Power.** Application in: N-th Root. Root. Rationalization. Logarithm. Base. **Logical knowledge.** **7. True**. Rationality.Application in: Knowledge of Powers. Rational Choice of Algorithm. **Functions. Measuring.** **8. Trigonometric** Functions. Application in: Trigonometric Functions. Trigonometric Identities. Figures. Solids. **9. Triangle. Application in:** Sin and Cosine Theorems. Solving Triangles. Solving Parallelogram and Trapezoid. Logical knowledge. **10. True. Rationality.Application in**: Solving Triangles. Abilities for Discovering and Creating of Paths for Solving Triangles **Modelling.** **11. Modelling with Linear and Quadratic** Equations.Application in: Geometric Problems. Interpretation and Evaluating of Obtained and Estimated Result.	Principles of Presenting Statistical Data. Terminology and Measures of Location and Dispersion. Bar Diagram and Histogram. **Logical Knowledge.`** **5. Choice in Concrete** Situations. Application in: Average. Choice of Statistical Methods. Effectiveness of the Data. Modelling. **6. Evaluating of Results.** Application in: Analyzing of Statistical Data. Interpretation of Finished Information. **Functions. Measuring.** **7. Trigonometric Functions. Application in:** Basic Trigonometric Identities. Even Function. Odd Function. Periodical function. Figures and Solids. **8. Solving an Arbitrary Triangle. Application in:** Basic Triangle Identities. Modelling by Using Trigonometry. Logical knowledge. **9. Contradiction. Application in:** Contradiction of Assertions. Choice of Rational Methods for Solving Triangle.	Spheres. Figures and Solids. **4. Trigonometry in Solid Geometry.** Application in: Geometric Problems with Rotation Solids and Polyhedrons. Extreme Problems with Solids. **Logical knowledge.** **5. Concretization of General Assertion**. Application in: Necessary and Sufficient Condition. Plane Problems as Components of Solving Solid Problems. **Probability and Statistics.** **6. Statistics**. Application in: Generally Combination. Statistical Frequency. Average Value. Quadrate Deviation. Dispersion. Normal Curve. **Numbers. Algebra** **7. Complex** Numbers.Application in: Algebraic and Trigonometric Representation. Operations with Complex Numbers. Moavre’s Formulae. Zeros of Polynomials. **Modelling.** **8. Elements of Analytic Geometry in the Plane.** Application in: Vectors. Length of Segment. Angle Between Two Lines. Descartes Equation of a Line in the Plane. Elements of Triangle.Canonical Equations of Cone Curves and Its Graphics .	

	Numbers. Algebra. **7. Irrational Equations**. Applications in: Irrational Equations. Extraneous Root. Equivalence Equations. Figures and Solids. **8. Similarity.** **Applications in**: Ratio of the Segments and Proportional Segments. Similar Triangles and Coefficient of Similarity. Fourth Proportional Segment. **Logical Knowledge.** **9. Necessary and Sufficient Conditions.** Applications in: Theorem and Conversely Theorem. Contradiction. Typical Situations With Similar Triangles. **Modelling.** **10. Modelling with Linear and Quadrate Equations.** Applications in: Modelling Geometrical Problems with Equations and Systems of Equations. Interpretation of Obtained and Estimated Result. **Functions. Measuring.** **11. Rectangular Triangle.** Applications in: Trigonometric Functions. Metrical Dependencies. Trigonometric Identities. Figures and Solids. **12. Solving Rectangular Triangle.**	**Functions. Measuring.** **12. Areas of Plane Figures.** **Application in:** Triangle. Rhombus. Square. Regular Polygon. Logical knowledge. **13. True. Rationality.Application in:** Areas of Plane Figures. Modelling. 14. Modelling with Linear and Quadratic Equations and Systems. Application in: ♦ Geometric Problems. ♦ Interpretation and Evaluating of Obtained and Estimated Result. **Probability and Statistics.** **15. Combinatorics.** Application in: Combinatorial Problems. Classical Probability. Logical knowledge. **16. Logical Conceptions.** **Application in:** Combinations. Algorithms for Counting.		**Logical knowledge.** **9. Properties Of Relations and Operations.** Application in: Representation of Number Multitude to Point Multitude in Fixed Coordinate System. Evaluating of Estimated Result.	

	Applications in: Finding the Basic Elements of Rectangular Triangle. Finding the Elements of Isosceles Triangle and Isosceles Trapezium. **Logical Knowledge.** **13. The Meaning of "Necessary Condition",** "Sufficient Condition" and "Necessary and Sufficient Condition". Applications in: Rectangle Triangle (Theorem and Conversely Theorem). Discovering and Creating of Situations Solving with Rectangular Triangle. Using Logical Knowledge In Concrete Situations Connected with Rectangular Triangles. **Modelling.** **14. Modelling with Linear and Quadrate Equations**. Applications in: Modelling Situations for Rectangular Triangles with Equations and Systems. Interpretation of Obtained and Estimated Result**.**				
CZECH REPUBLIC	**Fractional expressions**. Equations with the unknown in the denominator	**Number systems, properties of real numbers** Revision of basic algebraic concepts (factorisation of quadratic polynomials,	**Simplifications of algebraic expressions – revision** Mappings and functions – basic concepts. Use of	**Vectors – revision** **Line in plane**. Plane in space. Mutual position of a line and a plane **Conics** –	**Sequences – revision**. Applications of geometrical sequence Mathematical induction Limit of a sequence. Sum

	Systems of linear equations with two unknowns Functions (range and domain, dependent and independent variable; increasing and decreasing functions; constant function, linear function, quadratic function, inverse proportion) **Similarity.** **Trigonometric functions** sine and tangent in a right-angled triangle (scale factor; similarity of triangles, theorems; two segments in a ratio; maps; trigonometric functions in a right-angles triangle as a ratio of sides) **Pyramid, cone, sphere** **Basics of financial mathematics** (interest, principal, simple and compound interest)	fractions) **Equations** (quadratic, with the unknown in the denominator) **Quadratic equations with the parameter** Powers, roots **Simplifications of algebraic expressions** **Equations with absolute value, with roots** **Basics of plane geometry** (plane shapes, pairs of angles, angles in polygons, in the circle, sets of points of a given property) **Inequality** (linear, with the unknown in the denominator) **Inequality** (with absolute value, with roots) **Systems** of equations, inequalities and their systems **Equations,** inequalities and systems - graphical solution **Geometrical mappings** in the plane (symmetries and their composition, homothety, similarity) **Constructions** **Pythagoras theorem, Euclid theorems** Sets and propositions **Elementary number theory** (divisibility, prime numbers, least common multiple, highest common factor)	tables of points to construct graphs General properties of functions. Linear and quadratic functions **Composite function** Polynomials and rational functions (fractions of polynomials) Introducing the concept of inverse function Logarithms. Exponential and logarithmic functions Trigonometric functions Sine and cosine theorems, Applications Solution of all the practised types equations and inequalities – revision Graphical solutions of simple systems Vectors. Scalar product. Line in plane and in space Plane in space. Mutual position of a line and a plane. Applications	general properties Conics – mutual position of a line and a conic. Tangents **Plane geometry** – volumes and surfaces of bodies Plane cuts of bodies **Complex numbers –** algebraic and trigonometric form, basic operations. Moivre theorem n-th roots. Binomial equations Solutions of equations in the domain of complex numbers Sets of a given property in **C** **Sequences** – basic properties Mathematical induction	of a number series. Applications Limit of a function Asymptotes of a function graph **Derivative** Local extremes, convexity and concavity of functions L'Hospital rule Behaviour of a function Primitive function. Substitution method, integration per partes. Simple problems Definite integral Geometrical and physical applications of definite integrals **Combinatorics –** permutations, combinations Binomial theorem Basics of probability
GERMANY	**Systems of linear equations** Solving linear equations with two variables (ax + by = c) according to y = … Graphical representation of the solution set Determination of the solution set of linear equations with two variables	**Measurement of bodies** (Volumes) Volume and surface area of (orthogonal) pyramids Orthogonal cone cuts Volume and surface of the ball (in R^3) Application tasks **Powers** Powers with exponents of natural numbers Priority rules	**Discussion of functions without differential calculus** Linear functions (y = ax + n) Standard functions (powers, sinus etc.) Determination of zeros of polynomials Polynomial division Bisection, Regula falsi **Sequences and limits**	**Differential calculus and continuous functions** Theorems about differentiable and continuous functions Rules for differentiation Extreme values (Fermat) Intermediate value theorem Theorem of Rolle Mediate value theorem Problems using extreme values methods	**Systems of linear equations** Solution with the Gauss algorithm Homogeneous and inhomogeneous equations Linear independence of vectors Rank of a system of linear equations Vector spaces

	Real numbers and roots Introduction of irrational numbers Interval algorithm Infinite sequence of intervals Sequence of the right points (decreasing) Sequence of the left points (increasing) Difference sequence is a null sequence Irrational numbers are containing rational numbers Quadratic and cubic roots Laws for the products and quotients of quadratic roots **Theorems around Pythagoras** (theorems for triangles with 90 degree angle) Theorem of Pythagoras a^2 + b^2 =c^2 Katheten-theorem: a^2= p *c Theorem for the height: h^2=p*q Application tasks (length of line segments) Constructions based on this material **Quadratic equations** Pure quadratic equations Quadratic equations without absolute term Formula for the solutions of a quadratic equation Equivalence computations	Powers with exponents of entire numbers Laws for powers Power functions Power with rational exponents Computations of terms with roots **Calculus in contexts** Problems dealing with questions: money/currencies, measure units, taxes, costs, rates, elections **Trigonometry** Measurement for angles Sinus and cosinus at the unit circle Tangens function, tangens at the triangle with 90 degree angle Sinus- and Cosinus-Theorem Computation at arbitrary triangles **Exponential and logarithmic function** Exponential functions and their graphs Logarithmic functions as inverse functions of the exponential function Laws for logarithms Application tasks for the laws Application tasks (percent and rates)	**(theorems for limits)** arithmetic, geometric and infinite sequence limit of a sequence null sequences Limit theorems: sum, difference, product and quotients **Derivatives (of entire-rational functions)** Derivative as limit of differential quotients Derivatives in context of applications Derivatives of entire-rational functions **Discussion of functions with differential calculus** Discussion of functions Symmetry (y(-x) =y (x)) Zeros y(x) = 0 Points with horizontal tangents (y' (x) = 0), critical points Convexity (points with y'' (x) = 0) Applications for extreme value methods Methods for computing zeros II (Newton method) **Introduction to Analytic Geometry** Points in coordinate systems Distance Circle < ball **Vectors in the Cartesian coordinate system** Computations with vectors Addition, Anti-vector (-x), null vector, multiplication Length of a vector, unit vector Midpoint of a line segment	Problem of determination of areas Approximation of areas Area as limit Definite integral -- area Calculus with definite integrals **Integration theory** Mediate value theorem of integration theory Connection between Differential calculus and integration theory Primitive functions -- indefinite integral Computation of definite integrals Area between two graphs Partial integration Integration via substitution Numerical integration Volume or rotational bodies Length of curves Indefinite integrals **Rational functions** Set of definition Poles Asymptotic behavior Discussion of rational functions Definite integrals of rational functions Problems using extreme values methods **Exponential function/Logarithmic function** Derivative of the inverse function Root functions Applications The Euler number e Rule of de l'Hospital Discussion of functions and determination of area Differential equations	**Analytic Geometry** Scalar product (Applications, computation of the area, length of a vector, distance, angle) Vector product Equations for hyper planes (Hessian normal form, coordinate form, parameter form) Hyperplanes in the space (how they are situated) Matrices (Addition, multiplication, laws, inverse) Cone cuts (circle, ellipse, parable, hyperbole) Focusing points, construction of intersection points for cone cuts Equations for cone cuts in Cartesian coordinates **Foundations of probability theory** Probability defined by statistical data Stochastic processes (Galton-model, Lotto) Basic definitions (random experiment, events, set of events, relative probability, probability) Simulations Models (combinatory model, trees, tables) Rules: product and sum rule Independence of two events Conditional probability , Total probability **4. Distributions and statistics** Random variable as a function of a probability distribution

	Applications tasks **Theorems with proportions** ("Strahlen-satz") and similarity Central dilation Invariants Center, proportions of line segments Proportions of area Clockwise direction for angles First and second "Strahlensatz" %%% "Strahl" = half line, "Satz" = Theorem Similarities of triangles Application tasks **Area and volume measurement;** circle, cylinder Length of the circle and area of the disk (in the plane) Circle segment and sector of a disk Application tasks Volume and surface area of the (orthogonal) cylinder Application tasks **Calculus in contexts:** (anti-) proportionality, calculus for percents and rates (anti-) proportionality, calculus for percents and rates Application tasks		Equation of a line in vectorial form **Lines (and circles)** in the plane cutting point for two lines cutting points of a line and circle		Expectation value Variance Inequality of Tchebychev Bernoulli chains Binomial distribution Descriptive statistics Tests for hypotheses Significance level Confidence intervals Markov chains
GREECE	**Real numbers** (operations, roots, order). Algebraic expressions	**Algebra** **Real numbers** (operations, order, absolute value, solution of linear	**Algebra** **Trigonometry** (basic trigonometric equations, trigonometric	a) (only for students in "Science" or in "Technology") **Complex numbers**	

	(polynomials, basic identities, factorisation, rational functions). Equations (first and second degree). Functions and their graphs (y = ax^2+bx+c, y=a/x). **Statistics** (frequency, variance, probability). **Similarity** (Thales' Theorem, similar polygons, comparison of areas and volumes of similar figures). **Trigonometry** (relation between trigonometric numbers, sine rule, cosine rule in a triangle) Linear systems and systems of linear inequalities (including graphical solution). **Vectors** (addition, subtraction, multiplication by scalar). **The sphere** (various properties, Earth, longitude, latitude).	equations and inequalities). **Functions and their graphs. Systems of linear equations with two or more unknowns.** Second degree equations and inequalities. **Trigonometry** (trigonometric ratios of general angles). **Euclidean Geometry** Axioms. **Triangles** (criteria for equality). **Parallel lines.** Sum of angles of a triangle. Properties. **Parallelograms.** Concurrency of various lines in a triangle. **Circle.** Angles in circles. Tangency. Inscribed figures. Loci. Proportion. Theorem of Thales. Bisectors of a triangle. Circle of Apollonius. **Similarity.** Criteria for similarity.	numbers of double angle, the function y = asinx + bcosx, solution of triangles). **Polynomials** (division of polynomials, polynomial equations). **Progressions** (arithmetic, geometric, applications in e.g. interest). Exponential and logarithmic function. **Euclidean Geometry** Metric relations (Pythagoras, medians, transversals of a circle, geometric constructions). Areas (Heron's formula). Measurement of a circle (inscription of regular polygons, circumference, area, length of arc, area of sector). c) (only for students in "Science" or in "Technology") **Vectors.** Inner product of vectors. Cartesian form of a straight line. Analytic geometry of circle and Conic Sections. **Elements of number Theory** (Euclidean algorithm, divisibility, prime numbers, linear Diophantine equations, modular arithmetic).	(operations, modulus, trigonometric form). **Limits and continuity of functions. Differential Calculus** (rate of change, rules of differentiation, Mean Value Theorem, local extremes, l' Hospital's rule, graphs of functions). Integration as inverse of differentiation (techniques of integration, integration by parts and by change of variable), differential equations (separation of variables, first order linear), proper integral as area (Fundamental Theorem of Calculus, Mean value Theorem). **Basic Probability and Statistics.** (Average, mean, variance, charts, collection of data).	
ITALY 13-18 years In this moment in Italy there	**Classic High School** **Algebra:** Simple cases of decomposition of polynomials in factors. Algebraic fractions; to calculate with them. Equations and first degree problems to an	**Classic High School** **Algebra:** Systems of first degree equations. Concept of real number. Calculation of the radicals: sign on the powers with fractional exponent. Second degree equations and easily reducible to the first	**Classic High School** ***Algebra:*** Arithmetic and geometric progressions. Exponential equations and logarithms. Use of the logarithmic tables and application to the calculation of numerical	**Classic High School** ***Trigonometry:*** The goniometric functions: breast, cosine and share. Formulas for the addition, the subtraction, the duplication and the bisection of the matters. Use of the	

are no equal ministerial programs for all the public schools. The subdivision for year is served as the teachers of the single schools. The programs here brought have been taken by the more diffused books. The programs have been select for the High school "Classic" and the High school "Scientific	unknown. **Geometry:** Circumference and circle. Relationship of straight lines and circumferences: an outline of relationship of circumferences (in the same plain). Angles in the circle (to the center or to the circumference). Regular Polygons. Some fundamental graphic problem. Equivalent polygons. Theorem of Pythagoras. **Scientific High school** Concept of real number. Calculation of the radicals; an outline of the powers with fractional exponents. Equations of 2° degree or to them referable. Examples of systems of solvable equations of superior degree to the I° with equations of I° and 2° degree. An outline of the arithmetic and geometric progressions. Cartesian coordinates orthogonal in the plan. Functions of a variable and them particular graphic representation; in the functions ax + b; ax^2; - x. Proportions among magnitude, similitude of the triangles and the polygons, theory of the measure, area of the polygons.	degree. Simple examples of systems of equations of superior degree to the first one. **Geometry:** Proportions among magnitude. Simile of the triangles and polygons, theory of the measure (outline), area of the polygons. **Scientific High school** Exponential equations and logarithms. Use of the logarithmic tables and application to the calculation of the value of numerical expressions. An outline of the use of the calculating rule. Rectification of the circumference and quadrature of the circle. Straight lines and plans in the space: orthogonally and parallelism. Dihedral, Trihedron. Polyhedron, particularly prisms and pyramids. Cylinder, cone, sphere.	expressions. ***Geometry:*** Rectification of the circumferences and quadrature of the circle. Straight lines and planes in the space: orthogonally and parallelism. Dihedral, trihedron. Polyhedron (particularly prisms and pyramids). Cylinder, cone, sphere. **Scientific High school** Geometric functions. Curves of the sine and the cosine. Formulas for the addition, the subtraction, the duplication and the bisection of the argument. Some simple goniometric equation. Resolution of the rectilinear triangles. The notion of limit of a function. Derivation of a function of a variable and his geometric and physical meaning. Derivation of x^2, of senx, cosx, tgx. Exercises of derivation. Notions of equivalence of the solid figures. Equivalence of prisms pyramids. Practical rules for the determination of the areas and the volumes of the studied solid.	goniometric tables and application to the resolution of the rectilinear triangles. **Geometry:** An outline of the equivalent polyhedron, on the base, eventually, of the principle of Cavalieri. Practical rules for the determination of areas and volumes of the studied solid. ***In the three classes:*** simple exercises of application of the algebra to the geometry. **Scientific High school** Maximum and Minimum with the method of the derivation, applications. Notion of integral with some application. Dispositions, permutations and simple combinations. Binomial of Newton. ***In the last four classes:*** applications of the algebra to the geometry of I° and 2° degree with relative discussion.	

