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Helping students of primary and secondary school to make invisible thoughts visible in Euclidean geometry

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Abstract. In this study we investigate a way to help pupils in the last class of a primary school (12-year old pupils) and students in the last class of a Gymnasium (15-year old students) to express geometrical thoughts, not expressed by them although in their mind, when solving a problem or constructing a proof. To empirically test our theoretical conjectures, an empirical research was conducted in a primary school and in a Gymnasium in Athens, Greece. The sample consisted of a total of 122 children (55 pupils in primary school and 67 students in Gymnasium). A pre-post research design in each group was selected to examine for possible improvements of each group separately between two different teaching methods applied to them: a traditional one and the proposed one. Normality tests gave us evidence that our data followed the normal distribution, thus enabling us to use parametric statistical tests. Descriptive statistics (i.e. relative frequencies, means, standard deviations) were used to shape a broad view of the sample characteristics. Also, inferential statistical methods (i.e. bivariate correlations, paired samples t-tests) were used to check for possible statistically significant differences between pre- and post- measures of each group's scores. Research results indicated that students in both grades significantly improved their performance in solving mathematical problems in Primary School, or in constructing geometrical proofs in Gymnasium, when they used the proposed method. Despite the fact that students were in different educational grades, ages, places and time, the same obstacles were found to be responsible for their difficulties in solving a problem or proving a geometrical proposition.
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1. Introduction

Nowadays, it is well known to mathematics teachers and researchers that students have difficulties in geometry. Particularly, students deal with difficulties in proving. There is a rich literature about it (Chazan, 1993; Hart, 1994; Martin and Harel, 1989; Senk, 1985; Harel and Sowder, 2007; Herbst, 2002; Hoffer, 1981).
De Villiers (2007, p.189) refers: *"A problem may at first glance look quite challenging. Where does one start?"* The question "*where does one start?*" is a very important one. Often, primary school pupils nor understand neither know how and where from to start the solution of a problem. Also, Gymnasium students do the same, and particularly those in the last class of Gymnasium (aged 15), who mainly learn to make intuitive proofs, do not feel the need to construct a proof. So, de Villiers (1990) wonders: *Who has not yet experienced frustration when confronted by students asking* "why do we have to prove this?" Moreover, they do not know nor why neither how to start constructing a proof. We must note that although students in Gymnasium (12-15 years old) are not taught how to construct proofs, those students in the last class of Gymnasium (15-years old) learn to compare triangles, i.e. to prove if two triangles are equal. However, a way to start a solution of a problem or to construct a proof is to write down the given and hypotheses and make some thoughts on them. Then, students can discuss these thoughts with their classmates or with themselves, if they are alone. The important is

that they have to bring out their thoughts to another or to themselves. In other words students must finally learn to discuss with themselves although they don’t seem to understand it.

2. Children refuse to express their thoughts

Here, we describe how we helped pupils in the last class of a primary school to express geometrical thoughts, not expressed by them, although in their mind when solving a problem. In particular, it seemed to us that some students don’t understand the need to express these thoughts. Also, we applied the same method to students in the last class of Gymnasium (aged 15). These students are not taught to prove geometry propositions yet, but they often construct simple, intuitive proofs. We noticed that, although they knew the answer to a question, in fact they did not express their thoughts, in the way pupils of last class of primary school had done earlier.

We must note that below we describe two case studies that took place in two different grade schools. The first one was a school of the primary education and the second one was a Gymnasium that belongs on secondary stage in Greek educational system. The pupils of primary school were 12 year-old and the students of Gymnasium were 15-year old. We were interested in understanding whether children use the inferences that result from the given in order to accomplish their task, i.e. the primary school pupils to solve a geometrical problem and the students in the last class of Gymnasium to construct a proof. Also, we must note that these case studies took place in different time. Participants were 122 children of primary and secondary school.

It is worth mentioning that despite the different educational stages, ages and time that these case studies took place we noted some similarities that did not allow students to carry a solution or a proof through. We describe these similarities later in this article.

2.1 Pupils in Primary School refuse to express their thoughts and what we proposed

Pupils of the last class of Primary school learn to solve problems like the one that follows:

What are the missing angles x, y and z in the following figures?

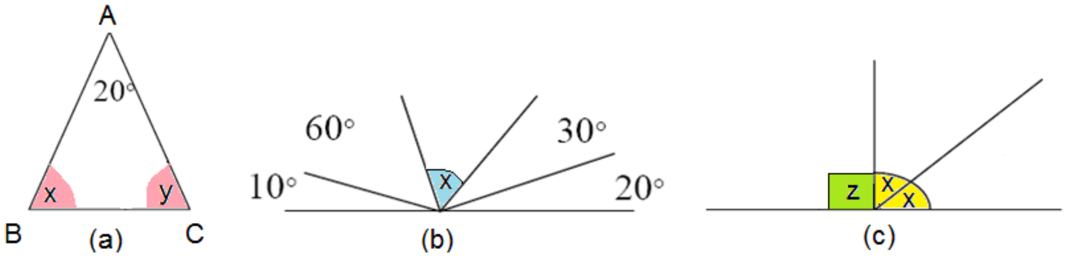


Figure 1.

We noticed that the majority of pupils could not find a point to start the solution of the problem. Pupils read the given of the problem several times, but they did not do anything more. After that, we discussed with some of these pupils about the geometrical figures (a), (b) and (c) ‘figure 1’. We asked them to describe what they observed after looking carefully at the figure for many times. The only thing they described was what it was written on the figure. For example, regarding figure (a) they observed that the triangle ABC was an isosceles one and that the angle A was 20^o, and although we expected that they would tell us that angles x and y are equal, they stopped and they did not say anything else. We wondered if these students knew that the angles on the base of an isosceles triangle are equal. But when we asked them if they knew anything about the angles x and y they answered that these angles were equal, so we understood that they actually knew that the angles x and y were equal. We asked them to explain why they didn’t say that the angles x and y were equal when we asked them to describe figure (a). Actually, we were surprised by the answer of the majority of pupils, who replied that they knew that the angles x and y were equal but they didn’t know their degrees and so they thought it was redundant to refer to these angles. The following dialog between the teacher and Thanassis is very characteristic:

> Teacher: *Tell me what you can see in this figure?* (Teacher refers to figure (a)).
> Thanassis: *It is a triangle…. an isosceles triangle. The angle A is 20^o.*

Teacher: *Which are the equal sides?*
Thanassis: *The equal sides are … the side AB is equal to side AC.*
Teacher: *Then?*
Thanassis:….. (he does not reply)
Teacher: *You told that the triangle ABC is an isosceles triangle with side AB equal to AC. Do you know anything else about the isosceles triangle?*
Thanassis:….. (he does not reply)
Teacher: *Don't you know anything about the angles of an isosceles triangle?*
Thanassis: *Yeah! The angles on the basis of the isosceles ABC are equal.*
Teacher: *Very well. Why don't you say it?*
Thanassis: *Well…Because… I don't say anything about angles B and C, because I do not know anything about them.*
Teacher: *But before you said that the angles B and C of are equal*
Thanassis: *Yes, they are equal but I am looking for them.*