”. **Change this one and the distribution of programs in the columns !!!!**					
ROMANIA	**Algebra** **1. Operations with real numbers** -forms for writing real numbers **powers** with integer exponents roots of order 2 and 3; *roots of oder n* **powers with rational exponent** operations with real numbers; absolute value; ***inequalities*** -integer part, fractional part; rounding; approximations **Solving equations** Equations of the form $ax+b=0,\ a,b\in$ □ . Equations of the form $ax^2+bx+c=0,\ a,b,$. Irrational equations (involving roots of order 2 and 3) Equations reducible to equations already studied	**Progressions:** arithmetic, geometric **Functions:** exponential; logarithm; equations and inequations involving exponentials and/or logarithms; systems of exponential or logarithmic equations Monotonicity, convexity, concavity, asymptotic behaviour (intuitive) injectivity, surjectivity, bijectivity **Inverses of trigonometric functions;** trigonometric inequations (graphic solution) **Elements of geometry (plane and space)** -complex numbers in algebraic form; operations -geometric interpretation of +, -, * -complex numbers in trigonometric form: product, power (Moivre formula), roots of order *n* of unity, geometric interpretation **geometric transforms* (all) -scalar product of two vectors in plane and space; conditions of perpendicularity	Elements of linear algebra and analytical geometry Cartesian system in plane and space. Equations of a line in the plane; equations of a plane and line in the space. conditions for parallelism and perpendicularity **Loci** **Matrices,** operations (+, multiplication by a number, *), properties Determinant (matrix of order $\leq$ 4), properties; Inverse of a matrix; **Matrix equations** Applications: area of a triangle; coliniarity of three points **volumes etc**. **Loci:** circle, elipse, hiperbola, parabola **Linear systems** ($\leq$ 4 unknowns); matrix form Rank of a matrix. Methods for solving linear systems: matrix method; Cramer rule; Gauss method; Geometric interpretation of a *3x3* linear system Applications: halfplanes,	Elements of algebra Relations of equivalence. Partitions Operations: internal, table of operations, properties **Group:** definition, examples: numerical groups; matrix groups **other groups* Morphisms and isomorphism of groups **Finite groups* **Subgroup* Rings: definition, examples (□ , □ $_n$, rings of functions, polynomials, square matrices) **Subrings** Fields: definition, examples (□ ,□ ,□ ,□ $_p$, p prime) **Morphism and isomorphism** Vector spaces and linear operators Vector space over a commutative field, definitions, examples, basis Vector spaces □ ^{2}, □ ^{2},	

	Elements of mathematical logic -statement, proposition, value of truth -predicat, quatifiers -elemenatry logic operations -_*necessary conditions; sufficient conditions_ -various kinds of mathematical reasonments: reduction ad absurdum, mathematical induction **Functions** -Cartesian product, representation by points -function: definition, equality, graph etc. -linear function -monotonicity, sign, graph representation -linear systems of two equations -linear inequations -separation of the plane into regions -square function: monotonicity, graph, sign -applications to inequations of second degree Symmetric systems of the form $\begin{cases} x+y=s, \\ xy=p \end{cases}$. Systems of the form $\begin{cases} mx+n=y, \\ ax^2+bx+c=y \end{cases}$ _*solving systems reducible to systems already studied;_	-relative positions in the space: lines and planes in space, two planes in space (incident, parallel, perpendicular) -angle of two lines, of two planes, of a line and a plane. Finding distances, areas or volumes using vectors or without _*Problems_ (sections, Cavalieri principle; inscribed, circumscribed bodies) Polynomials algebraic form; degree, +, *; root; polynomial function, equations division; fundamental theorem; division by _x-a_ _*Divisibility of polynomials_ factor decomposition. Bezout theorem relations between roots and coefficients (Vieta formulas) Roots of polynomials with real, rational, integer coefficients Elements of combinatorics -Problems of counting -permutations, arrangements, combinations -Newton binomial Elements of statistics and probabilities -Statistical data; representations -frequency, means; -operations with events; probability _*Classical schmes_ (Poisson, Bernoulli) and others	elements of linear programming Elements of Calculus Sequences of real numbers real line monotonicity, boundedness; bounds of a set Limit; convergent sequences; examples; Operations with sequences having a limit; excepted cases Limit of monotonic sequences. Criterion Number _e_; sequences in this class _*Comparison criteria_ Limit of a function: at a point, geometric interpretation Elementary functions; Operations with limits. Computation of limits. Excepted cases. Methods of eliminating undeterminations Continuous functions at a point; discontinuity; operations with continuous functions; properties; applications (equations, inequations) _Derivatives_ Differentiability at a point; on an interval; derivative geometric interpretation; differential Rules of derivation; derivation of elementary functions; of composed functions	Canonical basis Linear operators, projections _*other_ Matrix form of a linear operator. Operations Elements of calculus The notion of integral (area, examples) Primitives. Undefinite integral. Common primitives. Methods for finding primitives Integration by parts Change of variable Integration of rational functions Definite integral Fundamental theorem of calculus (formula Leibniz-Newton) Liniarity of integral, aditivity with respect the domain of integration _*Riemann integral of functions with a finite number of discontinuities_ Methods for computation of definite integrals Integration by parts Change of variable Applications -Areas formed by two curves -Volumes _*Others: length of graph; Area of rotating surface; center of gravity_ _*Differential equations_ (elemetary): examples, differential equations with separable variables; linear differential equations of order I and II.	

	systems of inequations -operations with functions: +, -, ° -**inverse*; other examples of functions **Geometry and trigonometry** **5. Parallelism and vector algebra** -vectors, equality, sum -product of a vector by a real number -decomposition over two given directions -coliniarity of two vectors; problems of coliniarity -Cartesian system, coordinates; distance between two points -coordinates of a vector, sum, product -equation of a line determined by a point and a line; by two points -recognizing parallelism and congruency of two lines -**translation*; applications -**homotety*; applications **6. Metric relations in the plane** -solving the right angle triangle -trigonometric circle; sin, cos, tan, ctan -reduction to the first quadrant; fundamental trigonometric formulas: sin $\sin^2 x + \cos^2 x = 1; \cos$		Fermat, Rolle, Lagrange theorems; Applications L'Hopital rules Graphic representation of function using derivatives the role of first derivative; monotonicity, extremal points the role of second derivative: convexity and concavity, inflexion point Drawing the graph (including asymptotes) Graphic representations of conics (circle, elipse, parabola, hiperbola) **Geometric properties of them*		

	-other trigonometric formulas (sin 2x, cos 2x, tan 2x, **sin x/2* etc.) -**transforming sums in products, expressing sin x, cos x, tan x as a function of tan(x/2)* -various ways for computing the length of a segment and measure of an angle -sine theorem, cosine theorem, solving arbitrary triangles -**angles and distances in the space* -trigonometric functions sin, cos, tan, ctan: **parity*, periodicity; graph -solving fundamental trigonometric equations; **equations reducible to them*				

1.10 General Pedagogical Approach

The growing popularity of educational programs tailored to the special needs of gifted students in Mathematics makes it especially important that educational research findings be used to support the rationale for providing such programs.

One of the major challenges in gifted education is convincing policymakers of the need for specialized personnel and differentiated learning models to serve gifted students (Gallagher, 1997; Renzulli, 1982; Renzulli & Reis, 1998) by challenging the hackneyed idea that “gifted students can make it on their own”. Communication of related research findings must create an understanding as to *why* traditional teaching methods in regular classrooms are inadequate for serving the needs of gifted students (Park, 1989; Westberg, Archambault, Dobyns, & Salvin, 1993).

Giftedness in Mathematics

The basis for any discussion regarding teaching at a level appropriate for mathematically gifted students begins with a general understanding of giftedness before moving to a specific understanding of mathematical giftedness, which is difficult because there is no universally accepted definition of general giftedness (Gagne, 1995; Morelock, 1996; Sternberg, 1993). This fundamental lack of agreement extends to mathematics, where differing descriptors of high mathematical performance and ability are evident in the literature (Sowell, I993) documented a variety of literature-based adjectives to describe exceptional mathematics students. These descriptors include “promising”, “high-end learners”, “gifted and talented”, and “academically superior”. This multiplicity of descriptors within the specific domain of mathematics parallels the plurality of descriptors of giftedness in general.

Despite such different descriptions of mathematics students with high potential, the literature discussing these students (Sowell et al., 1990) agrees that mathematically gifted students are able to do mathematics typically accomplished by older students or engage in qualitatively different mathematical thinking than their classmates or chronological peers. This literature also frames a picture of mathematical talent that corresponds to an understanding of giftedness as a dynamic and emerging trait. The NCTM Task Force on Mathematically Promising Students (Sheffield, 1999) recognized that mathematically gifted students come in all sizes, ages, and levels of academic achievement and noted that they may not possess identical traits. Furthermore, the task force avoided defining mathematical promise as giftedness. Instead, they defined mathematically promising students as "those who have the potential to become the leaders and problem solvers of the future" (Sheffield, p. 9).

It is widely accepted that mathematically gifted students have needs that differ in nature from those of other students. Mathematics gifted students differ from their classmates as far as it concerns the pace at which they learn, their depth of understanding and the interests that they hold. They, also, differ from the general group of students in the following abilities: spontaneous formation of problems, flexibility in handling data, mental agility of fluency of ideas, data organization ability, originality of interpretation, ability to transfer ideas, and ability to generalize. Mathematics gifted students can handle high levels of abstraction, and they have strong critical thinking skills. They are likely to understand mathematical ideas quickly and take a creative approach to solving problems. They are able to see relationships between mathematical concepts and procedures, work systematically and accurately and apply their knowledge and skills to new and unfamiliar situations. Moreover, they persist in completing tasks. As a result, they require some differentiated instruction, defined by Tomlinson, Callahan, Moon, Tomchin, Landrum, Imbeau, Hunsaker, and Eiss (1995) as "consistently using a variety of instructional approaches to modify content, process, and/or products in response to learning readiness and interest of academically diverse students." Yet recent studies have found few instructional or curricular modifications in regular elementary classrooms. Two different didactical approaches have been used as a differentiation mode for mathematics gifted students that is acceleration and enrichment programs. However, most experts recommend a combination of them.

Motivating Gifted Students

The question of *motivating* talented students has become a leading concern for educators in mathematics in many countries. The motivation of students becomes especially relevant to mathematics education in Europe in light of recurring challenges how to attract more talented students to remain in the European continent and contribute to the European Union's vision to achieve the highest economic and scientific development.

Challenge is a means that the teacher can use to increase talented students' intrinsic motivation. No matter where they obtain their education, mathematically talented students need an appropriately differentiated curriculum designed to address their individual characteristics, needs, abilities, and interests. Overall, there seem to be some important aspects in creating motivating curriculum programs for mathematically talented students (Miller, 1990; Stanley, 1991). Motivation and positive attitude towards Mathematics in general is a kind of internal drive that leads talented students to pursue a course of action. There has been extensive research on the role of attitudes and motivation in learning mathematics. The findings show that positive attitudes and motivation are related to success in learning. Unfortunately, the research cannot indicate precisely how motivation affects learning. That is, we do not know whether it is the motivation that produces successful learning or successful learning that enhances motivation.

Mathematics Gifted Students Curriculum

Although mathematics is generally considered a strand in the theory of intelligence (Gardner, 1999; Sternberg, 1985), the nature of being mathematically gifted and how the needs of mathematically gifted students can be met are relatively unexplored areas. Thus far, research studies have demonstrated the need for gifted students to have access to advanced mathematical content (Johnson & Sher, 1997) and exposure to authentic and challenging mathematics problems (Johnson, 1993; Kolitch & Brody, 1992).

However, mathematics curricula and instructional modifications made for gifted students are often inappropriate because of the highly repetitive nature of the courses and their lack of depth (Johnson & Sher, 1997; Kolitch & Brody, 1992; Park, 1989; Westberg et al., 1993). Thus, there is a strong need for research about the kinds of educational experiences that should be provided for mathematically gifted students, as well as research into the use of technological tools that could effectively and appropriately enhance instruction.

The mathematics gifted students curriculum should bring students to work collaboratively (Tomlinson et al., 1995). Students will benefit greatly, both academically and emotionally, from this type of experience. They will learn from each other, reinforce each other, and help each other over difficulties. Talented students learn best in a nurturing, emotionally safe, student-centred environment that encourages inquiry and independence, includes a wide variety of materials, is generally complex, and connects the school experience with the greater world.

The following are suggestions for differentiating by using (1) assessment, (2) curriculum materials, (2) instructional techniques, and (4) grouping models. These opportunities should be made broadly available to any student with interest in taking advantage of them.

- Pre-assessments should be given to students so that gifted students who already know the material do not have to repeat it but may be provided with instruction and activities that are meaningful. In the elementary grades, gifted learners still need to know their basic facts. If they do not, don't hold them back from other more complex tasks, but continue to work concurrently on the basics.
- One of the characteristics of talented learners is that they themselves strive for mastery of mathematical topics and techniques and for their in-depth understanding. They should, however, tackle challenging work regularly because standard work in class does not suffice. Talented learners need to solve harder problems, beyond the requirement of the standard curriculum, so teachers should create assessments that allow for differences in understanding, creativity, and accomplishment; give students a chance to show what they have learned. Ask students to explain their reasoning both orally and in writing.

- Teachers should choose textbooks that provide more enriched opportunities. Math textbooks often repeat topics from year to year in the grades prior to algebra. Since most textbooks are written for the general population, they are not always appropriate for the gifted. Multiple resources are necessary. No single text will adequately meet the needs of these learners.
- The mathematics curriculum should stress mathematical reasoning and develop independent exploratory skills (Niederer & Irwin, 2001). For instance, this is exemplified by using problem solving and discovery learning, engaging in special projects in mathematics, discovering formulas, looking for patterns, and organizing data to find relationships. Activities should help students to develop structured and unstructured inquiry, reinforce categorization and synthesis skills, develop efficient study habits, and encourage probing and divergent questions. Mathematically talented students need more time with extension and enrichment opportunities. The scope of the mathematics curriculum should be extensive so that it will provide an adequate foundation for students who may become mathematicians in the future. The mathematics curriculum should be flexibly paced (on the basis of an assessment of students' knowledge and skill). Curricula for mathematically talented students should promote self-initiated and self-directed learning and growth. Content, as well as learning experiences, can be modified through acceleration, compacting, variety, reorganization, flexible pacing, and the use of more advanced or complex concepts, abstractions, and materials.
- Inquiry-based, discovery-learning approaches that emphasize open-ended problems with multiple solutions or multiple paths to solutions are extremely effective. Students can design their own ways to find the answers to complex questions. A lot of higher-level questions in justification and discussion of problems should be posed.
- An effective instructional technique for gifted students that promotes self-initiated and self-directed learning is the use of a-didactic situations. In the "Theory of the Situations" of G. Brousseau (1997), the a-didactic situations have three phases: phase of action, phase of formulation and phase of validation. The phase of action corresponds to mathematics in reality and consists of making proper the decisive strategies in a situation of concreteness. The phase of communication consists of finding a code of communication to communicate the strategy being used. Finally, the situation of validation is that in which the participants decide who came up with the optimal strategy. In order to answer this question, the students have to formulate "theorems in action" that allow the optimisation of possible solutions. Thus, from a pedagogic point of view, the "game" assumes a very important role. The student learns to move from the phase of action to the public negotiation (in class and without the direct intervention of the teacher) of all the possible strategies (the theorems in action). The teacher prepares the a-didactics situation and remains arbiter of the rules that need to be respected. All the phases are directly managed by the students.
- Technology can provide a tool, an inspiration, or an independent learning environment for any student, but for the gifted it is often a means to reach the appropriate depth and breadth of curriculum and advanced product opportunities. Calculators can be used as an exploration tool to solve complex and interesting problems. Computer programming is a higher level skill that enhances problem solving abilities and promotes careful reasoning and creativity. The use of a database, spreadsheet, graphic calculator, or scientific calculator can facilitate powerful data analysis. The World Wide Web is a vast and exciting source of problems, contests, enrichment, teacher resources, and information about mathematical ideas that are not addressed in textbooks. Technology is an area in which disadvantaged gifted students may be left out because of lack of access or confidence. It is essential that students who do not have access at home get the exposure at school so that they will not fall behind the experiences of other students.

PART I. IDENTIFICATION

1.11 Aims of identification

One widespread assumption is that mathematically talented students are born that way and eventually blossom (Marjoram & Nelson, 1985).[1] However, this is not always the case. Some may never be identified as mathematically talented individuals. The usual method for identifying such students in European countries is through competitions but it is generally accepted that many talented students in mathematics are never discovered simply because they do not participate in competitions or simply because they were not among the top ten during the competition process or they cannot perform under strict time limits.

Furthermore, some talented students may find themselves in mathematically "poor" learning environments and never reach their highest potential. Still others are not motivated enough and find that other things are far more rewarding and they lose their interest in mathematics for pursuits that offer tangible rewards (Mingus, 1999). Only a few find themselves in mathematically "rich" learning environments in which the teacher is well educated in mathematics, the school is supportive, they have ample opportunities to develop their abilities, and the public/private organizations and universities reward and promote mathematical achievement (Perleth & Heller, 1994). These children require appropriate and challenging learning experiences to facilitate their cognitive and emotional development (Henningsen & Stein, 1997; Hoeflinger, 1998).
First, mathematically talented students need to be identified in early stages and in a systematic way (Kissane, 1986). The information available on mathematically talented children is mostly based on research conducted on children at high-school level (Niederer et al., 2003). However, researchers and educators alike emphasize the value of early identification of talented children (Clark, 1997). While talents have been recognized in many cases at an early age, doubts about the accuracy of identification and the objectivity of parents and teachers linger (Buescher, 1987).

The Elements of Mathematical Talent
Mathematical talent refers to an unusually high ability to understand mathematical ideas and to reason mathematically, rather than just a high ability to do arithmetic computations or get top grades in mathematics (Miller, 1990; Stanley & Benbow, 1986). According to many scholars, terms such as *mathematically talented*, *mathematically gifted*, and *highly able in mathematics* are generally used to refer to students whose mathematics ability places them in the top 2% or 3% of the population. Some characteristics that may yield important clues in discovering mathematically talented individuals are the following:

1. An unusually keen awareness of and intense curiosity about mathematics.
2. An unusual quickness in learning, understanding, and applying mathematical ideas.
3. A high ability to think and work abstractly and the ability to see mathematical patterns and relationships.
4. An unusual ability to think and work with mathematical problems in flexible, creative ways rather than in a stereotypic fashion.
5. An unusual ability to transfer learning to new, untaught mathematical situations. (Krutetski, 1976; Maitra, 2000; Miller, 1990)

Clearly, not all students who achieve the highest test scores or receive the highest grades in mathematics class are necessarily mathematically talented. Besides, many of the mathematics programs in our schools are heavily devoted to the development of computational skills and provide little opportunity for students to demonstrate the complex types of reasoning skills that are characteristic of truly talented students (Miller, 1990; Span & Overtoom-Corsmit, 1986). While high achievement in school certainly can be a clue to high ability in mathematics, additional information is needed (Buescher, 1987).

[1] Some scholars (e.g. Gagné, 1985) prefer the term "mathematically gifted" to "mathematically talented" because they view giftedness as an inherent ability rather than, as the term "talent" might suggest, the manifestation of some outstanding performance. Obviously, there are certain advantages and disadvantages from using either of these two terms.

On the other hand, some mathematically talented students do not demonstrate outstanding academic achievement or enthusiasm toward school mathematics programs (Tirosh, 1989). Many mathematically talented students do not appear to be challenged fully by their schoolwork and are in need of special guidance and attention to help reveal and develop their full potential (Grassl & Mingus, 1999). In such cases their ability in mathematics can be easily overlooked, even though they may exhibit other clues suggesting high ability in mathematics. Therefore, it is highly important to develop successful ways of identifying the abilities of these students.

1.12 Methods of Identification

In general, the ways that are being used to identify mathematically talented students can be divided into two major categories: (a) standardized tests; and (b) in-depth interviews and attitude surveys supplemented with data gathered from administering special mathematical tasks. All this evidence can lead to the creation of a *Talent Portfolio*. The two methods of identification of talented students are described below. Then we combine these methods to propose a systematic process that identifies mathematically talented students.
A. Standardized Tests (adapted from Miller, 1990; Clark, 1997)

Intelligence Tests
IQ test results may provide some clues to the existence of mathematical talent. Used alone, however, these tests are not sufficient to identify mathematical talented students. Mathematical talent is a specific aptitude, while an IQ score is a summary of many different aptitudes and abilities.

Creativity Tests
There are differing opinions on how the results of creativity tests can be used to help identify mathematical talent. Although mathematically talented students display creativity when dealing with mathematical ideas, this is not always apparent in creativity test results. However, high creativity assessments, along with indications of high interest in mathematics, may provide significant clues of mathematical talent.

Mathematics Achievement Tests
Mathematics achievement tests also can provide valuable clues in identifying mathematical talent, but the results of these tests have to be interpreted carefully. Mathematics achievement tests are often computation-oriented and give little information about how a student actually reasons mathematically. Also, the tests seldom have enough difficult problems to appropriately assess the upper limits of a talented student's ability or show that this ability is qualitatively different from that of other very good, but not truly mathematically talented, students. If these limitations are kept in mind, the results of mathematics achievement tests can be useful. Students scoring above the 95th or 97th percentiles on national norms may have high ability in mathematics, but more information is needed to separate the high achievers from the truly gifted. It should not be assumed that there are no mathematically talented students among those scoring below the 95th percentile; those students will have to be recognized through other methods.

Mathematics Aptitude Tests
Standardized mathematics aptitude tests may be used in basically the same way as mathematics achievement tests. Aptitude tests have some of the same limitations as achievement tests except that, because they are designed to place less emphasis on computational skills and more emphasis on mathematical reasoning skills, the results from these tests may often be more useful in identifying mathematically talented students.

Out-of-Grade-Level Mathematics Aptitude Tests
Finally, many of the limitations associated with mathematics achievement or aptitude tests can be addressed by administering out-of-grade-level versions of aptitude tests. This process is supposed to be used only with students who already have demonstrated strong mathematics abilities on regular-grade-level instruments or those who show definite signs of high mathematics ability. The advantage of these tests is that they provide a much better assessment of mathematical reasoning skills because the student must find ways to solve problems, many of which he or she has not been taught to do. As it is reported in the literature (particularly in the US over the past two decades), the out-of-grade-level testing procedure has been used successfully in several mathematics talent searches and school mathematics programs with junior and senior high school students. More recently, similar programs have been used successfully to identify mathematically talented students in the elementary grades.
B. In-depth Interviews and Attitude Surveys Supplemented With Data Gathered from Administering Special Mathematical Tasks

In this category, the following methods of identifying mathematically talented students may be used: in-depth interviews with students, parents and teachers to gather information about students' self-confidence, attitudes, and interests (see *Interview Questions List*); attitude surveys; and, carefully constructed mathematical tasks in which the students have the opportunity to explain to the interviewer their reasoning methods for problem-solving. This data collection methodology may eventually lead to the creation of a *Talent Portfolio*, a vehicle for systematically gathering and recording information about a student's abilities, interests, and learning style preferences.

C. A Systematic Process to Identify Mathematically Talented Students

Obviously, identifying mathematically talented students is not a simple task, and there is more than one way to go about it. Some common features of successful identification processes are combined in the following model (adapted from Miller, 1990; Clark, 1997).

Phase 1: Screening

The objective in phase one is to screen the students suspected of having high ability in mathematics. These students will be further evaluated in the next phase.

Step One. An *identification checklist* may be created to gather some clues that suggest mathematical talent. For example, students scoring above the 95th percentile on a mathematics aptitude test are entered first. Next, those scoring above the 95th percentile on mathematics achievement tests (who are not already on the list) are added. In a similar manner, students who have high IQ scores; students who are creative and have high interest in mathematics; and students nominated by parents, teachers, self, or peers can be added.

Step Two. The checklist information for each student is reviewed and initial interviews with students are conducted. If the information collected for a particular student suggests that out-of-grade-level testing is not advisable, that student's name should be removed, because phase two testing may damage the egos of students who do not really excel in mathematics. However, caution should be exercised not to eliminate talented students in this process. Parent involvement in these decisions is also recommended.

Phase 2: Administration of out-of-grade-level mathematics abilities assessment (or a combination of other standardized tests)

The objective in phase two is to separate the mathematically talented students from those who are merely good students in mathematics and to begin assessing the extent of the ability of the mathematically talented students.