Apparently, pupils believe that is redundant to use all the information that is contained in the given data of the problem. We explained to pupils that they had to write down all the information about the subject matter which they dealt with. This information exceeds the information that is contained in the given data of the problem. To help them to do this we separated the information about the subject matter in two categories. The first category included information that is described as 'given', e.g. angle A is 20^o, sides AB and AC are equal and the second category included information that is the 'result of the given', e.g. angles x and y are equal, the sum of the angles in a triangle is 180 degrees, etc. Although it seemed that students had understood these categories they did not achieve to write down all the information about the subject matter, i.e. the two categories. Particularly, they totally omitted to write the second category or part of it. There was a block in their mind that it was responsible for their difficulty in mentioning all the information that was the result from given. Rather, they believed that they should not use the whole information. This exactly reveals Thanassis' answer: "*I don't say anything about angles B and C, because I do not know anything about them… they are equal but I am looking for them*". Thanassis thought that he should not use anything of the angles B and C. According to Clements & Sarama, (2000) and Gagatsis & Patronis (1990) young children's conceptions remain constant after six years of age, without necessarily being accurate. These constant conceptions, lacking accuration, function as the cognitive obstacles that Bachelard (1968) and Brousseau, (1983) are referred to and they are responsible for the creation of the aforementioned block in pupils' mind. We tried to find a way to exploit this 'block' and we seized the opportunity that this block provided us. So, we called 'blocked information' the information that results from the given. We also suggested students to write their thoughts using two color pens. Students should write down the given of the problem using the first color (blue), and they were supposed to use the other color (red) to write the blocked information. In this way, we succeeded to make pupils write all the information contained in the given geometrical propositions.

2.2. Students in Gymnasium refuse to express their thoughts and what we proposed
The Curriculum of Gymnasium does not include teaching of proofs of geometrical propositions. When students come across some proofs these are mainly intuitive proofs. Although students of Gymnasium generally aren't taught proving, students of the last class of Gymnasium (15-years old) are taught to construct proofs that refer only to equality of triangles. Also, according to the Curriculum, students of Gymnasium are taught isosceles triangles, equilateral triangles and their properties, some initial knowledge about the notion of circle (radius, arc, tangent, etc.). Also they are taught parallel lines and the related angles with parallels lines.
According to Hanna (1990) a formal proof of a given sentence is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last sentence is the one to be proved.
Students are beginners in proving formal proofs in last class of Gymnasium. Although, teacher teaches certain examples of proofs, he/she doesn't teach students how to construct a proof comprehensively because the Curriculum of Gymnasium does not include teaching of formal proofs. This fact makes things worse. Anyways, the aforesaid are included in traditional teaching and it is generally the prevalent situation. Knowing this situation we were careful on how to teach students of last class of Gymnasium to construct proofs. We

used "Reasoning Control Matrix for the Proving Process" (RECOMPP) 'Appendix A'. RECOMPP[1] is a reusable matrix pattern that helps students produce reasoning production. Its layout and its filling technique are predefined. More analytically, it consists of six discrete sections and its layout consists of rows, columns, and cells that may contain figures, hypotheses or conclusions, proofs, and partial proofs (Dimakos, et al., 2007). Section 2 of RECOMPP is where the hypotheses and the conclusions of the problem must be written. Here, the student is given a table (consisted of two rows and two columns), 'figure 2' where he/she must write down, in two separate lines, the hypotheses, and the conclusions of the problem, respectively.

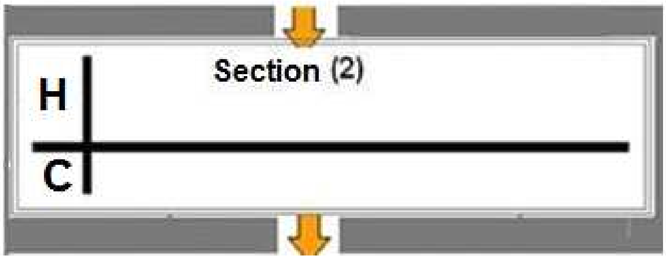


Figure 2. Section two of RECOMPP

Section 5 of RECOMPP is the most important section of all, because it is where students produce reasoning. Section 5 is where the student is motivated to reason, collect, and write those statements and relationships among the elements of the sketch prepared before that will lead him/her to the successful writing of the proof. Here, the student is given a table (consisting of just two columns and several rows). The student must write a statement e.g. "Statement A", that needs to be proved, in this table, in the first column, labelled "To prove that … ". The student must write a statement e.g. "Statement B", that is necessary in order to prove "Statement A", in the second column, labelled "It is required to prove that". Then they write the proof in the next section 6 'figure 3'.