Step One. Students who are scheduled to take the out-of-grade-level test, along with their parents, should be informed about the nature of this test and the reason it is being given. The out-of-grade-level test would then be administered with student and parent consent. It is reported in the literature that the out-of-grade-level test is usually designed for students one and one-third times the age of the child being tested. A sample testing schedule is provided below (Figure 1)

Current Grade (Fall)	Out-of-Grade-Level Test
1st	3rd grade - Fall
2nd	4th grade - Fall
3rd	5th grade - Spring
4th	7th grade - Fall
5th	8th grade - Fall
6th	9th grade - Spring
7th	11th grade - Fall
8th	12th grade – Fall

Figure 1. Sample Testing Schedule

Step Two. The results of each student's out-of-grade-level test should be evaluated in conjunction with the results of phase one screening. The student's out-of-grade-level score will provide an indication of degree of mathematical talent. Scores above the 74th percentile represent a degree of mathematical talent. This level of talent places the student in the upper 1% of the population in mathematics ability. Scores above the 64th percentile denote a level of talent that most likely places the student in the upper 3% of the population. Students in these two groups would be identified as mathematically talented.

Phase 3: Finally, the objective in phase three is to confirm the talents and needs of mathematically gifted students. Here we can use in-depth interviews with each student identified as talented, and interviews with parents and teachers to gather information about the student's self-confidence, attitudes, interests etc. We may administer attitude surveys supplemented with carefully constructed mathematical tasks in which each student has the opportunity to explain to the interviewer his or her reasoning methods for problem-solving. All the data collected can be included in a *Talent Portfolio*.

INTERVIEW QUESTIONS LIST

For Students

1. What types of mathematical activities do you like most? Why?

2. What are your primary goals and expectations from being engaged in mathematics?

3. How have these goals or expectations been fulfilled or how have they changed since the beginning of this school year? Last year? A few years ago?

4. How often and when do you have the opportunity to engage in mathematical activities that you find *challenging*? Why do you find these activities challenging?

5. What do you think makes someone good in mathematics?

6. Which skills and abilities do you think you possess in mathematics? What do you identify as your strengths and special characteristics? Do you have a favorite way of learning mathematics?

7. Are you good at solving problems? Why? What do you do when you solve a problem? Are there any steps you follow?

8. When you have finished doing mathematics, do you usually think about what you did and what worked well while you were engaged in mathematics?

9. Do you think you can get better in doing mathematics? How?

10. Tell me a story about an exciting time when you were engaged in doing mathematics. Why was this important for you?

11. How have your peers influenced your experiences at learning mathematics?

12. What do you think are the main strengths of your mathematics program? What are its principle weaknesses or areas for improvement?

13. How has your class/school/teachers added to or changed your understanding of mathematics?

14. What are the most interesting or important things you have learned about mathematics in your class/school?

15. Has being engaged in mathematics influenced your interest in higher education and/or career options? How?

16. Anything else that you want to add?

For Teachers

General questions:

1. Is there a well-established mathematics curriculum that is followed in your school? How does it apply to mathematically talented students? What opportunities or specialized services does the school provide specifically for mathematically talented students?

2. Does the current procedure in your school (system) adequately identify mathematically talented students? Are the criteria and procedures flexible enough for students with a range of skill levels in mathematics? In your opinion, what is the strongest component and what is the weakest component of these procedures?

3. Does the current mathematics program meet the needs of talented students in your school? How? How could it be done differently?

4. How much flexibility do you have when it comes to developing your students' mathematical talents – not only in terms of intellectual needs but also in terms of psychological or social needs?

5. How do you document the needs of mathematically talented students? How do you respond to those needs? What is the most common way that you evaluate these students (e.g. paper-pencil tests, homework, portfolios, etc.)? How do you monitor the progress of these students? How could you do this differently?

6. To what extent are students encouraged to work with other students (of the same grade level/of a different grade level?) When and under what circumstances might this happen?

7. Are mathematically talented students assigned specialized projects? How are the projects developed? Do these students work independently or with you (or other teachers) to complete these projects?

8. Do you have opportunities for curriculum planning and professional development in how to identify and motivate mathematically talented students? What kinds of such opportunities have you had so far? Are you satisfied? What new opportunities do you need to be able to do this job more successfully?

9. Are there teachers who are trained in or have experience working with mathematically talented students? Do you feel comfortable working with mathematically talented students?

10. Are students permitted to leave your class for specialized study (e.g. to take classes at a local college, to participate in an advanced program)?

11. To what extent are mathematically talented students encouraged to apply for various scholarships? What is the process?

12. How do you work with mathematically talented students to develop their organizational skills, discipline and work habits?

Specific questions concerning a student suspected to be mathematically talented:

1. What is your general feeling about Student A? What are his/her strengths and weaknesses?

2. What makes you think that Student A is a mathematically talented student? What are some of the characteristics that you think make this student mathematically talented? How do you know this?

3. How well does Student A work independently? What kinds of strategies does he/she use to solve problems? What kinds of concerns does he/she express about being engaged in mathematics? How do you handle these concerns?

4. What strategies do you use to: (a) identify the needs of Student A and (b) support and motivate this student in order to develop his/her mathematical abilities? Do these strategies differ from student to student? If so, how?

5. What kind of communication/collaboration do you have with Student's A parents? How do you handle their concerns?

6. Anything else that you want to add about Student A?

For Parents

1. What are you child's interests, goals, strengths, and weaknesses in mathematics?

2. What makes you think that your child is a mathematically talented student? What are some of the characteristics that you think make your child mathematically talented? How do you know this?

3. How well does your child work independently? What kinds of strategies does he/she use to solve problems? What kinds of concerns does he/she express about being engaged in mathematics? How do you handle these concerns?

4. What strategies do you use to: (a) identify the needs of your child and (b) support and motivate your child in order to develop his/her mathematical abilities? What assistance/information do you need to provide better support to your child?

5. What kind of communication/collaboration do you have with your child's mathematics teacher? What are your requests/suggestions to his/her teacher?

6. Does the current school mathematics program meet the needs of your child? How? How could it be done differently?

7. What kinds of support/resources are provided by the teacher/school to your child to motivate his mathematical interests and abilities? Are you satisfied? Why? How could this support be improved?

8. Anything else that you want to add about your child?

1.13 Tests

IDENTIFICATION OF MATHEMATICAL TALENTED STUDENTS
by Mircea Becheanu

Part I (Age: 10-11.)

GEOMETRY.

Problem 1.
On a sheet of paper a square of area $4cm^2$ is drawn. Using a new sheet of paper and a pair of scissors, to construct a square of area $9cm^2$.

Solution.
Cut the given square into four unit squares. Then construct the new square from nine unit squares.

Problem 2.
Three segments arising from the point M , say $MA,\ MB,\ MC$ are constructed such that

$\angle AMB = \angle BMC = 45°$ *and* $MA^2 = MC^2 = \frac{1}{2}MB^2$. Find $\angle MAC$.

Solution.
The figure $MABC$ is a square. Hence AC is a diagonal and $\angle MAC = 45^0$.

Problem 3.
The triangle BXC is isosceles such that $BX = CX = 4$. How many equilateral triangles ΔABC whose sides have integer lengths can be constructed, such that X lies inside these triangles.

Solution.
Since BC should be an integer, it is one of the numbers $1,2,...,7$. Taking into account the condition that X is inside the triangle ABC it follows that $BC < 4$, so there are three triangles.

Problem 4.
An arbitrary triangle is given. Show that it is possible to cover the plane with infinitely many triangles identical with the given triangle such that two arbitrary triangles do not overlap.

Solution.
Two triangles identical with the given triangle are joined to obtain a parallelogram. Infinitely many such parallelograms are joined to obtain a strip bounded by two parallel lines. The plane can be covered by infinitely many strips.

Problem 5.
Let M be an arbitrary point on the side BC of the square $ABCD$. The angle bisector of the angle MAD meets the side CD in the point K . Show that $BM + DK = AM$.

Solution.
Take the point N on the line CD such that $DN = BM$ and D is between C and N . Then the triangles ABM and ADN are congruent (equal). It follows that $AN = AM$, $\angle DAN = \angle BAM$. Now it is easy to see that the triangle AKN is isosceles and therefore $AM = AN = NK = ND + DK = BM + DK$.

NUMBER THEORY

Problem 6.
Find the least positive integer which has three digits and whose remainder by division to 2, 3 and 5 is 1.

Solution.
The number 1 satisfies the divisibility conditions. Any other number which satisfies the conditions is obtained from it by adding a multiple of $2.3.5.=30$. So, the required number is 121.

Problem 7.
How many numbers are divisible by 13 among the first 1000 positive integers? How many numbers are relatively prime to 13 in the same number set?

Solution.
There are $\left[\frac{1000}{13}\right]=76$ numbers which are divisible by 13. A number is relatively prime to 13 if and only if it is not divisible by 13. So, there are 1000-76=924 numbers relatively prime to 13.

Problem 8.
Find all three-digit numbers with sum of digits equal to 5, and such that the numbers:
a) do not contain any zero
b) may contain zero.
Find the smallest of these numbers in each of the cases a) and b).

Solution.
The partitions of 5 with at most 3 parts are:

$$1+1+1;\quad 1+2+2;\quad 2+3;\quad 1+4;\quad 5.$$

Starting from these one may construct all required numbers, either containing or not containing a zero. The least number in case a) is 113 and in case b) is 104.

Problem 9.
a) Show that 27 is the sum of three consecutive integers.
b) Show that 3^5 is the sum of three consecutive integers.
c) Show that any number divisible by 3 is the sum of three consecutive integers.
d) Show that 3^{100} is the sum of nine consecutive integers.

Solution. The answer is based on the identity:

$$3k=(k-1)+k+(k+1).$$

Problem 10.
Find the largest integer, which does not have two identical digits and the product of its digits equals 72.

Solution.
The possible factorisations of 72 with no repeating factors are: $1\cdot8\cdot9$, $1\cdot2\cdot4\cdot9$, and $1\cdot3\cdot4\cdot6$. The greatest number is then 9421.

Problem 11.
Show that the equation $x^3+y^4=7$ has no solution in positive integers.

Solution.

Taking the equation modulo 13 we have for x^3 the possible residues 0,1,5,8,12 and for y^4 the possible residues 0,1,3,9. So, we can not obtain a sum equal to 7.

ALGEBRA.

Problem 12..

Show that

$$\frac{1}{2.7}+\frac{1}{7.12}+\frac{1}{12.17}+....+\frac{1}{47.52}<\frac{1}{10}.$$

Solution.

One may use the identity:

$$\frac{1}{n(n+5)}=\frac{1}{5}\left(\frac{1}{n}-\frac{1}{n+5}\right).$$

Then the sum can be computed by telescoping it into $\frac{1}{10}-\frac{1}{5.52}$. Hence the required inequality is obvious.

Problem 13..

We are given the numbers

$$\frac{a_1}{b_1}=1;\ \frac{a_2}{b_2}=2;\ ....\ \frac{a_{100}}{b_{100}}=100.$$

Find the number

$$A=\frac{a_1+a_2+...+a_{100}}{b_1+2b_2+...+100b_{100}}.$$

Solution.

From $a_1=b_1,\ a_2=2b_2,...,\ a_{100}=100b_{1000}$ it follows that

$$a_1+a_2+...+a_{100}=b_1+2b_2+...+100b_{1000}$$

and $A=1$.

Problem 14.

We are given two natural numbers. If we add the triple of first number to the half of the second we get the same result as when we add first number and twice of the second. Find two such numbers.

Solution.

Let a and b the numbers. From $3a+\frac{b}{2}=a+2b$ one gets $4a=3b$. So, $a=3k$ and $b=4k$.

These are all numbers with the stated property.

Problem 15.
Let x, y, z be positive numbers such that

$$x=\frac{2y}{y+1},\ \ y=\frac{2z}{z+1},\ \ z=\frac{2x}{x+1}.$$

Prove that $x=y=z=1$.

Solution.
Assume by way of contradiction that $x>y$. Since $x-y=\dfrac{2(y-z)}{(y+1)(z+1)}$ it follows that $y>z$.

Then we repeat the argument to obtain $z>x$. We have a contradiction. Hence $x=y=z$. It is easy to see that their value is 1.

Problem 16.
We are given seven distinct positive integers which add up to 100. Show that there are three of them whose sum is not less than 50.

Solution.
Assume that the numbers are $a_1<a_2<\ldots<a_7$.
If $a_4\geq 15$ we have

$$a_5+a_6+a_7\geq 16+17+18=51$$

and we are done.
If $a_4\leq 14$ we have

$$a_1+a_2+a_3+a_4\leq 11+12+13+14=50.$$

In that case,

$$a_5+a_6+a_7\geq 100-50=50.$$

Problem 17.
Let x, y be positive numbers. Show that

1) $\dfrac{x^2}{x+y}\geq\dfrac{3x-y}{4}$;

2) $\dfrac{x^3}{x+y}+\dfrac{y^3}{y+z}+\dfrac{z^3}{z+x}\geq\dfrac{1}{2}\left(x^2+y^2+z^2\right)$.

Solution.
The point (1) can be proved by direct computation:

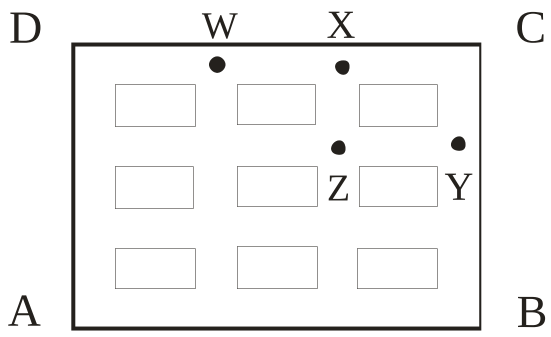
$$\frac{x^2}{x+y}-\frac{3x-y}{4}=\frac{(x-y)^2}{4(x+y)}\geq 0.$$

For the point (2) one use the previous point and summing up the inequalities we obtain:

$$\sum\frac{x^3}{x+y}\geq\frac{1}{4}\left(3x^2-xy+3y^2-yz+3z^2-zx\right)\geq\frac{1}{2}\left(x^2+y^2+z^2\right).$$

COMBINATORICS.

Problem 18.
In the figure you can see blocks of houses. There are streets between them. How many different ways are there to go from A to C if we walk through the streets only up and to the right?



Solution. Every walk will pass either through point X or Y. Every walk to X pass either through point Z or W. If one reconsiur in this way all possible walks from A to X one finds 10 walks. So, from A to C we get 20 possible walks. S

Problem 19.
John built a construction from cubes. Figure 1 shows how it seen from the front and Figure 2 from above.
a) Draw a possible side elevation of the construction.
b) Find the minimum and maximum number of cubes necessary to build the construction.

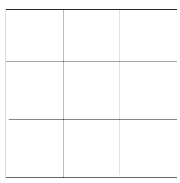


Figure 1

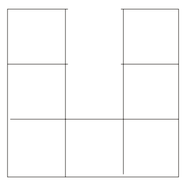


Figure 2

Solution.
The minimum number of cubes is $9+4=13$. The maximum number of cubes is $9+6+6=21$, depending on number of levels on the east and west side in Figure 2.

Problem 20.
Divide the set of integer numbers from 1 to 100 into six groups. Prove that one can always find a group containing two different numbers such that one is a divisor of the other.

Solution.
We have all together seven different powers of 2 in the given set. Consequently, by dividing them into six groups, there will be at least one group which contains two different powers of two, the smaller dividing the greater.

Problem 21.
Three people, say A, B and C, have a conversation.
A says that B is lying.
B says that C is lying.
C says both A, B are lying.
Who is lying, who is telling the truth?

Solution.
If A is telling the truth then B is lying and C tells the truth, which contradicts what C says. If A lies, then B says the truth, and consequently C lies, and what C says is not true indeed, as only A lies. Consequently A lies, B says the truth and C lies.

MIXED PROBLEMS.

Problem 22.

A horse race between three teams is 10 *km* long. Each horse runs with a constant velocity. When the winner finished the race he was 2 km. ahead of the second and *4 km* ahead of the third. How far was the second ahead of the third when it finished the race?

Solution.

Let *x,y,z* be the constant velocity of the first, second and third horse, respectively. Then $\frac{10}{x}=\frac{8}{y}=\frac{6}{z}$. The required distance *d* between the second and the third should satisfy the proportion: $\frac{10}{y}=\frac{10-d}{z}$. Since $\frac{y}{z}=\frac{4}{3}$ it follows d = 2.5*km*.

Problem 23.

The telephone numbers in a small city consist of two digit numbers between 00 and 99. Possibly, not all numbers are used. If a subscribed number is read in the reverse order one obtains either an unused number or a number used by the same subscriber. Find the greatest number of telephone subscribers from the city.

Solution.

We consider a 100x100 array of square boxes and assign in each box one of the numbers 00, 01, 02, . . . ,99, in ascending order, from left to right and from the upper row to the bottom row. Since numbers *ab* and *ba*, which are symmetrical with respect to the diagonal, are used by the same person, it follows that only the numbers above the diagonal are used. So the maximum number of subscribers is

$$1+2+3+...+50=\frac{50\times 51}{2}=1275$$

Problem 24.

On a street there are 150 houses. Every morning three different newspapers, say *A, B, C*, are distributed in the houses. We know that newspaper *A* is distributed in 40 houses, *B* in 35 houses and *C* in 60 houses. Also, in 7 houses are distributed both *A* and *B*, in 10 houses *B* and *C*, and in 4 houses are distributed *A* and *C*. Also, no newspapers are distributed in 34 houses. How many houses receive all three newspapers?

Solution.

Let us denote by *a* the number of houses where the news paper *A* is distributed, by *ab* the number of houses where the newspapers *A* and *B* are distributed, etc. Using the inclusion-exclusion principle (or, alternatively, a Euler-Ven diagram) one has:

$$150-34=a+b+c-ab-bc-ca+abc.$$

Taking in account that $a=40,\ b=35,\ c=60,\ ab=7,\ bc=10,\ ca=4$, one obtains

$$abc=116-\left(40+35+60\right)+\left(7+10+4\right)=2.$$

Part II (Age:13-14)

NUMBER THEORY.

Problem 1.

Show that among the numbers $a_n = 2^4.3^{16} + 5^2.3^{14} + 3^n$, where $n \geq 1$, none is a perfect square.

Solution.
All numbers a_n are even. An even number which is a perfect square is congruent to 0 mod4. We have $a_n \equiv (-1)^{14} + (-1)^n \equiv 1 + (-1)^n \left(mod\, 4\right)$. Hence, n should be odd.

Since $2^4.3^{16} + 5^2.3^{14} = 169.3^{14}$ it follows that $a_n = 169.3^{14} + 3^n = b^2$. If $n < 14$ and odd it follows that a_n contains an odd power of 3 and then, it can not be a perfect square. If $n > 14$ it follows that the number $169 + 3^{n-14}$ is a perfect square, which is impossible. This can be seen from the Diophantine equation $3^x = y^2 - 13^2$.

Problem 2.
Solve in integer numbers the equation: $x + y = x^2 - xy + y^2$.

Solution.
Multiply the equation by 2 to obtain the new equation: $(x-1)^2 + (y-1)^2 + (x-y)^2 = 2$.

Problem 3.
Show that the number $n^2 + 3n + 5$ can not be divisible by 121, for any $n, n \in \mathbb{N}$.

Solution.
From the identity $n^2 + 3n + 5 = (n+7)(n-4) + 33$ we obtain that $n \equiv 4 \left(mod\, 11\right)$.

.

Writing $n = 11k + 4$ and one obtains: $n^2 + 3n + 5 = 121k(k+1) + 33$, which is not divisible by 121.

Problem 4.
Find the least positive integer n with the following property: $\frac{n}{2}$ is a square, $\frac{n}{3}$ is a cube and $\frac{n}{5}$ is a 5^{th} power.

Solution.
The required number is $2^{15}3^{10}5^6$. Indeed, *n* is necessary divisible by 2,3 and 5, so it has the form $n = 2^x3^y5^z$. Moreover, *x* is odd and divisible by 3 and 5, etc.

Problem 5.
Consider all possible products of 2 consecutive integers among which the greatest is a perfect square. Find the largest number which is a common divisor of all these numbers.

Solution.
The numbers are $n^2, n^2 - 1$. There product is $(n-1)(n+1)n^2$. Since $(n-1),\ (n+1),\ n^2$ are three consecutive integers, their product is divisible by 3. If *n* is even, n^2 is divisible by 4. If *n* is odd, $(n-1)(n+1)$ is divisible by 4. Hence, all these numbers are divisible by 12. Since $12 = 3 \times 4^2$ it follows that the required number is 12.

GEOMETRY.

Problem 6.
We are given an arbitrary triangle *ABC.* Construct a new triangle in which an angle equals an angle of *ABC* and whose area is twice the area of ΔABC.

Solution.
Extend the side *AB* with a new segment *BD* such that *AB* = *BD* . Then *ABD* is the required triangle.

Problem 7.
Let *ABC* be a triangle and *D* , *E* be points on the sides *AC* , *AB* respectively. The lines *BD* and *CE* meet in the point *F* such that the areas of triangles *BEF*, *BFC* and *CFD* are 5, 12, and 4 units, respectively. Find the area of *ABC*.

Solution.
Let us denote by *x, y* the area of the triangles *AEF*, *AFD*, respectively. One has the following equalities:

$$\frac{EF}{FC}=\frac{x}{4+y}=\frac{5}{12},$$

$$\frac{BF}{FD}=\frac{5+x}{y}=\frac{12}{4}=3$$

This is a system of equations. By solving it one obtains $x=\frac{85}{31}$ and $y=\frac{80}{31}$. So, the required area is $F_{\Delta ABC}=5+12+4=\frac{85}{31}+\frac{80}{31}=\frac{816}{31}$

Problem 8.
Construct a parallelogram whose area is the same as the area of a given triangle and which has an angle equal to an angle of the triangle.