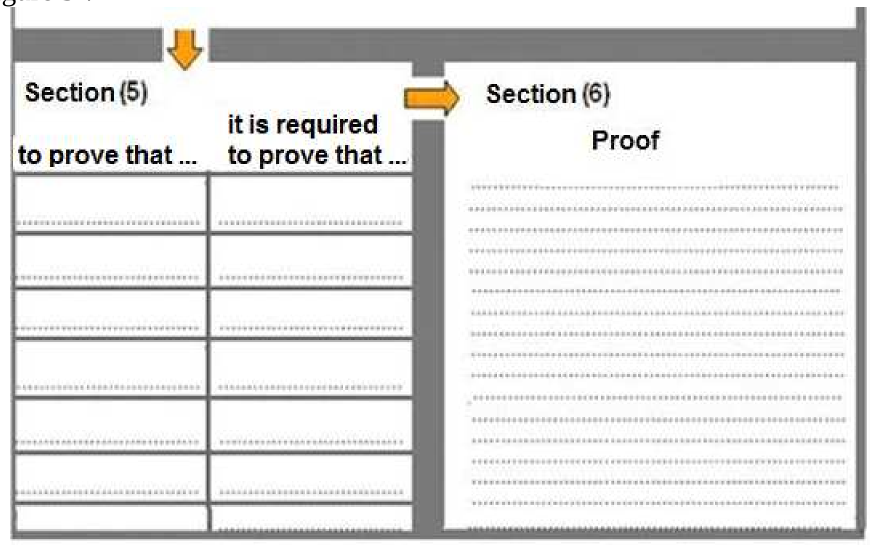


Figure 3. Section 5 and Section 6 of RECOMPP

In this case we noticed that students of last class of Gymnasium reacted similarly to the pupils of the last class of primary school. In particular, most of these students drew the geometrical figure, (some of them faced difficulties while drawing geometrical figures), next they filled section 2 of RECOMPP with the hypo-

[1] A detailed description of RECOMPP is given in the article *Developing a proof-writing tool for novice Lyceum geometry students* (Dimakos et. al., 2007).

theses and they stopped. Some students who completed section 2 stopped at section 5. Below we describe a case where at first most students stopped in section 2 and next they stopped again in section 5.
We gave students the following proposition:

Extend the base BC of an isosceles triangle ABC from both sides. Take on extensions two points such BD = CE. Prove that the triangle ADE is an isosceles one.

We noticed that students drew the figure and marked the equalities on it (e.g. they put a tick on the equal sides of the isosceles triangle ABC, and they put a tick on segments BD and CE). Also, they completed section 2 of RECOMPP and they stopped ‘see figure 4’.
It seemed to us that they did not have the required knowledge to construct the proof, or they didn’t know how to go further to the solution. After discussing with them we realized that students had all the necessary pieces of knowledge required for the solution, i.e. that the base angles of an isosceles triangle are congruent (equal), that they had to compare triangles, and also all the related theorems, etc., but they had stopped. They had done nothing from what they had told us and they were supposed to do. We must note that they took into account the given, but they could not understand that they had to use certain inferences resulting from the given. We analyze these topics below.
Initially, students stopped when they filled the given in section 2 of RECOMPP. Here, our first observation was made. Students after taking into account the proposition they filled section 2 of RECOMPP by writing the hypotheses, but they didn’t translate them into mathematical relations.

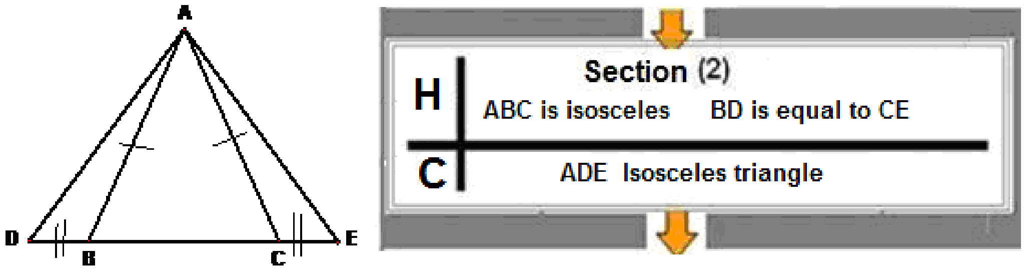


Figure 4.