Solution.
Let *ABC* be the given triangle and fix the vertex *A*. There exists a point *D* in the plane such that *ABDC* is a parallelogram. Draw a line which connect the midpoints of the sides *AB* and *CD*. One obtains the required parallelogram.

Problem 9.
Divide a regular hexagon into 8 equal parts. Define the kind of each part.

Solution.
Denote the hexagon by *ABCDEF* and let *O* be its center. If *M, N, P, K, L, R* are the midpoints of the segments *AO, BO, CO, DO, EO, FO,* respectively. The figures *ABNM, BCPN, CDKP, DELK, EFRL, FAMR, RPNM* and *RPKL* are equal trapezoids.

Problem 10.
The internal bisectors of the angles *A, B, C* of a triangle *ABC* meet the opposite sides at *D, E, F* respectively. Show that if the perpendiculars to the sides at *D, E, F* respectively, are concurrent, then the triangle is isosceles.

Solution.
By the Pythagoras's theorem, concurrency implies: $BD^2+CE^2+AF^2=DC^2+EA^2+FB^2$.

But the segments from above can be estimated as *BD* = $BD=\frac{ca}{b+c}$, etc. Thus

$$\frac{c^2a^2}{(b+c)^2}+\frac{a^2b^2}{(c+a)^2}+\frac{b^2c^2}{(a+b)^2}=\frac{a^2b^2}{(b+c)^2}+\frac{b^2c^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}.$$

Clearing denominators this can be brought into the form: $(a-b)(b-c)(c-a)(a+b+c)=0$
From which the result follows.

Problem 11.

Let *ABC* be a triangle which has the following properties:

- $BC = 2(AC - CB)$;
- there exists a point *K* on the side *BC* such that $\angle ABK = 2\angle AKB$.

Show that $BC = 4CK$.

Solution.

For shortness we denote $AB = c, BC = a, CA = b$. Let *D* be the foot of the altitude from *A* and *E* be a point on the segment *BC* such that $BD = DE$. It follows that the triangles *BAE* and *AEK* are isosceles such that $AB = AE = EK = c$.

.

Now, we turn to the condition $a = 2(b-c)$. Since

$$b^2 - c^2 = AC^2 - AB^2 = CD^2 - BD^2 = a(CD - BD)$$

it follows that $b + c = 2(CD - BD)$. The last equality, together with the condition $a = 2(b-c)$ gives $CE = c + \frac{a}{4}$. Then, $CK = CE - EK = \frac{a}{4}$.

Problem 12.

In a given triangle *ABC* the medians *AD* and *BE* are perpendicular. Determine the length of the third median *CF* in terms of the lengths $BC = a$ and $AC = b$.

Solution.

Let *G* be the common point of the medians and let denote *AB* = *c*. The triangle *AGB* is a right triangle and *GF* is its median corresponding to the right angle. Then $GF = \frac{1}{2}AB = \frac{1}{2}c$. Since $CF = 3GF$, it is sufficient to find *GF*. It is known that the length of a median, say *CF* is given by the formula:

$$CF^2 = \frac{2(a^2+b^2)-c^2}{4}$$

Therefore, we have the equality:

$$\frac{9c^2}{4} = \frac{2(a^2+b^2)-c^2}{4}$$

From this one obtains $c = \frac{\sqrt{a^2+b^2}}{5}$. The result is: *CF* = $\frac{3}{2} \cdot \sqrt{\frac{a^2+b^2}{5}}$.

ALGEBRA.

Problem 13.

Show that the number $\sqrt{n^2 + n + 1}$ is an irrational number, for all positive integers n.(VB)

Solution 1.

The number $\sqrt{n^2+n+1}$ is rational if and only if n^2+n+1 is a perfect square: $n^2+n+1=m^2$, where $m \in \square$. Multiply the above equality by 4 to create a perfect square in the lhs:

$$\left(2n^2+1\right)^2+3=\left(2m\right)^2.$$

From this one obtains the decomposition: $\left(2m+2n+1\right)\left(2m-2n-1\right)=3$. Since 3 is a prime number we have necessary the equalities: $2m+2n+1=3$ and $2m-2n-1=1$. This is a system from which one has $n=0$ and $m=1$.

.

Solution 2. (smarter).

From the inequalities $n^2<n^2+n+1<\left(n+1\right)^2$ it follows that n^2+n+1 can not be a perfect square.

Problem 14.

Let α be a positive real number and a, b be the roots of the equation: $x^2-\left(\alpha+1\right)x+\alpha=0$. Show that if $2, a, b$ are the sides of a triangle, then $1<\alpha<3$.

Solution.

The roots of the equation are 1 and α. So, 1 , 2 , α are the sides of a triangle. By the triangle inequality we have α < 1 + 2 and 2 < 1 + α. The result follows.

Problem 15.

Solve the equation:

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+...+x}=\frac{200}{101}.$$

Solution.

We use the formula: $1+2+...+x=\frac{x(x+1)}{2}$. The equation becomes

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{x(x+1)}=\frac{100}{101}$$

Using the formula $\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$ one obtains the equation $1-\frac{1}{x+1}=1-\frac{1}{101}$, whose solution is $x=100$.

Problem 16.

Find the functions $f:\square\to\square$ which satisfies the condition $f\left(x+1\right)+3f\left(-x\right)=2x+1$ for all $x\in\square$.

Solution.

Set -*x* - 1 instead of *x* in the given condition and obtain the relation:

$$f\left(-x\right)+3f\left(x+1\right)=-2x-1$$

Together with the given condition one obtains a system of linear equations. Solving the system we get $f\left(x\right)=-x-\frac{1}{2}$

Problem 17.

It is given the number $m=\frac{8}{9}+\frac{9}{11}+\frac{11}{13}+\frac{13}{15}$. Find in terms of *m* the number

$$M=\frac{37}{9}+\frac{13}{11}+\frac{41}{13}+\frac{17}{15}.$$

Solution.

We have $\frac{37}{9}=5-\frac{8}{9},\frac{13}{11}=2-\frac{9}{11},\frac{41}{13}=4-\frac{11}{13},\frac{17}{15}=2-\frac{13}{15}$. Hence, *M* = 13 - *m*.

Problem 18.

How many digits does the number

$$N=1234...200320042005$$

have?

Solution.

There are 9 one digit numbers, there are 90 two digit numbers, 900 three digit numbers and 1006 four digits numbers. There are 6913 digits in all.

Problem 19.

How many 0 digits are at the end of the number

$$N=1.2.3...(2004).(2005)?$$

Solution.

We have to count the powers of 2 and 5 in N. Since $N = 2^a5^bM$ where $a > b$ it is sufficient to compute b. The exponent b appears in the multiples of $5,5^2,5^3,5^4$. The number of these multiples is

$$\left[\frac{2005}{5}\right]+\left[\frac{2005}{25}\right]+\left[\frac{2005}{625}\right]+\left[\frac{2005}{625}\right]=401+80+16+3=500.$$

Problem 20.

The function f is defined by $f(t)=\frac{t}{1-t}$ for all $t\neq 1$. Let $f(x)=y$. Find x in terms of f and y.

Solution.

From $\frac{x}{1-x}=y$ follows $x=y-xy$ and finally, $x=\frac{y}{1+y}=-f(-y)$

Problem 21.

Find all real solutions of the equation:

$$(x-y-3)^2+(y-z)^2+(z-x)^2=3.$$

Solution.

Note $x-y-3=a, y-z=b, z-x=c$. Then $a+b+c=-3$ and $a^2+b^2+c^2=3$. By squaring the equality $a+b+c=3$ and setting $a^2+b^2+c^2=3$ one obtains $\sum ab=3$. Since $\sum a^2=\sum ab$ it follows $a=b=c$. From $3a+3=0$ it follows $a=b=c=-1$. Setting these values in the system giving x,y,z one obtains $x-y=2;$ $y-z=-1;$ $z-x=-1$. This linear system has no unique solution. Taking *z* as a parameter one has the solution $(x,y,z)=(z+1,z-1,z)$.

COMBINATORICS.

Problem 22.
Find the number of all possible representations of the number 2005 as a sum of 1's and/ or 2's? Two representations are considered to be identical if they differ by the order of terms in the sums.

Solution.
The number of representations is given by the number of appearances of the number 2 in the representation. Since $2005 = 1 + 2 \times 1002$ it follows that the following representations are possible:

$$\begin{array}{lcl} 2005 & = & 1+1+1+\ldots \quad \ldots +1+1+1+1+1 \\ 2005 & = & 1+1+1+\ldots \quad \ldots +1+1+1+2 \\ 2005 & = & 1+1+1+\ldots \quad \ldots +1+2+2 \\ \ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \quad \ldots \ldots\ldots\ldots\ldots \\ 2005 & = & 1+2+\ldots+2 \end{array}$$

the number of 2's in the last row being 1002. There are 1003 representations in all.

Problem 23.
There are nine distinct points inside a square of side 1, no three on a line. Show that one can find three of them such that the area of the triangle they form is not greater than 1/8.

Solution.
The square is divided into four equal squares of area 1/4. By the pigeonhole principle, there are three points in one of these small squares. The required result follows from the following lemma:
Lemma. The area of a triangle inscribed in a square cannot exceed 1/2 of the area of the square.
Proof. The triangle can be inscribed in a bigger triangle whose vertices are on the sides of the square.
Alternative solution. The given square can be divided into four rectangles of dimensions $(1/4)\times 1$. By the pigeonhole principle there are three points in one of these rectangles. The area of a triangle inscribed in a rectangle cannot exceed 1/2 of the area of rectangle.

Problem 24.
We are given a square of side 1 and 61 points inside it. Show that one can find two points such that the distance between them is not longer than 1/5.

Solution.
One divides each side of the square into five equal segments. These points are connected by lines which are parallel to the diagonals of the square. One obtains 20 isosceles triangles and 40 squares of dimensions $\frac{1}{5\sqrt{2}} \times \frac{1}{5\sqrt{2}}$. So, the square is divided into 60 parts, triangles and squares. By the pigeonhole principle, there are two points in one of these parts. The distance between them is not greater that 1/5.

Problem 25.
Remove a square from the corner of an 8×8 chess-board. How can we tile the 63 squares of the remaining board using 21 tiles of size 1×3 (in blocks)?

Solution.
Take the main diagonal of the chess-board containing the block already removed. Colour the blocks in lines perpendicular to this diagonal with three different colours alternating, say starting beside the removed block (2 blocks) and keeping the order of the colours. You will get 22, 21 and 20 blocks of the same colour. We cannot tile with 21 tiles of size 1×3, because each tiling would contain tiles coloured in all three colours, but their number differs from 21.

Problem 26.
There are three gods sitting in an oracle: the god of Truth, the god of Wisdom and the god of Lies. The god of Truth always tells the truth, the god of Lies always lies and the god of Wisdom sometimes tells the truth and other times lies. One day a philosopher came to visit them. The gods were sitting next to each other and the philosopher wanted to know in what order they are sitting, so he asked the following questions. He asked the one on the left: Who is sitting next to you? The answer was: The god of Truth. He asked the middle one: Who are you? The answer was: I am the god of Wisdom. Then he asked the one on the right: Who is sitting next to you? The answer was: The god of Lies. In what order were the gods sitting in?

Solution.
The one on the left cannot be the god of Truth, as he said the next to me is the god of Truth. He cannot be the god of Lies either, because then Wisdom must be in the middle and the right one the god of Truth, who said the one next to him was the god of Lies, and he tells the truth. Consequently the one on the left is the god of Wisdom. In this case the middle one lies, so he is the god of Lies, and the right one the god of Truth.

Problem 27.
Two kinds of people live on an island, the ones that always tell the truth and the ones that always lie. Once we asked five of them, who knew each other: How many of you are truthful? We got the following answers: 0,1,2,3,4. How many out of these were liars?

Solution.
If at least two truth tellers had been asked, we would have two equal answers. So we can have 0 or 1 truth teller. But if we have 0, then the one who answered 1 would say the truth, contradiction. Consequently we have exactly 4 liars, and one islander is telling the truth.

Problem 28.
Two people are playing a game on a round table. They are each placing an 1 Euro coin in alternate turns. They are not allowed to put a coin over another one nor move a previously placed coin. The loser is the person who cannot complete his move. Who wins and how (supposing they use a good strategy)?

Solution.
The winner is the player who starts, and the winning strategy is to put the first coin in the middle, then place the next coin in a symmetrical position (respectively to the centre of the table) to the last coin placed by the second player.

MIXED PROBLEMS.

Problem 29.
We are given a cube $ABCDA'B'C'D'$ of side $AB=1$. Find the length of the shortest closed path across the faces of the cube and which passes through the vertices $AC'BD'A$ in that order.

Solution.
The path consists of two diagonals of rectangles and of length $\sqrt{5}$ and two diagonals of squares of length $\sqrt{2}$. So, the answer is $2\left(\sqrt{5}+\sqrt{2}\right)$.

Part III (Age:15-16+)

GEOMETRY

Problem 1.
Given a right triangle with legs a and b, construct a square and an equilateral triangle which have the same area as the given triangle.

Solution.
To construct the square it is necessary to construct the segment $\sqrt{ab}$. This is the altitude of a right triangle inscribed in a circle of diameter *a* + *b* and whose foot divide the diameter in segments of lengths *a* and *b*. The same segment can be used to construct the equilateral triangle.

Problem 2.
Let ABC be an isosceles triangle such that $CA = CB$. Find the locus of the points X in the plane such that

$$AX^2 + BX^2 = CX^2.$$

Solution.
We work in coordinates. Take the points $A(-a,0), B(a,0), C(0,c), X(x,y)$.

The condition which defines the locus gives us the equation: $x^2 + y^2 + 2cy = c^2$. This is a circle.

Problem 3.
Let A_1, A_2, A_3 be a triangle and T_1, T_2, T_3 be the tangency points of its sides with the excribed circles. We denote by H_1, H_2, H_3 the orthocenters of the triangles $A_1T_2T_3,\ A_2T_3T_1,\ A_3T_1T_2$, respectively. Show that the lines H are concurrent.

Solution.
We shall use the following preliminary result: if ABC is a triangle, *P* is an interior point of it and *Q, R, S* are the segments of P along the midpoints of the sides *BC, CA, AB*, respectively, then the lines *AQ*, *BR* and *CS* are concurrent at a point *M* which is the common midpoint of the segments *AQ*, *BR*, *CS*. The proof of this statement follows from the fact that *ABQR*, *BCRS*, *ACQS* are all parallelograms and *AQ*, *BR*, *CS* are their diagonals.

In our problem, it is sufficient to prove that H_1 is the reflection of *O* along the midpoint of the segment T_2T_3. Then apply the statement to the triangle T_1, T_2, T_3.

Remark. A solution by complex numbers is also available. It uses the following property: on each side of a triangle the tangency points of the inscribed circle and excribed circle are symmetric with respect to the midpoint of the side.

Problem 4.
Let ABC be a triangle and *M* be the midpoint of the segment *BC*. Consider an interior point *N* such that $\angle ABN = \angle BAM$ and $\angle ACN = \angle CAM$. Prove that $\angle BAN = \angle CAM$.

Solution.
We will show that *N* lies on the reflection *AP* of the line *AM* in the angle bisector of $\angle BAC$. Clearly, *N* is uniquely defined by its properties; therefore, if one takes a point $S \in AP$ with these properties the conclusion follows.

Take D the reflection of A in M and define S by the ratio: $\dfrac{AS}{BS}=\dfrac{AC}{AD}$ Then, the triangles BAS and DAC are similar and hence we obtain $\angle ABS=\angle ADC=\angle MAB$. In the same way one deduces $\angle ACS=\angle ADB=\angle MAC$, and this concludes the proof.

Problem 5.

Let $(O,\ r)$ be a circle, AB a chord of it, Q the projection of O on AB, C an internal point of OB and M the midpoint of AC. Show that

$$OM\le\frac{OB-OC}{2OB}r+\frac{OC}{OB}OQ$$

Solution.

Introduce complex numbers a and b such that $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$. Moreover, $|a|=|b|=r$. If $t=\dfrac{OC}{OB}$, so that $0 < t < 1$, then $\overrightarrow{OC}=tb$ and $\overrightarrow{OM}=\dfrac{1}{2}(a+tb)$. Extend OM until it meets AB in D. Writing the complex number representing the vector $\overrightarrow{OD}$ in two different ways and equating we have, for suitable real scalars λ, v the following equality:

$$\frac{\lambda}{2}(a+tb)=va+(1-v)\frac{a+b}{2}$$

Equating the coefficients of each of a,b on both sides and solving the system we find:

$$\lambda=\frac{2}{1+t},\quad v=\frac{1-t}{1+t}$$

Hence $OD=\left|va+(1-v)\dfrac{a+b}{2}\right|\le\dfrac{1-t}{1+t}|a|+\dfrac{2t}{1+t}OQ$.

Using now $OM=\dfrac{1}{\lambda}=\dfrac{1+t}{2}OD$ and $|a|=r$, the result follows.

ALGEBRA.

Problem 6.

Let $x_1,x_2,...,x_n$ be real numbers, such that $x_i\in[0,1]$ for all $i=1,2,...,n$. Find the maximum of the expression

$$E=x_1^2+x_2^2+...+x_n^2-x_1x_2-x_2x_3-...-x_{n-1}x_n-x_nx_1.$$

And determine all $x_1,x_2,...,x_n$ for which this maximum is obtained.

Solution.

The expression is cyclic in $x_1,x_2,...,x_n$ and it is a quadratic function in x_1. Let $f:\mathbb{R}\to\mathbb{R}$ be the function

$$f(x)=x^2-(x_2+x_n)x+x_2^2+...+x_n^2-x_2x_3-...-x_{n-1}x_n$$

This function is decreasing on $\left(-\infty,\dfrac{x_2+x_n}{2}\right]$ and increasing on $\left[\dfrac{x_2+x_n}{2},\infty\right)$. Since $x_2,x_n\in[0,1]$ it follows that $\dfrac{x_2+x_n}{2}\in[0,1]$, whence the maximum value of f on the interval $[0,1]$ is either $f(0)$ or $f(1)$, that is $x_1\in\{0,1\}$. Observing again the symmetry we conclude that the maximum of *E* is obtained only when $x_i\in\{0,1\}$, for all $i=1,2,...,n$. Now, observe that

$$2E=(x_1-x_2)^2+(x_2-x_3)^2+...+(x_n-x_1)^2\le n.$$

If n is even the maximum of 2*E* is *n* and it is attained when all squares are 1, that is if $(x_1,x_2,...,x_n)$ is either $(1,0,1,0,...,1,0)$ or $(0,1,0,1...,0,1)$. If *n* is odd the maximum number of parenthesis which are 1 is $n-1$, that is $2E\le n-1$. It is clear when it is attained.

Problem 8.

Find all ordered systems of positive rational numbers (x,y,z) such that the numbers

$$x+\frac{1}{y},\ y+\frac{1}{z},\ z+\frac{1}{x}$$

are all integers.

Solution.

Compute the product

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)=\left(xyz+\frac{1}{xyz}\right)+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Since $(x+y+z)+\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$ is an integer, it follows that $\left(xyz+\dfrac{1}{xyz}\right)$ is an integer too. The only positive rational *q* for which $q+\dfrac{1}{q}$ is an integer is $q=1$. It follows that $xyz=1$. Using this supplementary condition one obtains for (x,yz) the possibilities $(1,2,5)$, $(1,3,3)$, $(1,5,2)$ and those which can be obtained by cyclic permutations.

Problem 9.

Let A be a set of real numbers which satisfies the conditions:

a) $1\in A$

b) if $x\in A$ then $x^2\in A$

c) if $(x^2-4x+4)\in A$ then $x\in A$.

Show that $\left(2004+\sqrt{2005}\right)\in A$.

Solution.

Let *x* be a number from A. Then $x^2\in A$. Since $x^2=((x+2)-2)^2=(x+2)^2-4(x+2)+4$, it follows by c) that $(x+2)\in A$. So, we have proved the following property:

(*) if $x\in A$ then $(x+2)\in A$..

Since $1 \in A$ it follows by induction that *A* contains all odd positive integers. Therefore, $2005 = \left(\left(\sqrt{2005}+2\right)-2\right)^2$. By condition c) it follows that $\left(\sqrt{2005}+2\right) \in A$. Now, using again (*), we obtain the result by induction.

Problem 10.

Let $n \geq 3$ be an integer and x be a real number everywhere such that x, x^2 and x^n have the same fractional part. Show that x is an integer.

Solution.

We have $x^2 = x + k$ and $x^n = x + \ell$ for some integers k, ℓ. It follows that:

$$x^3 = x^2 + kx = (k+1)x + k\,,$$

$$x^4 = (k+1)x^2 + kx = (2k+1)x + k^2 + k\,,$$

and by mathematical induction,

$$x^p = a_p x + b_p, \quad \forall p \geq 2,$$

where a_p, b_p are integers and $a_{p+1} = a_p + b_p,\ b_{p+1} = a_p$.

Since x is a real number and $x^2 - x - k = 0$ we have $\Delta = 1 + 4k \geq 0$, therefore $k \geq 0$. If $k = 0$ we have $x^2 = x$ and x is an integer. If $k \geq 1$, we have $a_p > 1$ for $p \geq 3$ and $x^n = a_n x + b_n = x + k$ shows that x is rational. In that case Δ must be a perfect square and x an integer.

Problem 11.

Find all real numbers x such that $2^{2x-1} = x^2$.

Solution.