In order to help them to translate hypotheses into mathematical relations we improved section 2 of RECOMPP. Particularly, we divided section 2 in two parts ‘figure 5’. In the second part students had to analyse and translate hypotheses of propositions into mathematical relations. Moreover, we motivated students to do so. Students went further to the solution writing the analysis of hypotheses but they stopped again in Section 5 of RECOMPP.

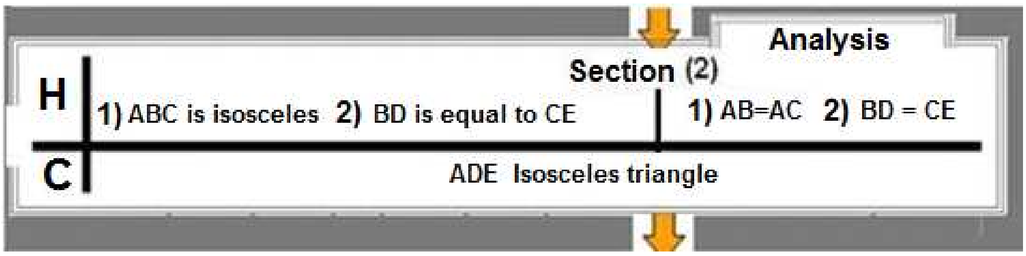


Figure 5.

In Section 5 students got confused. Although, they begun the filling of columns and rows in Section 5, they stopped. Some of them stopped filling section 5 at the first step or they didn’t even begin. In this point we understood that students did not know what they have to fill because they did not take into account the aforesaid ‘blocked information’. Next, we discuss the case of a student that could not fill section 5 because she didn't use derived information from the given. Teacher in order to understand what happened with Kelly had the following characteristic dialog with her. After the dialog the teacher understood that Kelly didn't know how to start to construct a proof and that she would stop filling section 5 when she would have to write a sentence that derived from the given.

Teacher: *Why did you stop filling this section? (Teacher refers to section 5).*
Kelly: *I do not know what I am supposed to write now…*
Teacher: *Tell me the proposition by looking only at the geometrical figure, not reading it.* (It is worth mentioning that the figure had been drawn by her).
Kelly: *We have an isosceles triangle ABC. Sides AB and AC are the equal sides….*
(Although she was looking at the figure, where she had marked that segments BD and CE were equal, she could not remember that they were equal).
Teacher: *What is it that you have to prove?*
Kelly: *….hmm…I do not remember…*
Teacher: *Well. Now, read the proposition again for several times, and look at the figure very carefully. Before beginning to write a solution you should be able to tell the proposition by only looking at the figure.*
(After some minutes)
Kelly: *We can start the solution.*
Teacher: *Are you sure?*
Kelly: (She smiled) *Yes…*
Teacher: *O.K. I am listening to you.*
Kelly: *The triangle ABC is an isosceles triangle, sides AB and AC are the equal sides and the extensions BD and CE are equal, too. We join points D and E with point A and we must prove that the triangle ADE is an isosceles one.*
Teacher: *Excellent! Well, what must we do now?*
Kelly: *We must compare the triangles….. ah…. ABD and ACE*
Teacher: *O.K., Compare the triangles ABD and ACE.*
Kelly: *The triangles ABD and ACE are congruent, because they have AB = AC and BD = CE,…*
Teacher: *Well…*
Kelly:……(she does not speak)
Teacher: *Which theorem do you need?*
Kelly: *I am not sure …..*
Teacher: *Which theorem do you believe that you need?*
Kelly:……(she does not speak)
Teacher: (teacher supposes that she doesn't know the related theorems). *Do you remember the related theorems?*
Kelly: *Yes.*
Teacher: (teacher wanted to be absolutely sure that Kelly knew the related theorems) *Do you want to tell me these theorems?*
Kelly: *Yes. ….If, in one triangle, two sides and the angle between them are equal to two sides and the angle between them of another triangle, then these triangles are equal. …….If one side and two adjacent angles of one triangle are equal to a side and two adjacent angles of another triangle, then these triangles are equal. ……If three sides of one triangle are equal in length to three sides of another triangle, then these triangles are equal*
Teacher: *Excellent! Compare the triangles ABD and ACE. Before, you told me that they are congruent.*
Kelly: *I am confused!*
Teacher: *So, you say that you are confused! Why?*
Kelly: *Because I have found two equal sides ….and all theorems contain two equal sides, and something else. So, I do not know which one I need in order to go on.*
Teacher: *Why don't you try one of them?*
Kelly: *Which one of them?*