The equation can be written in the form:

$$\left(2^{x-\frac{1}{2}} - x\right)\left(2^{x-\frac{1}{2}} + x\right) = 0$$

The function $f(x) = 2^{x-\frac{1}{2}} + x$ is strictly increasing on $\mathbb{R}$ and $f(-1/2) = 0$. Hence $x = -1/2$ is the unique solution of the equation $f(x) = 0$.

Let a be a solution of the equation $g(x) = 0$, where $g(x) = 2^{x-\frac{1}{2}} - x$. Then $a > 0$ and one has successively the inequalities:

$$a = 2^{a-\frac{1}{2}} > 2^{-\frac{1}{2}} \Rightarrow a = 2^{a-\frac{1}{2}} > 2^{\frac{1}{5}} > 1.1 \Rightarrow a = 2^{\frac{2a-1}{2}} > 2^{\frac{2.2-1}{2}} = 2^{\frac{3}{5}} > 1.5\,,$$

$$\Rightarrow a = 2^{\frac{2a-1}{2}} > 2^{1.5-\frac{1}{2}} = 2 \Rightarrow a = 2^{\frac{2a-1}{2}} > 2^{\frac{3}{2}} > 2.8 \Rightarrow a > 4\,.$$

Next step of the solution requires to show that there is no solution for $a>4$. It is easy to see that $g(n)>1$ for all integers $n\ge 4$. Then, for any a with $a\ge 4$, one has

$$4\le[x]\le x<[x]+1\Rightarrow 2^{x-\frac{1}{2}}\ge 2^{[x]-\frac{1}{2}}>[x]+1>x$$

This ends the proof.

NUMBER THEORY

Problem 12.
Find all natural numbers of the form *ABB* that are divisible by 4.(Digits *A* and *B* are distinct).

Solution.
Use the criteria of divisibility by 4 to obtain all numbers of the form A00, A44, A88

.

Problem 13.
We call an integer lucky, if its digits can be divided into two groups in such a way that the sum of the numbers in each group equal the same amount. For example, the number 34175 is lucky, because $3+7=1+4+5$. What is the smallest 4 digit lucky number, whose neighbour is also a lucky integer?

Solution.
The greater in a pair of lucky numbers must end with 0, else the difference of two consecutive numbers is 1, more the difference of the sums of their digits is also1 so one of them is even the other odd, and for the later one we cannot group the digits on two groups of equal sum. Concentrate on the greater one of the pair, and check its pair as well. The smallest possible ones are 1010, 1100, 1210, 1230, 1320, 1340, 1430, 1450, ... The first to have lucky pair is 1450, as 1449 is lucky as well.

Problem 14.
Show that there are infinitely many ordered systems of relatively prime positive integers (x,y,z,t) such that

$$x^3+y^3+z^3=t^4.$$

Solution.
Let consider the following identity:

$$(a+1)^4-(a-1)^4=8a^3+8a$$

where a is a positive integer. Take $a=b^3$, where b is an even positive integer. From the above identity one obtains:

$$\left(b^3+1\right)^4=\left(2b^3\right)^3+\left(2b\right)^3+\left(\left(b^3-1\right)^2\right)^2.$$

Since b is even, $\left(b^3+1\right)$ and $\left(b^3-1\right)$ are odd numbers. It follows that $x=2b^3,\ y=2b,$ $z=\left(b^3-1\right)^2$ and $t=b^3+1$ have no common divisor greater than 1.

Problem 15.

Show that there are infinitely many positive integers x, y, z, t such that $x^2 + y^2 + z^2 + t^2 = xyzt$.

Solution.
Observe that $(2,2,2,2)$ and $(6,2,2,2)$ are solutions. We look for solutions of the form $x = 2u$, $y = 2v$, $z=t=2$. One obtains the equations in two variables $u^2 + v^2 + 2 = 4uv$ or $(u - v)^2 = 2(1 + uv)$. This shows that $u - v$ is even, hence $u + v$ is even too. Then we may write $u = a + b$, $v = a - b$. One obtains that a, b should verify Pell's equation $a^2 - 3b^2 = 1$, which has infinitely many solutions.

Problem 16.
Find all prime numbers *p* and *q* for which $p + q = (p - q)^3$.

Solution.
From the equation, one has $p > q$. Hence $p - q \geq 1$. This gives $(p - q)^2 + 1 \geq 2$.
From the equation we have: $(p - q)\left((p - q)^2 + 1\right) = 2p$ and because $p - q < p$ it follows $p - q = 2$, $p = 5, q = 3$.

Problem 17.
Find all triples (x, y, z) of positive integers such that $2x^4 + 2y^4 = z^4$.

Solution.
Since the equation is homogeneous we may assume that gcd $(x, y, z) = 1$. By the Fermat's theorem $n^4 \equiv 1(mod\, 5)$, for all numbers *n* which are not divisible by 5. So we have either $z^4 = 0$ or $z^4 = 2$, modulo 5. But this can not be obtained, unless $x, y, z \equiv 0(mod\, 5)$. This contradicts the assumptions that *x*, *y*, *z* are relatively prime. So, the equation has no integer solutions.

COMBINATORICS.

Problem 18.
Find the number of 5-tuples (x, y, z, u, v) such that $1 \leq x < y < z < u < v \leq 90$ are integers and there are no consecutive pairs among them.

Solution.
If the conditions of the problems are respected, the solutions will satisfy

$$x + 1 \leq y,\ y + 1 \leq z,\ z + 1 \leq u,\ u + 1 \leq v$$

as well. In other words

$$x \leq y - 1,\ y - 1 \leq z - 2,\ z - 2 \leq u - 3,\ u - 3 \leq v - 4\ ,$$

in other words we can choose any five different numbers of the first 86, that is in $\binom{86}{5}$ different ways, and then take them as the values of $x, y - 1, z - 2, u - 3, v - 4$, and finally obtain of *x*, *y*, *z*, *u*, *v* satisfying the conditions of the problem.

Problem 19.
How many different ways are there to get to the top of flight of stairs which has 10 steps, if we can take 1 or 2 steps at a time?

Solution.
Observe that we can go to the first step in only 1 way, but for the second we have 2 different ways, 2 steps of 1 or a 2 step jump. Also, observe more, that to arrive on the step n, we can jump from step $n-1$, or step $n-2$. The total number of possibilities is the sum of the previous two possibilities, that is the number of possibilities for the step n-1 (and use a 1 step jump) plus those for step $n-2$ (and use a 2 step jump).
So we will have $1,2,(1+2),...,N_{n-2},N_{n-1},(N_{n-2}+N_{n-1})$. Recognize a Fibonacci type sequence. The answer is 1,2,3,5,8,13,21,34,55,89. For ten steps we have 89 different ways.

Problem 20.
Prove that the number of partitions of the set $\{1,2,...,n\}$ into *k* classes so that no classes contains two consecutive integers equals the number of partitions of the set $\{1,2,...,n-1\}$ into $n-1$ classes.

Solution.
Let denote $S(n,k)$ the first number and $T(n,k)$ the second number. It is easy to see that $S(n,2)=1$ and $S(n,k)=S(n-1,k-1)+(k-1)S(n-1,k)$, because *n* can produce himself a class or it can be added to any of the $k-1$ classes previously produced, which does not contain $n-1$.
On the other hand the numbers $T(n,k)$ verify the conditions $T(n,1)=1$ and $T(n,k)=T(n-1,k-1)+kT(n-1,k)$. Therefore the sequences $S(n,k),k\geq 2$ and $S(n,k-1),k\geq 2$, coincide.

Problem 21.
We have a 5×10 cm bar chocolate. What is the least and the greatest number of cuts needed to slice up the chocolate into 1×1 cm pieces (we can not overlap pieces when cutting)?

Solution.
As we always cut a piece to obtain two, after the first cut we shall have 2 pieces, after the second cut 3 pieces, and so on. We need altogether 49 cuts to have 50 pieces. The minimum and maximum numbers of cuts coincide.

Problem 22.
We have 1000 positive integers and we know that the sum of their reciprocals is greater than 10. Show that the integers cannot be all different.

Solution.
The reciprocal of a smaller number is greater, so if we want to have the greatest possible sum for the reciprocals of 1000 positive integers we must take the first 1000. Now

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\frac{1}{7}+\frac{1}{8}+...+\frac{1}{1000}\leq$$

$$\leq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\frac{1}{7}+\frac{1}{8}+...+\frac{1}{1023}\leq$$

$$\leq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\ldots+\frac{1}{512}+\frac{1}{512}\ldots+\frac{1}{512}=10.$$

We must have numbers repeated below 1000, else the sum cannot be greater than 10.

MIXED PROBLEMS

Problem 23.
Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC-BD| \leq |AB-CD|..$$

When does the equality hold?

Solution.
Let E, F be the midpoints of the diagonals $AC, BD,$ respectively. In every quadrilateral the Euler equality holds:

$$AC^2+BD^2+4EF^2=AB^2+BC^2+CD^2+DA^2.$$

Since $ABCD$ is cyclic, it also verifies Ptolemy's identity:

$$AB.CD+BC.AD=AC.BD.$$

Hence,

$$(AC-BD)^2+4EF^2=(AB-CD)^2+(AD-BC)^2.$$

.

It is sufficient to prove that $4EF^2 \geq (AD-BC)^2$. Let M be the midpoint of AB. In the triangle MEF, from the triangle inequality one has $EF \geq |ME-MF| = \frac{1}{2}|AD-BC|$. The equality holds if and only if $AB \parallel CD$, that is $ABCD$ is either an isosceles trapezoid or a rectangle.

Problem 24.
Consider all convex regular or star regular polygons with n sides inscribed in a given circle. We say that two such polygons are identical if they can be obtained one from another by a rotation about the center of the circle. How many distinct such polygons exist?

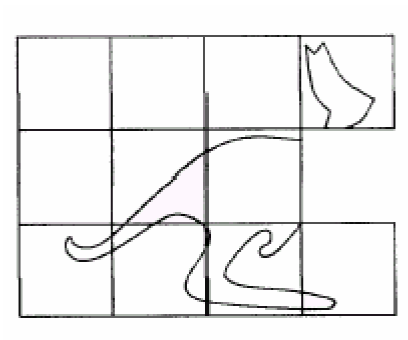
Solution.
Any star polygon is obtained by rotating about the center a vertex of a polygon by an angle of $\frac{2k\pi}{n}$, where $1 \leq k \leq n$ and *g.c.d*(*k*, *n*) = 1. So there are $\varphi(n)$ possibilities, where $\varphi(n)$ is the Euler function. Since there are clockwise and counter-clockwise rotations, the number of distinct polygons is $\frac{\varphi(n)}{2}$.

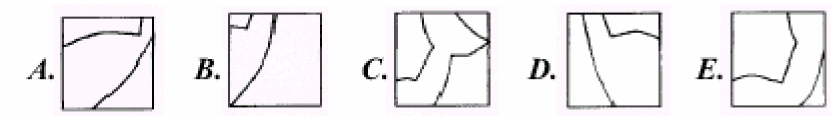
MULTIPLE CHOICE TESTS

LEVEL 1

Multiple choice test 1

1. Which one of the squares was removed from the picture of the Kangaroo below?





2. Costas is 27 years old and his son is 5 years old. In how many years will Costa's age be three times the age of his son?

A. 4 **B.** 6 **C.** 9 **D.** 15 **E.** 81

3. The sum of 101 units, 101 tens and 101 hundreds is:

A. 101 112 **B.** 10 211 **C.** 11 111 **D.** 11 211 **E.** 10 121

4. Which is the smallest number which when divided by four gives 1 as a remainder, when divided by 5 gives 2 as a remainder and when divided by 6 gives 3 as a remainder?

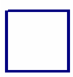
A. 57 **B.** 21 **C.** 67 **D.** 32 **E.** 42

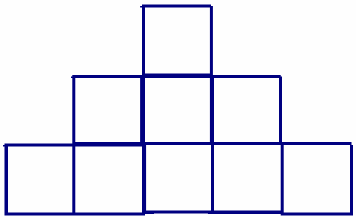
5. In a mathematics test students were given 10 problems. For every correct answer 5 points were given whereas for every wrong answer 2 points were deducted from the final score. Costas responded to all the problems and got 29 points. How many correct answers did he give?

A. 5 **B.** 6 **C.** 7 **D.** 8 **E.** 9

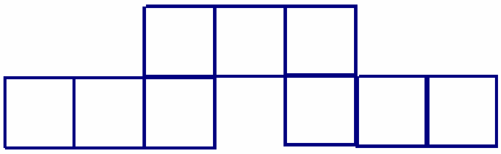
6. There are 78 seats in a bus. The bus leaves the station empty and takes one passenger from the first bus stop, two passengers from the second bus stop, three from the third bus stop etc. If no passenger gets off the bus, after how many bus stops will the bus be full?

A. 5 **B.** 8 **C.** 12 **D.** 13 **E.** 10

7. Shapes A and B have the same area. The perimeter of shape A is 48 cm. What is the perimeter of shape B? (The small shapes like this one , are squares).



A



B

A. 48 **B.** 60 **C.** 50 **D.** 80 **E.** 10

8. A, B and C are different numbers. If $\begin{array}{r} A \\ B\ + \\ \Gamma \\ \hline AB \end{array}$, what is the value of C?

A. 9 **B.** 2 **C.** 1 **D.** 8 **E.** 0

9. Which one of the following statements is **false**?

A. 28 ÷ 7 > 3 X 1

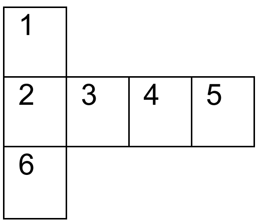
B. 9 X 6 < 7 X 8

C. 8 X 0 < 7 ÷ 7

D. 63 ÷ 7 > 64 ÷ 8

E. 48 ÷ 6 < 36 ÷ 9

10. If the figure below is folded, it becomes a cube. Which number will be on the bottom of the cube, if 5 is on the top of the cube?

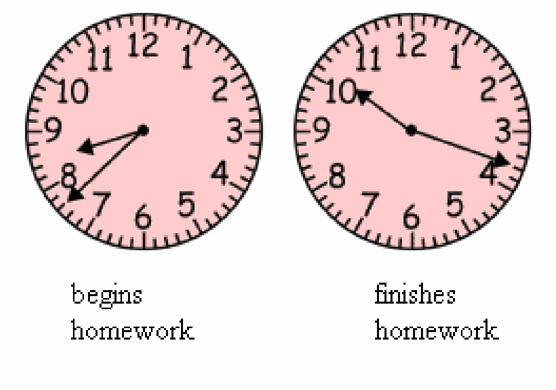


A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 6

11. If ((x – 6) ÷ 6) + 6 = 66 ÷ 6, find the value of x.

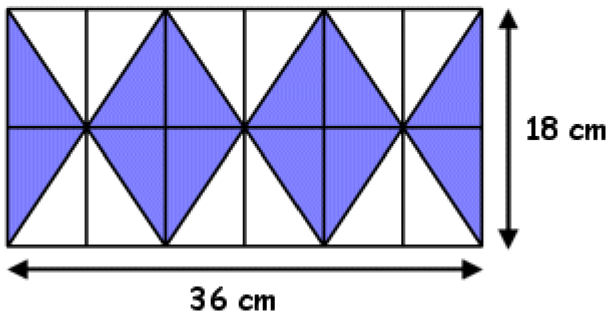
A. 16 **B.** 36 **C.** 56 **D.** 66 **E.** 366

12. How long did it take Costas to finish his homework?



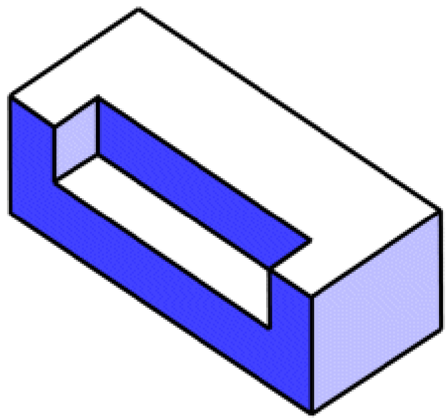
A. 90 minutes **B.** 40 minutes **C.** 100 minutes
D. 1 hour and 20 minutes **E.** 1 hour and 18 minutes

13. Find the area of the shaded part of the following figure.



A. 648 cm^2 **B.** 324 cm^2 **C.** 162 cm^2 **D.** 36 cm^2 **E.** 18 cm^2

14. How many faces does the shape below have?



A. 8 **B.** 7 **C.** 9 **D.** 10 **E.** 12

15. Find the whole number which is:
less than 100;
a multiple of 3;
a multiple of 5;
odd, and such that,
the sum of its digits is odd.

A. 30 **B.** 75 **C.** 36 **D.** 45 **E.** 25

16. In the subtraction shown, M and N each represent a single digit. What is the value of M + N?

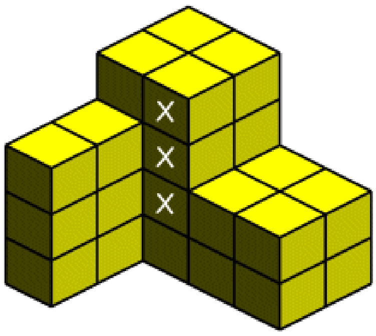
$$
\begin{array}{r}
\boxed{M}\,\boxed{4} \\
-\ \boxed{3}\,\boxed{N} \\
\hline
\boxed{1}\,\boxed{6}
\end{array}
$$

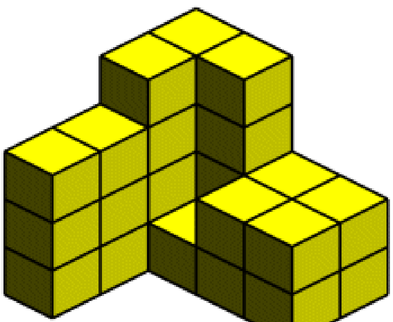
A. 14 **B.** 12 **C.** 15 **D.** 13 **E.** 11

17. What is the value of:
268 + 1375 + 6179 - 168 - 1275 – 6079=

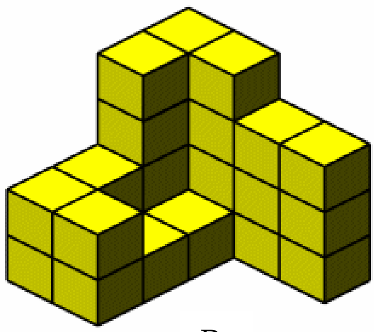
A. 300 **B.** 0 **C.** -100 **D.** 100 **E.** -300

18. If I from the construction below I remove the three cubes indicated with X, which one of the following will be the new construction created?

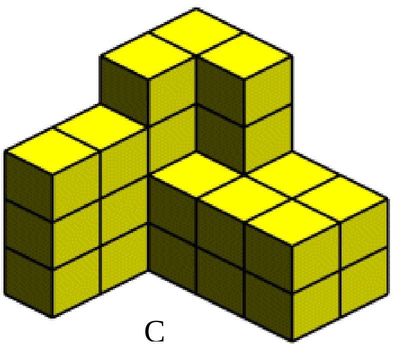




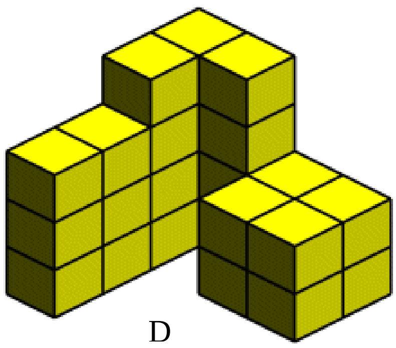
A



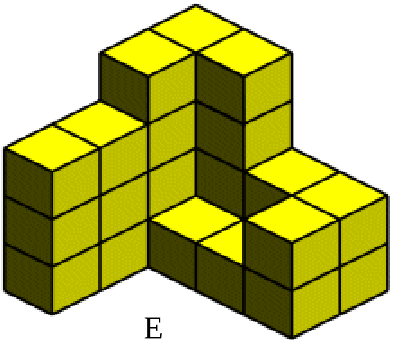
B



C

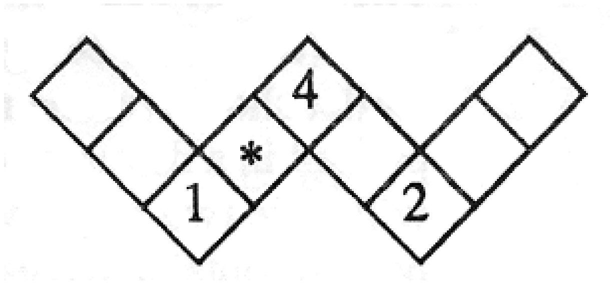


D



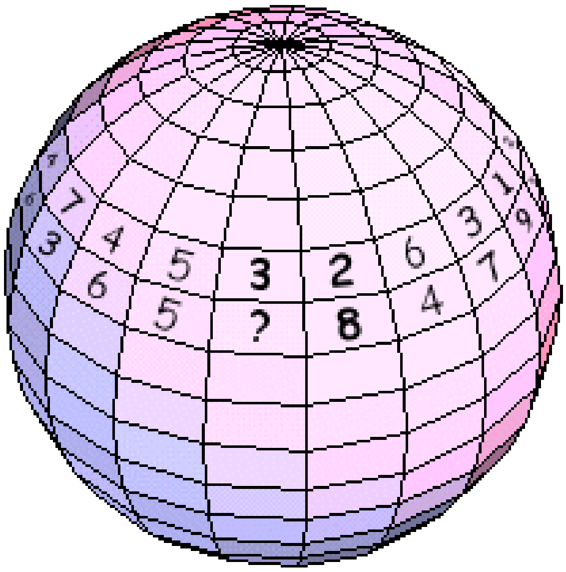
E

19. The digits from 1 to 9 inclusive are to be placed in the figure shown below. Only one digit goes in every square. If the sum in each of the four lines is the same which digit should replace *?



A. 8 **B.** 5 **C.** 9 **D.** 6 **E.** 7

20. Which is the missing number in the shape below;

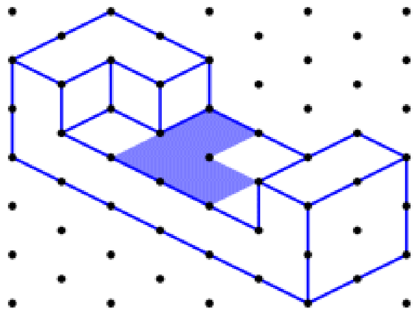


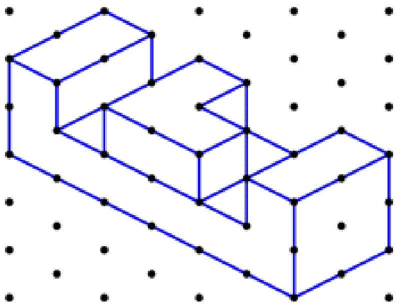
A. 1 **B.** 3 **C.** 5 **D.** 7 **E.** 9

21. Which of the following is an equivalent fraction for $\frac{3}{7}$?

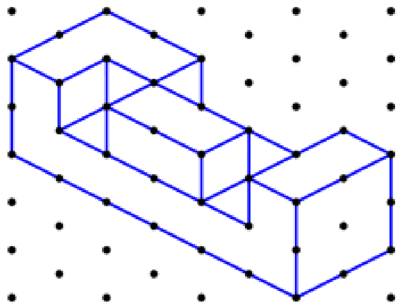
A. $\frac{18}{49}$ **B.** $\frac{27}{56}$ **C.** $\frac{33}{70}$ **D.** $\frac{42}{91}$ **E.** $\frac{48}{112}$

22. If three cubes are placed on the shaded part of the shape which one of the following will be the resulting shape?

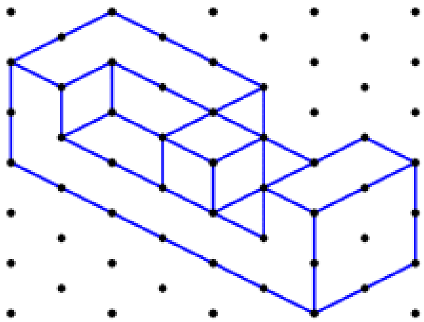




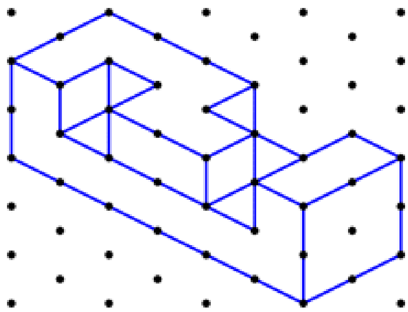
A.



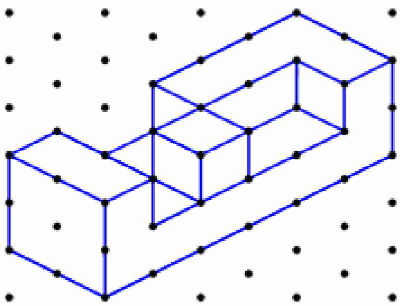
B.



C.

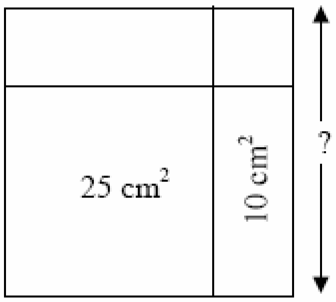


D.



E.

23. A big square is divided into four pieces: two squares and two rectangles. The area of two of these pieces is written inside: 25 cm^2 and 10 cm^2. What is the length of the side of the big square?



A. 5 cm **B.** 6 cm **C.** 7 cm **D.** 8 cm **E.** 9 cm

24. Which of the following figures has two circular bases?

A. Pyramid **B.** Sphere **C.** Cube **D.** Cylinder **E.** Cone

25. Find number X, if the number pairs in the table follow the same rule.

A	B
2	7
3	12
4	19
6	39
X	103

A. 8 **B.** 7 **C.** 6 **D.** 3 **E.** 10

26. Find the value of X.

(999+999+999+999+999) ÷ 999 = 9 - X

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7

27. 100 + 0,01 – 0,001 =

A. 100,09 **B.** 100,9 **C.** 99,09 **D.** 100,009 **E.** 100

28. Sixteen is called a square number, because 16 = 4 X 4. How many square numbers are there between 2 and 101?

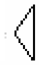
A. 7 **B. 8** **C. 9** **D. 10** **E. 11**

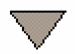
29. Aunt Anna is 42 years old. Eleni is 5 years younger than Niki, and Niki is half the age of Aunt Anna. How old is Eleni?

A. 15 **B. 16** **C. 17** **D. 21** **E. 37**

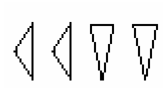
30. In Mesopotamia in 2500 B.C.,

 This sign was used to represent 1,

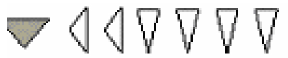
 This sign was used to represent 10 and

 This sign was used to represent 60.

Thus, 22 would be written like this:



How would 124 be written?

A  **B**  **C** 

D  **E** 

Multiple choice test 2

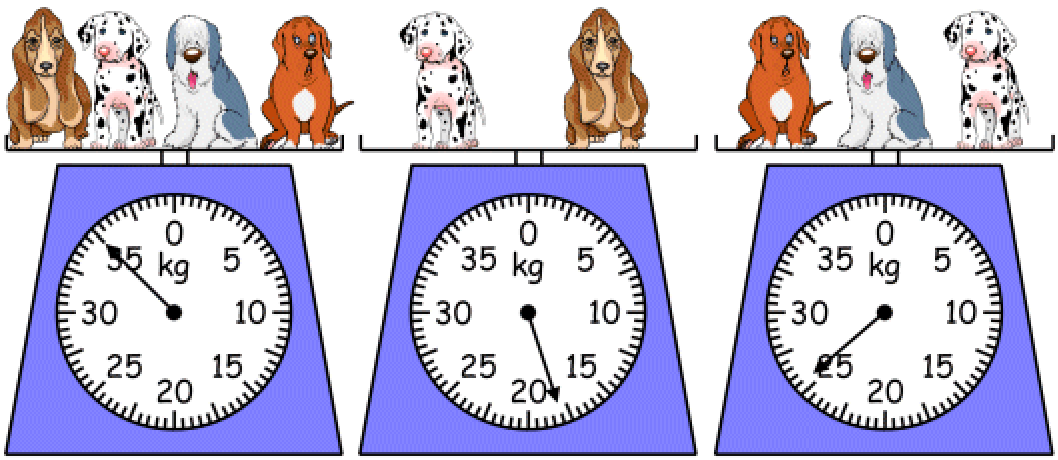
1. What is the sum of the angles in a parallelogram?

A. 180° **B.** 225° **C.** 270° **D.** 315° **E.** 360°

2. Which of the following fractions has the largest value?

A. $\frac{7}{8}$ **B.** $\frac{66}{77}$ **C.** $\frac{555}{666}$ **D.** $\frac{444}{555}$ **E.** $\frac{3333}{4444}$

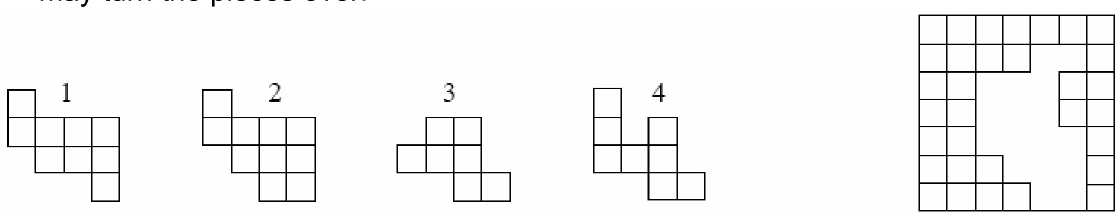
3. Look at the picture below:



What is the weight of the dog at the right? 

A. 8 kg
B. 8 kg and 500 g
C. 7 kg
D. 9 kg
E. 10 kg

4. The gap in the figure on the right has to be filled. Which two pieces will you use for that? You may turn the pieces over.



A. 1 and 3 **B.** 1 and 4 **C.** 2 and 3 **D.** 2 and 4 **E.** 3 and 4

5. Which one of the following numbers is not a divisor of 2002?

A. 14 **B.** 26 **C.** 42 **D.** 77 **E.** 91

6. An electric bell rings every 10 minutes. A second bell rings every 12 minutes. At 12:00 both bells rang simultaneously. In how many minutes will the two bells ring simultaneously again?

A. 22 **B.** 30 **C.** 60 **D.** 72 **E.** 120

7. Costas thought of a number, the number “a”. He adds 5 on “a” and then he doubles the result. Then he subtracts 6 from the new result and divides it by 2. Finally, he subtracts 2 and gets 5 as a result. Which one of the following was number “a”?

A. 4 **B.** 5 **C.** $\frac{5}{2}$ **D.** 7 **E.** $\frac{7}{2}$

8. Which of the following is not equivalent to $\frac{21}{8}$?

A. $2\frac{5}{8}$ **B.** $\frac{168}{64}$ **C.** 2,625 **D.** $2\frac{20}{32}$ **E.** $\frac{189}{81}$

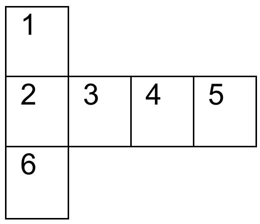
If $\frac{3}{2}=1{,}5$, then $\frac{0{,}03}{0{,}2}=$

A. 1,5 **B.** 0,15 **C.** 0,015 **D.** 0,0015 **E.** 0,00015

The sum of five consecutive even numbers is 320. Which is the smaller of these two numbers?

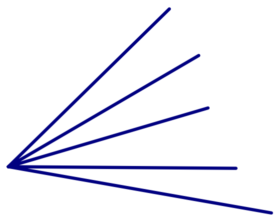
A. 2 **B.** 310 **C.** 312 **D.** 60 **E.** 160

If the figure below is folded, it becomes a cube. Which number will be on the bottom of the cube, if 5 is on the top of the cube?



A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 6

How many acute angles are there in the shape below?



A. 4 **B.** 5 **C.** 10 **D.** 11 **E.** 6

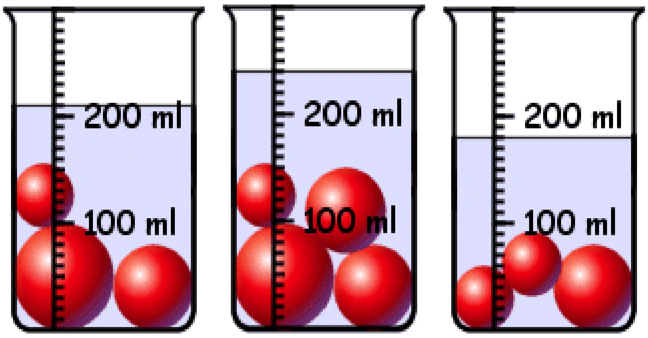
Harry has yellow, green and blue balls. In total he has 20 balls. 17 are not green and 12 are not yellow. How many are the blue balls?

A. 3 **B.** 4 **C.** 5 **D.** 8 **E.** 9

Five points are marked on a circle. How many different triangles can be formed by joining any three of those points?

A. 10 **B.** 6 **C.** 8 **D.** 15 **E.** 7

The following vessels have the same quantity of water. What would the reading be if in one of these vessels there were only two small spheres?

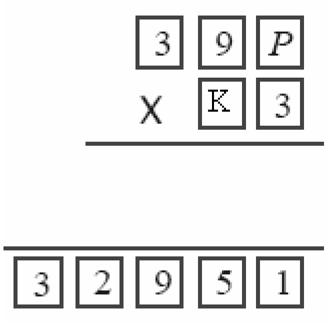


A. 130 ml **B.** 140 ml **C.** 150 ml **D.** 160 ml **E.** 170 ml

If $\frac{6}{x+1} = \frac{3}{2}$, then x equals

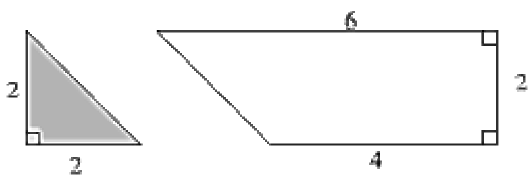
A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

In the multiplication shown, P and K each represent a single digit, and the product is 32 951. What is the value of P + K?



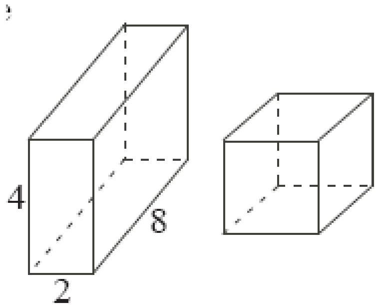
A. 14 **B.** 12 **C.** 13 **D.** 11 **E.** 15

In how many triangles of the same shape and size of the shaded triangle can the trapezoid below be divided into?



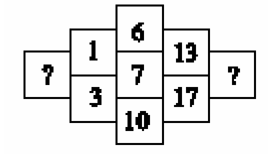
A. 3 **B.** 5 **C.** 4 **D.** 6 **E.** 3,5

In the diagram, the rectangular solid and the cube have equal volumes. The length of each edge of the cube is:



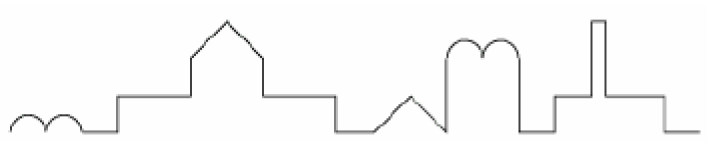
A. 2 **B.** 4 **C.** 8 **D.** 16 **E.** 32

What numbers should be in the boxes instead of the ?-signs?
A. 2 and 14 **B.** 2 and 30 **C.** 3 and 221 **D.** 4 and 14

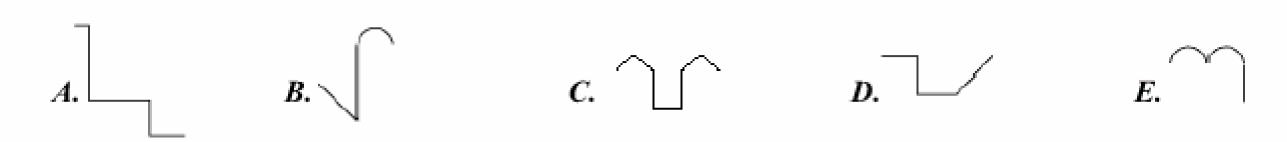


E. 4 and 30

In the distance we see the skyline of a castle.



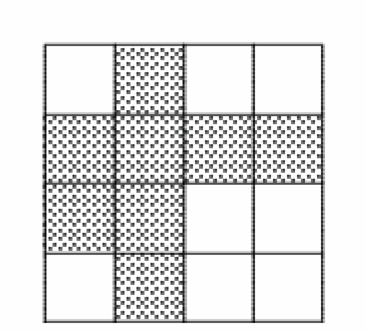
Which one of the following pieces does not belong to the skyline?



In Canada part of the population can only speak English, part of the population can only speak French, and part of the population can speak both languages. A survey shows that 85% of the population speaks English, 75% of the population speaks French. What percentage of the population speaks both languages?

A. 50 **B.** 57 **C.** 25 **D.** 60 **E.** 40

The area of the shaded shape is 200 cm^2. What is the perimeter of the shaded shape?



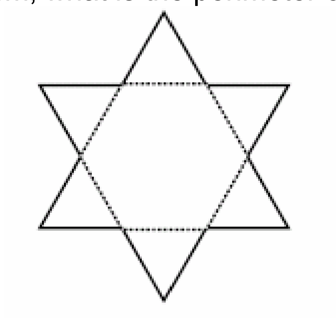
A. 50 **B.** 75 **C.** 80 **D.** 400 **E.** 16

Below is a magic square. What is the value of x?

	$\times$	$\frac{4}{5}$
	$\frac{1}{2}$	$\frac{1}{10}$
$\frac{1}{5}$		

A. $\frac{3}{10}$ **B.** $\frac{1}{10}$ **C.** $\frac{2}{5}$ **D.** $\frac{15}{10}$ **E. 1**

The star below was constructed by extending the sides of a regular hexagon (see dotted lines). If the perimeter of the star is 96 cm, what is the perimeter of the hexagon?



A. 60 cm **B.** 202 cm **C.** 64 cm **D.** 12 cm **E.** 48 cm

One weekend Costas had a lot of homework. If he did one fourth of the homework on Friday and one-sixth of the homework on Saturday, how much of his homework was left for Sunday?

A. $\frac{1}{2}$ **B.** $\frac{7}{12}$ **C.** $\frac{15}{24}$ **D.** $\frac{8}{12}$ **E.** $\frac{8}{10}$

If $2{,}125 - x = x + \frac{5}{8}$, what is the value of x?

A. 0,375 **B.** 0,5 **C.** 0,625 **D.** 0,75 **E.** 1

Which number follows in the pattern below?

20, 41, 83, 167, _____

A. 334 **B.** 250 **C.** 301 **D.** 335 **E.** 350

Which of the following is the largest product?

A. $9{,}999 \times 9$ **B.** $999{,}9 \times 99$ **C.** $99{,}99 \times 999$ **D.** $9{,}999 \times 9{,}999$
E. $0{,}9999 \times 99{,}999$

Which one of the following numbers is divisible by 2, 3, 4, and 5?

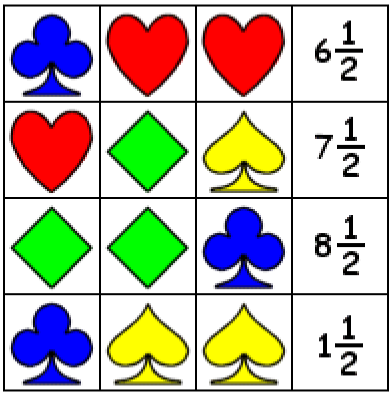
A. 60 **B.** 80 **C.** 100 **D.** 125 **E.** 160

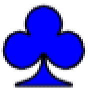
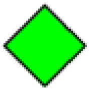
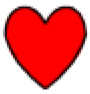
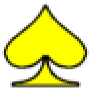
Multiple choice test 3

What is 40% of 50% of £60?

A. £7 **B.** £15 **Γ.** £8 **Δ.** £20 **E.** £12

Look at the picture below.



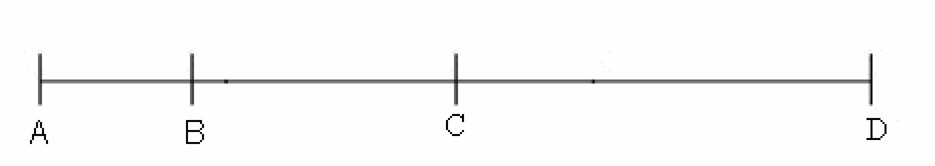
Find the sum of  +  +  + 

A. 8 **B.** 24 **C.** 12 **D.** 6 **E.** 16

If $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{X}{12}=2$, the value of X is

A. -4 **B.** -1 **C.** 1 **D.** 13 **E.** 18

In the following figure the ratio of the length AB to the length BC is 1 to 3. The ratio of the length of BC to CD is 5 to 8. What is the ratio of the length of AC to the length of CD?



A. 3:4 **B.** 3:5 **C.** 5:6 **D.** 4:5 **E.** 2:3

Find a two digit number such that if we take away 5, it is a multiple of 4, if we take away 6, it is a multiple of 5 and if we take away 7, it is a multiple of 6.

A. 21 **B.** 66 **C.** 61 **D.** 31 **E.** 25

In a basketball match there were 500 spectators. 30% of the spectators were not students. 30% of the students were in grade 6. 60% of the grade 6 students were boys. How many grade 6 girls attended the basketball match?

A. 18 **B.** 27 **C.** 42 **D.** 54 **E.** 63

How many different six-digit numbers can you construct, if you use the digits 1 , 2, 3, 4, 5 and 6 only once in each number .

A. 24 **B.** 720 **C.** 80 **D.** 100 **E.** 120

What is the value of K and M if $\frac{1}{3}=\frac{1}{K}+\frac{1}{M}$; (K and M represent different whole numbers)

A. K=12, M=4
B. K=2, M=1
C. K=0, M=3
D. K=15, M=2
E. K=12, M=6

The mean value of five numbers is 18. If I increase the first number by adding on 1, the second by adding on 2, the third by adding on 3, the forth by adding on 4 and the fifth by adding on 5, what will be the mean value of these new five numbers?

A. 3 **B.** 15 **C.** 21 **D.** 33 **E.** 18

Express 222% of $\frac{1}{2}$ as a decimal.

A. 111 **B.** 11.1 **C.** 11 **D.** 1.11 **E.** 0.11

Which common fraction is equivalent to 0,004375?

A. $\frac{4375}{1000}$ **B.** $\frac{4375}{10000}$ **C.** $\frac{7}{10000}$ **D.** $\frac{7}{1600}$ **E.** $\frac{7}{4000}$

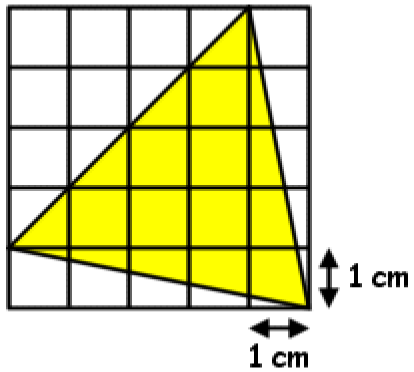
Which one of the following calculations is **incorrect**?