Teacher: *Is there anything else in the proposition that could help you go on?*
Kelly:……(she does not speak)
Teacher: *You told that the triangle ABC is an isosceles triangle with side AB equal to AC. Do you know anything else about the isosceles triangle?*
Kelly: *The angles on the basis of the isosceles ABC are equal*
Teacher: *Why don't you use it?*
Kelly: *Because it is not included in the given!*

At this point, Kelly seems to be confused and she did not think to use sentences that derive from the given. As we have already mentioned earlier in this paragraph students are taught to construct mainly intuitive proofs, but intuition may be immature (Baylor, 2001) and also students have developed limited geometric intuition (Nardi, 2009). The limited geometric intuition of beginners functions as an obstacle that blocks novice students' mind and it does not allow them to use inferences that derive from the given. Taking into account that Kelly and classmates are students in the last class of Gymnasium i.e. novice students in proving, who don't realize that they can use the results of the given, we applied the aforesaid 'blocked information'. In particular, we explained students that except for the given of the proposition, they must also write down all the other sentences that derive from the given. Also, we suggested them to write down the given of the problem using a blue color pen, and write the products that derive from the given (the blocked information) using a red color pen. In this way, we succeeded in making students write all the information of a geometrical proposition (see 'Appendix B').

3. Research

3.1 Participants
For this study, two groups of participants were used: one in a public primary school, and one in a public low secondary school (Gymnasium). Both schools that participated in this research were located in Peristeri, a fairly undeveloped suburb of Athens, Greece. Therefore, the majority of students were of low socioeconomic status. Students in primary school were 12-year old children, and those in Gymnasium were 15-year old children. Using a convenience sampling method, a total sample of 122 students was recruited. 55 out of 122 students (45.08%) were at primary school, and 67 out of 122 students (54.92%) were at Gymnasium. More analytically, the first group of participants (in primary school) consisted of 55 students, 24 boys (43.6%) and 31 girls (56.4%). The second group of participants (in Gymnasium) consisted of 67 students, 24 boys (35.8%) and 43 girls (64.2%).

3.2 Materials
A test was administered twice to each group of children. Below, we describe the tests (pre-test and post-test) given to each one of the groups.
Test for pupils of primary school
Each one of these tests (pre-test and post-test) consisted of two geometrical problems of last class of primary school. Students were given 90 minutes to solve the problems. Each problem was scored according to a scale from 0-5, so the total score ranged between 0 and 10.

The following two problems comprised the pre-test:

Problem 1: What are the missing angles x?

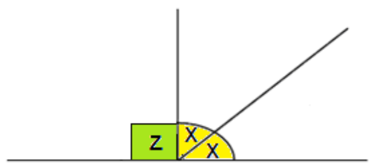


Figure 6.

Problem 2: The rectangles below have the same area. What is the perimeter of the green rectangle?

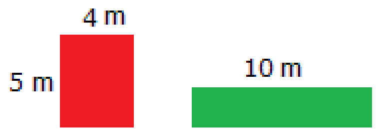


Figure 7.