A. 4 X 5 + 67 = 45 + 6 X 7 **B.** 3 X 7 + 48 = 37 + 4 X 8

C. 6 X 3 + 85 = 63 + 8 X 5 **D.** 2 X 5 + 69 = 25 +6 X 9

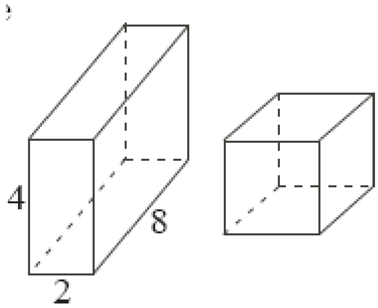
E. 9 X 6 + 73 = 96 + 7 X 3

Find the area of the shaded triangle.



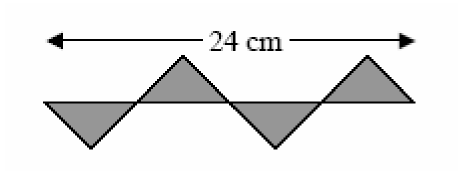
A. 14 cm^2 **B.** 10 cm^2 **C.** 13 cm^2 **D.** 12 cm^2 **E.** 24 cm^2

In the diagram, the rectangular solid and the cube have equal volumes. The length of each edge of the cube is



A. 2 **B.** 4 **C.** 8 **D.** 16 **E.** 32

The four triangles are half a square each. They all are equally big. How many cm^2 is the area of the four triangles together?

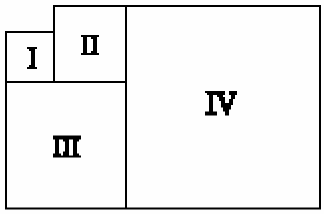


A. 20 **B.** 25 **C.** 30 **D.** 36 **E.** 45

In Canada part of the population can only speak English, part of the population can only speak French, and part of the population can speak both languages. A survey shows that 85% of the population speaks English and 75% of the population speaks French. What percentage of the population speaks both languages?

A. 50 **B.** 57 **C.** 25 **D.** 60 **E.** 40

Figures I, II, III and IV are squares. The perimeter of square I is 16 m and the perimeter of square II is 24 m.



Find the perimeter of square IV.

A. 56 m **B.** 60 m **C.** 64 m **D.** 72 m **E.** 80 m

Christian added 3gr of salt to 17gr of water. What is the percentage of salt in the solution obtained?

A. 20% **B.** 17% **C.** 16% **D.** 15% **E.** 6%

Three plates, A, B, and C are arranged in increasing order of their weight.

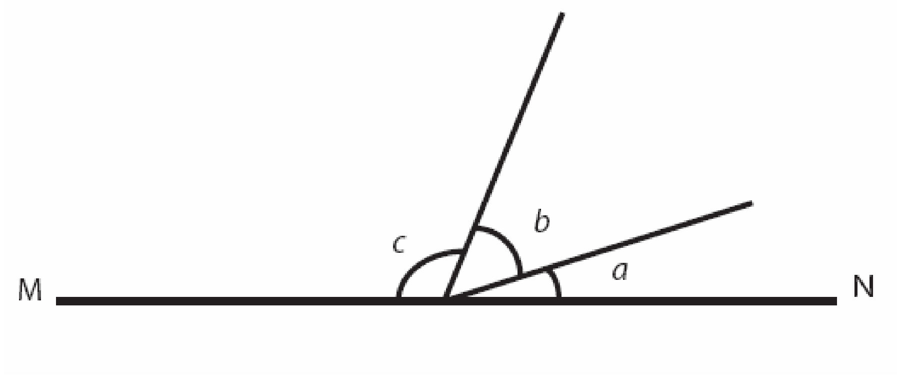
To keep this order, plate D must be placed:

A. between A and B **B.** between B and C

C. before A **D.** after C

D and C have the same weight

In the following figure, MN is a straight line. The angles *a, b* and *c* satisfy the relations, *a*:*b*=1:2 and *c*:*b*=3:1. Find angle *b*.



A. 120° **B.** 60° **C.** 40° **D.** 20° **E.** 8°

In a party, there are *n* persons. If everybody shakes hands once with every other person at the party and there are a total of 231 handshakes, what is the value of *n*?

A. 21 **B.** 22 **C.** 11 **D.** 12 **E.** 462

If $5^3 - 2^4 = 4^3 + n$, what is the value of n?

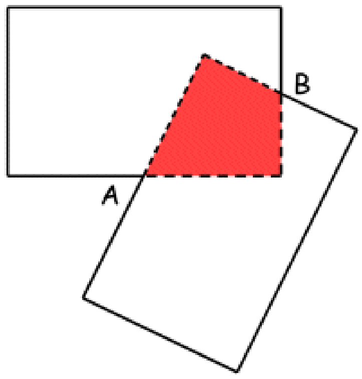
A. 29 **B.** 37 **C.** 45 **D.** 53 **E.** 61

Which number comes next in the following sequence?

$4\frac{7}{10}, 3\frac{2}{5}, 2\frac{1}{10}, \frac{4}{5}, \_\_\_\_$

A. $-\frac{3}{10}$ **B.** $-\frac{1}{2}$ **C.** $-\frac{1}{5}$ **D.** $-\frac{2}{5}$ **E.** $-\frac{3}{5}$

The shape below was constructed by two rectangles which have the same dimensions. The length of each rectangle is 16 cm and the width is 10 cm. A and B are points in the middle of rectangles' sides. What is the perimeter of the shaded quadrilateral?

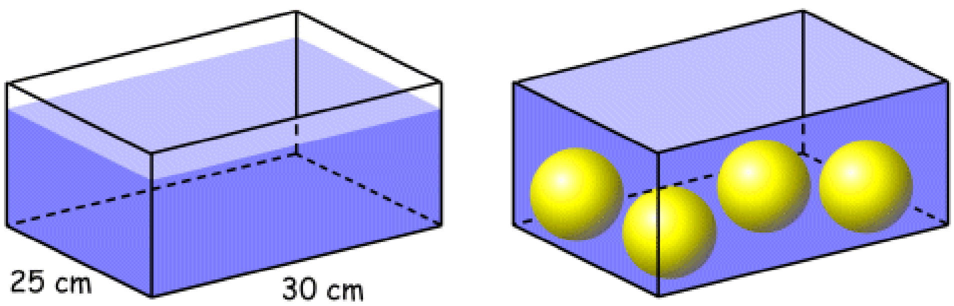


A. 13 cm **B.** 52 cm **C.** 8 cm **D.** 5 cm **E.** 26 cm

The areas of three squares are 16, 49 and 169. What is the average (mean) of their side lengths?

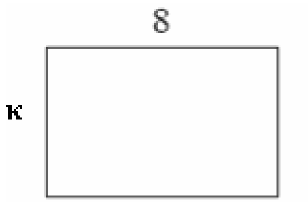
A. 8 **B.** 12 **C.** 24 **D.** 39 **E.** 32

There are two identical rectangular solids and their dimensions are as they appear on the picture below. If I add in one of them 12500 cm^3 water, then its $\frac{5}{6}$ will be filled. Then, if I place 4 balls, the vessel will be completely full. What is the volume of each ball?



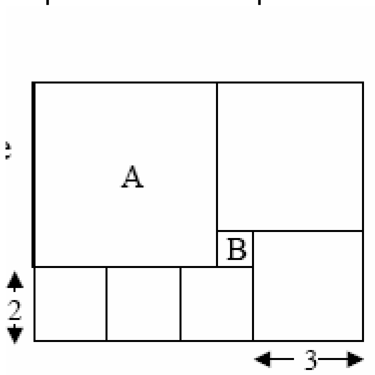
A. 2500 cm^3 **B.** 625 cm^3 **C.** 750 cm^3 **D.** $\frac{1}{6}$ cm^3 **E.** $\frac{1}{24}$ cm^3

In the diagram, the rectangle has a width of κ, a length of 8, and a perimeter of 24. What is the ratio of its width to its length?



A. 1:4 **B.** 1:3 **C.** 1:2 **D.** 3:8 **E.** 2:3

In the diagram below you see seven squares. Square A is the biggest and square B is the smallest. How many times does square B fit into square A?



A. 16 **B.** 25 **C.** 36 **D.** 49 **E.** 64

In the square shown, the product of the numbers in each row, column and diagonal is the same. What is the sum of the two numbers missing?

12	1	18
9	6	4
		3

A. 28 **B.** 15 **C.** 30 **D.** 38 **E.** 72

When the number 16 is doubled and the answer is then halved, the result is

A. 2^1 **B.** 2^2 **C.** 2^3 **D.** 2^4 **E.** 2^8

Multiple choice test 4

If $4a+8=32$, then $a+2=$

A. 4 **B.** 6 **Γ.** 8 **Δ.** 12 **E.** 16

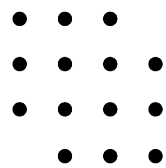
If $5^{\nu}+5^{\nu}+5^{\nu}+5^{\nu}+5^{\nu}=5^{25}$ where v is an integer, then v is equal to:

A. 2 **B.** 5 **Γ.** 10 **Δ.** 20 **E.** 24

Askas loves eating *"DAMA"* chocolates which cost 1 pound each. If with every 4 chocolate covers you get one free chocolate, how many *"DAMA"* chocolates can Askas eat if he spends 16 pounds;

A. 16 **B.** 19 **Γ.** 20 **Δ.** 21 **E.** 24

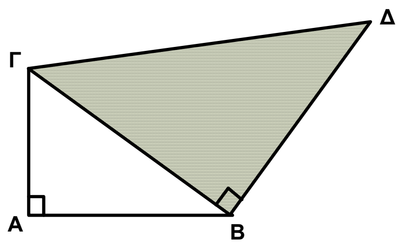
How many squares can be formed, by using 4 points as vertices;



A. 9 **B.** 11 **Γ.** 12 **Δ.** 13 **E.** None of the previous answers

---

In the figure below $\Gamma\hat{A}B=90^0$, $\Gamma\hat{B}\Delta=90^0$, $AB=5$, $A\Gamma=4$ and $\Gamma B=B\Delta$. The area of the $\Gamma B\Delta$ triangle is equal to:



A. 9 **B.** 4,5 **Γ.** 20,5 **Δ.** 41 **E.** 41^2

A square area is covered with 9 black square tiles of side α and 4 white square tiles of side 2α. The possibility that Harris stands in a white tile is equal to:

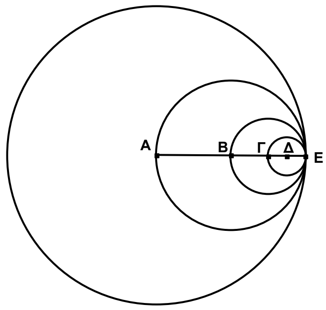
A. $\frac{4}{9}$ **B.** $\frac{16}{25}$ **Γ.** $\frac{8}{13}$ **Δ.** $\frac{4}{13}$ **E.** None of the previous answers

---

Number α is a prime number. The product of the factors of number $α^2$ is equal to:

A. α **B.** α^2 **Γ.** $2\alpha^2$ **Δ.** α^3 **E.** $3\alpha^3$

A, B, Γ and Δ are centers of the four circles that meet at point E as shown in the next figure. If the area of the circle with center Δ is π, then the radius of the circle with center A is equal to:



A. 4 **B.** 5 **Γ.** 8 **Δ.** 10 **E.** 12

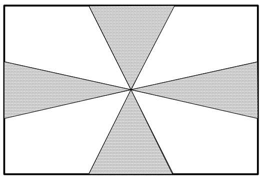
The remainder of the division of number $A = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 9 \cdot 10 + 50$ by the number 24 is equal:

A. 0 **B.** 4 **Γ.** 8 **Δ.** 12 **E.** None of the previous answers.

It takes 12 minutes for George to read from the beginning of the 12th page up to the end of the 17th page of a dictionary. If he starts from the 27th page at 6.20p.m., then at 6.55p.m. he will be reading page …

A. 42 **B.** 43 **Γ.** 44 **Δ.** 45 **E.** 46

Every side of the rectangle is divided into three equal segments. The line segments as appear in the figure are all passing through the center of the rectangle. The ratio of the area of the shaded region to the area of the unshaded region is equal to:



A. $1:1$ **B.** $1:2$ **Γ.** $1:3$ **Δ.** $2:3$ **E.** $3:4$

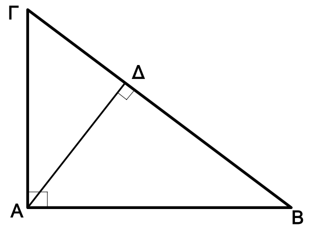
If x is increased by 25 percent, then by what percent is x^2 increased?

A. $6\frac{1}{4}$% **B.** 25% **Γ.** 50% **Δ.** $56\frac{1}{4}$% **E.** $156\frac{1}{4}$%

If $x + y = \frac{1}{5}$ and $x + \omega = \frac{1}{2}$, then the product $(2x + y + \omega)(\omega - y)$ is equal to: :

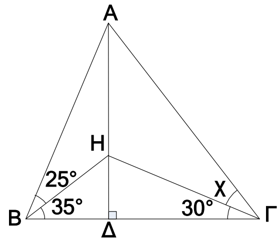
A. 0,04 **B.** 0,21 **Γ.** 0,7 **Δ.** 0,84 **E.** 1,7

ΑΒΓ is a right-angled triangle ($\hat{A} = 90°$), ΑΓ=6, ΑΒ=8 and $Α\Delta \perp ΒΓ$. The length of ΑΔ is equal to:



Α. 2,4 **Β.** 4 **Γ.** 4,8 **Δ.** 5 **Ε.** 6,4

ΑΔ is an altitude of the triangle ΑΒΓ. If Η is a point on ΑΔ such as $\widehat{ABH} = 25°$, $\widehat{ΗΒΔ} = 35°$ and $\widehat{ΗΓΔ} = 30°$, the measure of the angle $\widehat{ΗΓΑ}$ is equal to:



Α. 17,5° **Β.** 20° **Γ.** 22,5° **Δ.** 23,5° **Ε.** 25°

When a digital clock reads 3:38 p.m. the sum of the digits is 14. How many minutes after 3:38 p.m. will the sum of the digits be 20 for the first time?

Α. 42 **Β.** 132 **Γ.** 201 **Δ.** 251 **Ε.** 301

How many digits does the number $2^{12} \cdot 5^{8}$ have?

Α. 9 **Β.** 10 **Γ.** 11 **Δ.** 12 **Ε.** 13

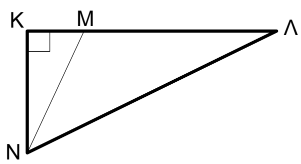
The sum of five consecutive integers is equal to A. The biggest of them in terms of A is:

Α. $\frac{A-10}{5}$ **Β.** $\frac{A+4}{5}$ **Γ.** $\frac{A+5}{4}$ **Δ.** $\frac{A-5}{2}$ **Ε.** $\frac{A+10}{5}$

$\frac{1}{4}\%$ of 2 is equal to:

Α. $\frac{1}{800}$ **Β.** $\frac{1}{200}$ **Γ.** $\frac{8}{100}$ **Δ.** $\frac{1}{2}$ **Ε.** $\frac{1}{8}$

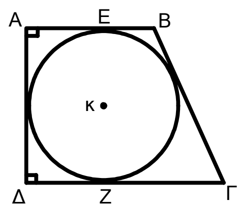
In the figure ΚΜ=$\frac{1}{4}$ΜΛ, $\hat{K} = 90^0$ and the area of the triangle ΜΚΝ is 80. The area of the triangle ΚΝΛ is equal to:



Α. 320 **Β.** 400 **Γ.** 480 **Δ.** 500 **Ε.** None of the previous answers

---

If $AE = \frac{a}{2}$ and $ΖΓ = 2 \cdot ΕΒ$ which of the follwing relations is the correct one?



A. $AE = \frac{3}{2} \cdot EB$ **B.** $AE = \sqrt{3} \cdot EB$ **Γ.** $AE = \frac{\sqrt{3}}{2} \cdot EB$

Δ. $AE = \frac{\sqrt{8}}{2} \cdot EB$ **E.** $AE = \frac{\sqrt{8}}{3} \cdot EB$

A square and an equilateral triangle have the same perimeter. The ratio of the area of the triangle to the area of the square is equal to:

A. $\frac{4\sqrt{3}}{9}$ **B.** $\frac{3}{4}$ **Γ.** $\frac{1}{1}$ **Δ.** $\frac{4}{3}$ **E.** Not possible to be completed using the given information.

An 8m length heavy – loaded truck covers a distance of 40m in 5 seconds. How many seconds does it take for the truck to pass over a bridge of 240m length?

A. 29 **B.** 31 **Γ.** 40 **Δ.** 41 **E.** 48

In a survey among 40 pupils, 13 pupils said they have a TV set in their bedroom, 18 pupils said that they have a P.C. in their, bedroom and 16 pupils said that they have neither TV set nor P.C. How many pupils said that they have got both of them in their bedroom?

A. 0 **B.** 3 **Γ.** 5 **Δ.** 6 **E.** 7

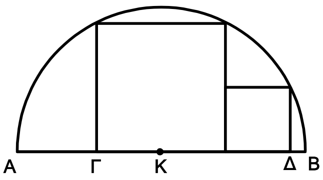
An integer number is called "*octanic*" if it is multiple of 8 or if at least one of its digits is 8. The number of "*octanic*" numbers between 1 and 100 is equal to:

A. 22 **B.** 24 **Γ.** 27 **Δ.** 30 **E.** 32

Demetris has 2 brothers more than sisters. His sister Maria has triple number of brothers than sisters. How many sisters does Demetris have?

A. 0 **B.** 1 **Γ.** 2 **Δ.** 3 **E.** 4

Two squares are inscribed in a semicircle of center K and radius $R = 2\sqrt{5}$ as it is shown in the diagram. The area of the bigger square is four times the area of the smaller square. The length of segment ΔB is equal to:



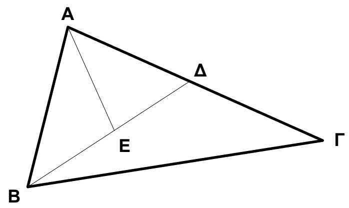
A. $2\left(\sqrt{5}-2\right)$ **B.** $2\sqrt{5}-2$ **Γ.** $\sqrt{5}-2$ **Δ.** $\sqrt{5}+2$ **E.** None of the previous answers

---

The sum of four consecutive integers can not be equal to:

A. 22 **B.** 202 **Γ.** 220 **Δ.** 222 **E.** 2006

If $\widehat{AB\Delta} = \widehat{EA\Delta} = \widehat{A\Gamma B}$ and $\widehat{AEB} = 100°$ as shown in the next figure, the measure of the angle $\widehat{BA\Gamma}$ is equal to:



A. 50° **B.** 60° **Γ.** 70° **Δ.** 80° **E.** Not possible to be completed using the given information.

The number $A = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{2006}$. Which of the following statement is true?

A. A is divisible by 7.
B. A is a prime number.
Γ. A has 3 as it last digit.
Δ. A is an even number.
E. A is greater than 2^{2007}.

LEVEL 2

A dairy industry, in a quantity of milk with 4% fat adds a quantity of milk with 1% fat and produces 1200 kg of milk with 2% fat.

The quantity of milk with 1% fat, that was added is (in kg)

A. 1000 B. 600 C. 800 D. 120 E. 480

The operation $\alpha * \beta$ is defined by $\alpha * \beta = \alpha^2 - \beta^2 \quad \forall \alpha, \beta \in R$.

The value of the expression $\mathrm{K} = \left[\left(1+\sqrt{3}\right) * 2\right] * \sqrt{3}$ is

A. 3 B. 0 C. $\sqrt{3}$ D. 9 E. 1

The domain of the function $f(x) = \sqrt{4+2x}$ is

A. $(-2, +\infty)$ B. $[0, +\infty)$ C. $[-2, +\infty)$ D. $[-2, 0]$ E. R

Given the function $f(x) = \alpha x^2 + 9x + \dfrac{81}{4\alpha}$, $\alpha \neq 0$.

Which of the following is correct, about the graph of f ?

A. intersects *x*-axis B. touches *y*-axis C. touches *x*-axis D. has minimum point E. has maximum point

If both of the integers α, β are bigger than 1 and satisfy $\alpha^7 = \beta^8$, then the minimum value of $\alpha + \beta$ is

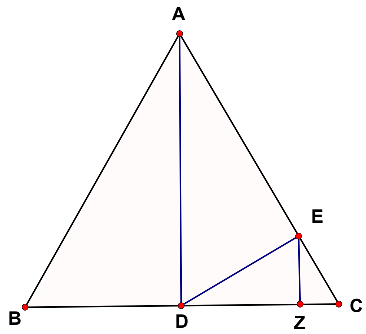
A. 384 B. 2 C. 15 D. 56 E. 512

The value of the expression $\mathrm{K} = \sqrt{19+8\sqrt{3}} - \sqrt{7+4\sqrt{3}}$ is

A. 4 B. $4\sqrt{3}$ C. $12+4\sqrt{3}$ D. -2 E. 2

In the figure, *ABC* is equilateral triangle and $AD \perp BC$, $DE \perp AC$, $EZ \perp BC$.

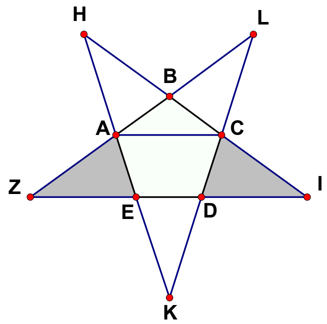
If $EZ = \sqrt{3}$, then the length of the side of the triangle ABC is



A. $\dfrac{3\sqrt{3}}{2}$ B. 8 C. 4 D. 3 E. 9

In the figure *ABCDE* is a regular 5-sided polygon and *Z*, *H*, *L*, *I*, *K* are the points of intersections of the extensions of its sides.