The following two problems comprised the post-test:

Problem 1: What is the area of the following figure?

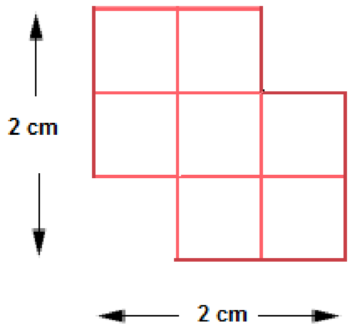


Figure 8.

Problem 2: What is the missing angle x?

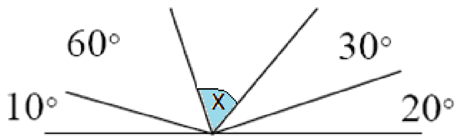


Figure 9.

Test for students of Gymnasium:
Each one of these tests (pre-test and post-test) consisted of four usual problems of medium difficulty of third class of Gymnasium. Students were given 90 minutes to solve the problems. Each problem was scored according to a scale from 0-5, so the total score ranged between 0 and 20.

The following four problems comprised the pre-test:

Problem 1: Prove that if a point lies on the bisector of an angle, then the point is equidistant from the sides of the angle.

Problem 2: The diagonal AC of a quadrilateral ABCD bisector the corners $\hat{A}$ and $\hat{C}$. Prove that AB=AD and BC = CD
Problem 3: Z and E are points on the sides BC and CD of a square ABCD. If BZ = DE prove that ZAE is an isosceles triangle.

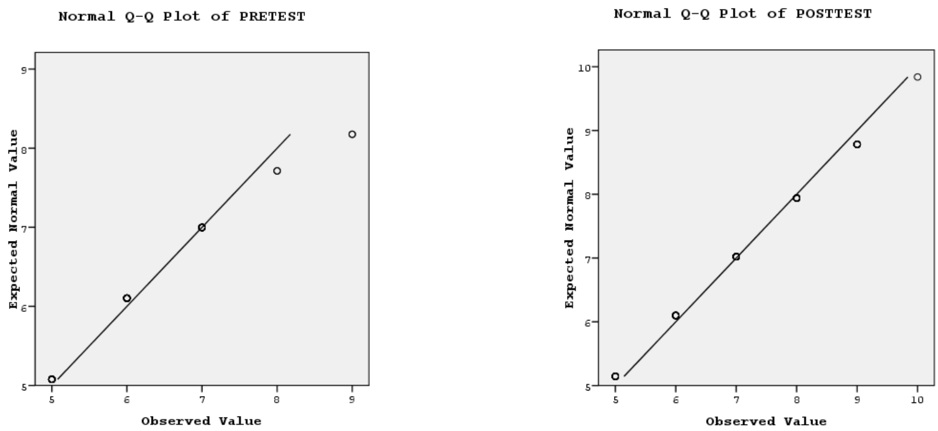
Problem 4: To be proved that the medians BE and CD of an isosceles triangle ABC such as AB = AC are congruent.

The following four problems comprised the post-test:

Problem 1: If AB and CD are two diameters of a circle (K, r), prove that chords AD and BC are equal.
Problem 2: In an isosceles triangle ABC, with AB = AC, the points M, N lie on the line segments AB, and AC respectively, such that M is the mid-point of AB, and N is the mid-point of AC. We equally extend the base BC of the triangle by the line segments BD, CE, such that BD=CE. Prove that DM=EN
Problem 3: Z and E are points on the sides AB and BC of a square ABCD. If AZ = BE prove that ΔZ=AE
Problem 4: First, draw an equilateral triangle ABC. Take on sides AB, BC and CA the points C΄, A΄, B΄ respectively and in such way that AC΄ = BA΄ =C B΄. Join the points C΄, A΄, B΄ and prove that the triangle A΄B΄C΄ is an equilateral triangle.