If the area of the «star» *AHBLCIDKEZA* is 1, then the area of the shaded quadrilateral *ACIZ* is



A. $\frac{2}{3}$ B. $\frac{1}{2}$ C. $\frac{3}{7}$ D. $\frac{3}{10}$ E. None of these

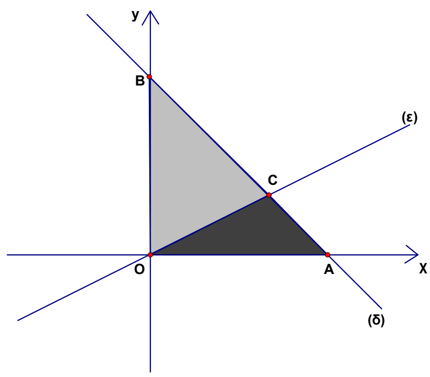
If $x=\sqrt[3]{4}-1$ and $y=\sqrt[3]{6}-\sqrt[3]{3}$, then which of the following is correct

A. $x=y$ B. $x<y$ C. $x=2y$ D. $x>y$ E. None of these

If $2^x=15$ and $15^y=256$, then the product xy equals

A. 7 B. 3 C. 1 D. 8 E. 6

The lines (ε): $x-2y=0$ and (δ): $x+y=4$ intersect at the point C. If the line (δ) intersects the axes Ox και Oy to the points *A* and *B* respectively, then the ratio of the area of the triangle *OAC* to the area of the triangle *OBC* equals



A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{3}{5}$ D. $\frac{1}{2}$ E. $\frac{4}{9}$

If $f(\alpha,\beta)=\begin{cases}\alpha & \text{if } \alpha=\beta \\ f(\alpha-\beta,\beta) & \text{if } \alpha>\beta \\ f(\beta-\alpha,\alpha) & \text{if } \alpha<\beta\end{cases}$, then $f(28,17)$ equals

A. 8 B. 0 C. 11 D. 5 E. 1

The sum of the digits of the number $10^{2006}-2006$ is

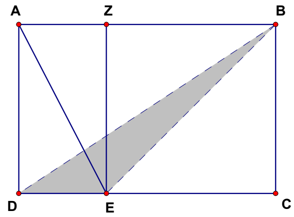
A. 18006 B. 20060 C. 2006 D. 18047 E. None of these

The rectangle *ABCD* is a small garden divided to

to the rectangle *AZED* and to the square *ZBCE,* so

that $AE = 2\sqrt{5}\,m$ and the shaded area of the triangle

DBE is $4m^2$. The area of the whole garden is



A. $24m^2$ B. $20m^2$ C. $16m^2$ D. $32m^2$ E . $10\sqrt{5}\,m^2$

The expression : $\dfrac{1}{2+\sqrt{7}}+\dfrac{1}{\sqrt{7}+\sqrt{10}}+\dfrac{1}{\sqrt{10}+\sqrt{13}}+\dfrac{1}{\sqrt{13}+4}$ equals

A. $\dfrac{3}{4}$ B. $\dfrac{3}{2}$ C. $\dfrac{2}{5}$ D. $\dfrac{1}{2}$ E. $\dfrac{2}{3}$

If x_1, x_2 are the roots of the equation $x^2 - 2kx + 2m = 0$, then $\dfrac{1}{x_1}, \dfrac{1}{x_2}$ are the roots of

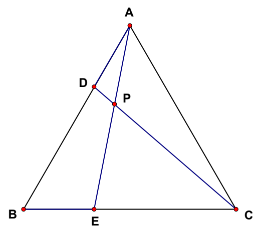
the equation

A. $x^2 - 2k^2x + 2m^2 = 0$ B. $x^2 - \dfrac{k}{m}x + \dfrac{1}{2m} = 0$ C. $x^2 - \dfrac{m}{k}x + \dfrac{1}{2m} = 0$ D. $2mx^2 - kx + 1 = 0$ E. $2kx^2 - 2mx + 1 = 0$

ABC is equilateral triangle of side α and

$AD = BE = \dfrac{\alpha}{3}$.

The measure of the angle $\angle CPE$ is



A. 60^0 B. 50^0 C. 40^0 D. 45^0 E. 70^0

K(k,0) is the minimum point

of the parabola and the parabola

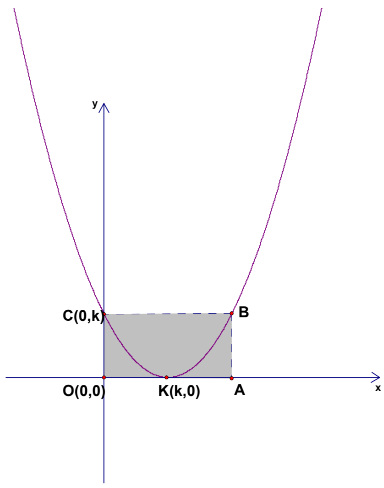
intersects the y-axis at the point

C (0, k).

If the area of the rectangle

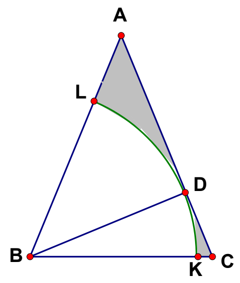
OABC is 8, then the equation of

the parabola is



A. $y = \dfrac{1}{2}(x+2)^2$ B. $y = \dfrac{1}{2}(x-2)^2$ C. $y = x^2 + 2$ D. $y = x^2 - 2x + 1$ E. $y = x^2 - 4x + 4$

In the figure ABC is isosceles triangle with $AB = AC = \sqrt{2}$ and $\angle A = 45^0$. If *BD* is altitude of the triangle and the sector *BLDKB* belongs to the circle (*B,BD*), the area of the shaded region is



A. $\dfrac{4\sqrt{3}-\pi}{6}$ B. $4\left(\sqrt{2}-\dfrac{\pi}{3}\right)$ C. $\dfrac{8\sqrt{2}-3\pi}{16}$ D. $\dfrac{\pi}{8}$ E. None of these

The sequence $f : \mathbb{N} \to R$ satisfies $f(n) = f(n-1) - f(n-2), \forall n \geq 3$.

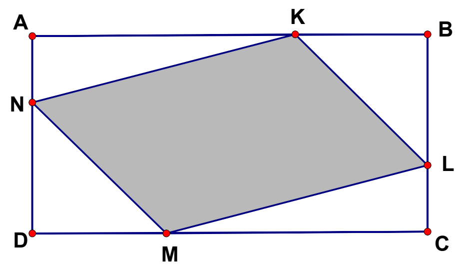
Given that $f(1) = f(2) = 1$, then $f(3n)$ equals

A. 3 B. -3 C. 2 D. 1 E. 0

A convex polygon has *n* sides and 740 diagonals. Then *n* equals

A. 30 B. 40 C. 50 D. 60 E. None of these

ABCD is rectangular and the points *K,L,M,N* lie on the sides *AB, BC, CD, DA* respectively so that $\dfrac{AK}{KB} = \dfrac{BL}{LC} = \dfrac{CM}{MD} = \dfrac{DN}{NA} = 2$. If E_1 is the area of *KLMN* and E_2 is the area of the rectangle *ABCD*, the ratio $\dfrac{E_1}{E_2}$ equals



A. $\dfrac{5}{9}$ B. $\dfrac{1}{3}$ C. $\dfrac{9}{5}$ D. $\dfrac{3}{5}$ E. None of these

Of 21 students taking Mathematics, Physics and Chemistry, no student takes one subject only. The number of students taking Mathematics and Chemistry only, equals to four times the number taking Mathematics and Physics only. If the number of students taking Physics and Chemistry only equals to three times the number of students taking all three subjects, then the number of students taking all three subjects is

A. 0 B. 5 C. 2 D. 4 E. 1

The number of divisors of the number 2006 is

A. 3 B. 4 C. 8 D. 5 E. 6

Using the musical instruments guitar, bouzouki and violin we'll make a 4 member orchestra which will consist of at least 2 different instruments. The number of such orchestras is

A. 12 B. 15 C. 11 D. 14 E. 13

The maximum number of points of intersection between three different circles and one line is

A. 9 B. 10 C. 11 D. 12 E. None of these

In the expansion of $\left(x^2+\frac{1}{x^2}\right)^4$ the value of the term which is independent of x is

A. 2 B. 6 C. 4 D. 10 E. 12

An isosceles triangle ABC has one obtuse angle, φ is the measure of each of its acute angles and $AB=AC=\alpha$. If *BD* is the altitude of the triangle, then *CD* equals

A. $\alpha(1+\cos\varphi)$ B. $\frac{\alpha(1-\cos 2\varphi)}{2}$ C. $\alpha(1+\cos 2\varphi)$ D. $2\alpha(1+\cos\varphi)$ E. $\alpha(1+\sin 2\varphi)$

Given the function $f:\mathbb{N}\rightarrow\mathbb{N}$ with $f(n)=\begin{cases} n+1 & ,if\ \ n\ is\ odd \\ n^2 & ,if\ \ n\ is\ even \end{cases}$.

The value of $f\left(f\left(f(3)\right)\right)$ is

A. 27 B. 81^2 C. 128 D. 64 E. 256

If $x=2^{100}$, $y=3^{75}$, $z=5^{50}$, then which of the following is correct

A. $x<y<z$ B. $x<z<y$ C. $y<z<x$ D. $y<x<z$ E. None of these

PART II-MOTIVATION

1.14 Introduction

After the correct identification of mathematically talented students, how to get the best performance from each student and motivate him or her is a challenging task, because mathematically talented students vary greatly in degree of talent and motivation. Clearly, there is no single approach for all of these students. The design of each student's instructional program should be based on an analysis of individual abilities and needs (Clark, 1997). For example, students with extremely high ability and motivation may profit more from a program that promotes rapid and relatively independent movement through instructional content; other students may do better in a program that is not paced so quickly (Miller, 1990).
However, it needs to be emphasized that to motivate mathematically talented students to reach the maximum of their abilities, all influential forces in their lives must be used wisely. These forces are (a) parental guidance and support; (b) teachers and curriculum programs; and (c) community support. In other words, there are three different levels on which mathematically talented students may be motivated (Fig. 1):

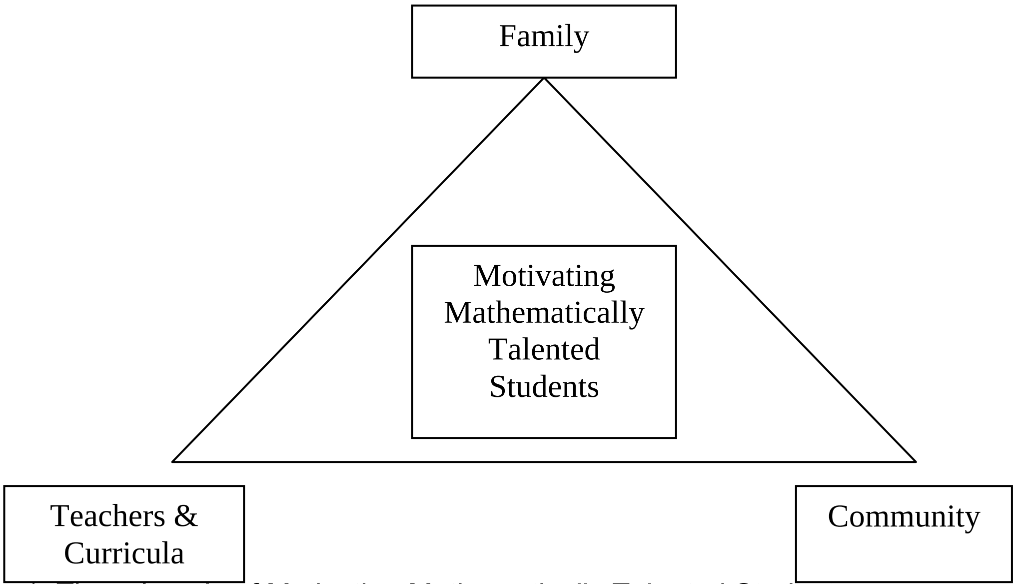


Figure 1. Three Levels of Motivating Mathematically Talented Students

Level 1: Parental Guidance and Support
Research has established the importance of parents' attitudes, guidance and support about the academic self-perceptions and achievement of their children. These findings have been confirmed in studies on parental influence on math self-concept of talented children (McGillicuddy-De Lisi, 1985; Parsons, Adler & Kaczala, 1982). Parents are viewed as the primary influence, although not because they could provide advanced mathematical training and assistance. Rather, parents play a crucial role in providing the supportive *cognitive and emotional framework* in which children can grow, learn mental discipline, focus their abilities, appreciate their own strengths and weaknesses, cultivate a desire to learn and take advantage of opportunities, and develop as thoughtful and kind people.

Level 2: Teachers and Curricula
As sources of encouragement, challenge, and support, teachers are a very powerful influence on mathematically talented students. Also, challenging curriculum programs explored in a nurturing classroom can encourage creativity and achievement. Each of these factors is discussed below.

The Role of Teachers: The teacher has two key roles in supporting gifted students' learning. First, the teacher needs to select tasks that are appropriately challenging and promote *cognition* (e.g. high-level thinking and reasoning such as exploring patterns and relationships, producing holistic and lateral solutions), *metacognition* (e.g. comparing and developing various methods of problem solving) *and motivation* (e.g. solving challenging tasks); second, to provide opportunities for students to engage in these tasks without reducing their complexity and cognitive demands (Henningsen & Stein, 1997). In particular, providing mathematically talented students with challenging tasks enhances their motivation and self-esteem (Bandura et al., 1996; Lupkowski-Shoplik & Assouline, 1994; Vallerand et al. 1994).

The teacher can provide *extrinsic* and *intrinsic* support to the students by engaging in the practice of *cognitive apprenticeship*, i.e. teaching strategies for developing expertise (Collins, Brown, & Newman, 1989). *Extrinsic support* is provided to the learners through *scaffolding, modelling, and coaching*; these are considered key factors in facilitating high-level thinking and reasoning (Henningsen & Stein, 1997). *Intrinsic support* is provided by the teacher facilitating the processes of exploration and reflection on ideas and by scaffolding the student's construction of meaning (Collins et al., 1989; Henningsen & Stein, 1997).

Curriculum Programs for Mathematically Talented Students: No matter where they obtain their education, mathematically talented students need an appropriately differentiated curriculum designed to address their individual characteristics, needs, abilities, and interests. Overall, there seem to be some important aspects in creating motivating curriculum programs for mathematically talented students (Miller, 1990; Rotigel & Lupkowski, 1999; Stanley, 1991; Velazquez, 1990).

1. The mathematics curriculum should bring mathematically talented students to work collaboratively (Tomlinson et al., 1997). Students will benefit greatly, both academically and emotionally, from this type of experience. They will learn from each other, reinforce each other, and help each other over difficulties. Talented students learn best in a nurturing, emotionally safe, student-centred environment that encourages inquiry and independence, includes a wide variety of materials, is generally complex, and connects the school experience with the greater world.
2. The mathematics curriculum should stress mathematical reasoning and develop independent exploratory skills (Niederer & Irwin, 2001). For instance, this is exemplified by using problem solving and discovery learning, engaging in special projects in mathematics, discovering formulas, looking for patterns, and organizing data to find relationships. Activities should help students to develop structured and unstructured inquiry, reinforce categorization and synthesis skills, develop efficient study habits, and encourage probing and divergent questions.
3. The mathematics curriculum should de-emphasize repetitious computational drill work (Velazquez, 1990). Mathematically talented students need more time with extension and enrichment opportunities. The scope of the mathematics curriculum should be extensive so that it will provide an adequate foundation for students who may become mathematicians in the future. In many programs the mathematics curriculum will have to be greatly expanded to meet this need. Providing an interdisciplinary approach is another way of meeting these students' needs. Researchers have found that talented students benefit greatly from curriculum experiences that cross traditional content areas, particularly when they are encouraged to acquire an integrated understanding of knowledge and the structure of the disciplines.
4. The mathematics curriculum should be flexibly paced (on the basis of an assessment of students' knowledge and skill). Curricula for mathematically talented students should promote self-initiated and self-directed learning and growth. Content, as well as learning experiences, can be modified through acceleration, compacting, variety, reorganization, flexible pacing, and the use of more advanced or complex concepts, abstractions, and materials. In particular, flexibility can be achieved in the following ways (Miller, 1990):
    - *Continuous progress*. Students receive appropriate instruction daily and move ahead as they master content and skill.
    - *Compacted courses*. Students complete two or more courses in an abbreviated time.
    - *Advanced-level courses*. Students are presented with course content normally taught at a higher grade.
    - *Mentoring*. Students participate in mentoring-paced programs.
    - *Grade skipping*. Students move ahead 1 or more years beyond the next level of promotion.
    - *Early entrance*. Students enter elementary school, middle school, high school, or college earlier than the usual age.
    - *Concurrent or dual enrolment*. Students at one school level take classes at another school level. For example, an elementary school student may take classes at the middle school.
    - *Credit by examination*. Students receive credit for a course upon satisfactory completion of an examination or upon certification of mastery.

Level 3: Community

Finally, the influence of the community is as equally important in motivating mathematically talented students as the previous two factors. For example, local organizations, governments, and universities can play a special role in supporting the growth of mathematically talented students. Below we provide some practical examples of how to promote intrinsic and extrinsic motivation.

1.15 Intrinsic Motivation

b. *The use of 'open' problems*

One extremely important instructional technique for mathematics gifted education is scaffolding. One way of promoting mathematical scaffolding is the use of 'open' problems (a-didactic situations, Brousseau, 1997). *Open problem* may be defined as the problem that does not require specific mathematical theorems, mathematical concepts or specific methods to be solved. Even in the case that a particular mathematical theorem or a specific mathematical concept or method is required, the student is unaware of this. Open problems demand from students to start exploring the problem, make hypotheses, propose ways of solving it, and validate their solutions.

Examples:
a. Two individuals A and B are on the same side of the river. Which is the shortest way for A to go into the river, pick up some water and take it to individual B?

This problem is open for students who haven't been taught the concept of axis symmetry and haven't done any problems related to it. It is noted that when students work on this problem, they try many ways to find the shortest way (situation of action); they explain these ways to other students or to the teacher (situation of formulation); and finally, they try to validate this process by providing a mathematical reasoning that justifies their solution (situation of validation). This third step of situation is very important because it does not only require personal validation but also validation within their peer group. This implies that the student should not simply convince his classmates but also take into consideration their point(s) of view.

In the above problem, for instance, many solutions may be proposed by the students during the situation of validation. However, the student who validates his/her solution (i.e. to find the symmetrical of A in relation to the river) should prove to his/her classmates that as the line segment is the shortest way between two points his solution is the correct one (given that the students do not have a ruler to measure the distances, it is observed that they propose erroneous procedures to solving this problem).

- A football player is running parallel to the line of the opponents' goal area. At which point along this line, the player has to shoot in order to have the highest possibility of scoring a goal.

c. *The use of historical topics in mathematics or historical mathematical problems (e.g. proofs of well known theorems; mathematical paradoxes; obstacles of well known mathematicians related to some mathematical concepts etc.)*

6. For example, the Pythagorean theorem (direct and inverse) has many solutions. It may be interesting to the students to present them solutions proposed by famous individuals (not only mathematicians) and to discuss the kind of thinking embedded in these proofs.
7. The students can explore famous mathematical paradoxes in the history of mathematics such as Zeno's Paradox and Cantor's Infinities.
8. Some mathematical concepts have been marked by obstacles experienced by famous mathematicians. For example, consider the rule of signs (-) x (-)=+ (G. Glaeser). The students may explore D' Alembert or Euler's attempts to prove this rule. The seeking out and employing of historical mathematical problems can be a rewarding and enriching experience.

d. *To stimulate students' curiosity, we use of problems related to well known historical and literary figures (e.g. Alexander the Great, Pinocchio etc.).*

1.16 Extrinsic Motivation

d) The first kind of extrinsic motivation is concerned with the organization of competitions and Olympiads, and the provision of prizes, rewards etc. All the pedagogues and educational researchers agree that game playing is an important tool for learning. Competitions and prizes are important ways that provide motivating challenges to mathematically talented students.
e) The second kind of extrinsic motivation is related to the organization of teaching (irrespective of the kind of problem presented to the students). For example, two students work independently on the same problem. After they finish, they compare and contrast their solutions. This is motivating to students not only because of the curiosity involved but also because it is a situation of validation of the solution for both students.
f) The third kind of extrinsic motivation is problem posing or ladder construction (such as the ladders provided in this manual). Here students construct problems or activities related to some data or mathematical concepts proposed to them.
g) Considerations on game and a-didactic situations. In the "Theory of the Situations" of G. Brousseau, the a-didactic situations have three phases: phase of action, phase of formulation and phase of validation. The phase of action corresponds to mathematics in reality and consists of making proper the decisive strategies in a situation of concreteness. The phase of communication consists of finding a code of communication to communicate the strategy being used. Finally, the situation of validation is that in which the participants decide who came up with the optimal strategy. In order to answer this question, the students have to formulate "theorems in action" that allow the optimisation of possible solutions. Thus, from a pedagogic point of view, the "game" assumes a very important role. The student learns to move from the phase of action to the public negotiation (in class and without the direct intervention of the teacher) of all the possible strategies (the theorems in action). The teacher prepares the a-didactics situation and remains arbiter of the rules that need to be respected. All the phases are directly managed by the students.
h) The use of animation in the teaching of mathematics

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