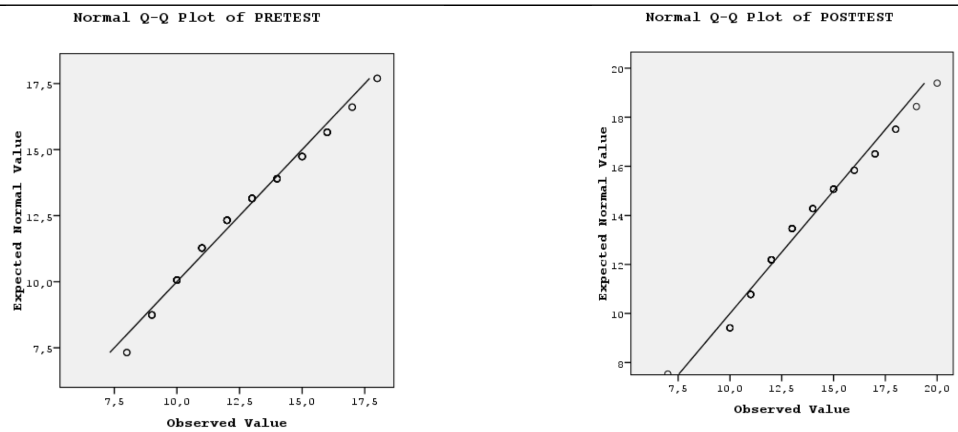
3.3 Procedure
After discussing with students of both groups, we conjectured that they had difficulty in expressing the ‘blocked information’ and using the whole information during problem solving (in primary school), or proving (in Gymnasium). To empirically test this theoretical conjecture we administered a test (pre-test) to each one of the groups. Results of this test corroborated our initial predictions that children both in Primary school and Gymnasium had serious difficulty in accomplishing their specific tasks.
After this test, students enrolled in a course, where we indicated them to follow another method using two color pens. Students were asked to use a blue pen to write down the given of the problems in Primary school, or the geometrical propositions in Gymnasium. They were also asked to use a red pen to write down the inferences derived from the given (‘blocked information’) of the problem in Primary School, or the proposition in Gymnasium. Then, students of both grades were given several examples to practice themselves in problems or proofs.
After that, they were given a test (post-test) to examine for possible differences due to the application of this new teaching method. Results revealed that children in both grades had significantly improved their performance in solving problems or constructing proofs.
Children of both grades enrolled in the research without any intervention by the researcher, regarding their classrooms settings, so as not to instantly disorder the way students used to collaborate with their classmates.
Researcher was in seamless cooperation with the instructors during the experiment. The researcher provided full explanation to every question posed by the instructors regarding the conditions of the instruction, and in their turn they gave valuable feedback to the researcher regarding the characteristics of students who participated in the instruction.

3.4 Data Analysis Methods
Descriptive statistics were used to produce data on the characteristics of the participants both from primary and secondary education. Percentages and frequencies were calculated for the numbers of boys and girls. Descriptive statistics were also used to report summaries of the numbers of individuals from each group.
Inferential statistics were used to examine whether statistical significant differences existed between pre- and post-test scores of each group. The scores across the sample were tested for normality and equal variance of samples in order to check if parametric tests could be used. The examination of Q-Q plots gave us evidence to consider that data regarding pre-test and post-test scores of both groups followed a normal distribution and so parametric tests were used. So, paired samples t-tests were used to check for possible statistically significant differences between mean scores of pre-test and post-test.



(a) (b)

Figure 10. First group (students in primary school) Q-Q plots for test of Normality



(a) (b)

Figure 11. Second group (students in Gymnasium) Q-Q plots for test of Normality

All test were carried out to the 5% significance level and considered significant when the 'p-values' were <0.05. All test statistics are reported to 2 decimal places and p values to 2 or 3 where necessary. Data collected from the tests was statistically analyzed using Statistical Packages for the Social Sciences (SPSS) version 11.0.

4. Results

Regarding the first group in primary school, a paired samples t-test was conducted to check for improvement of students in this group between pre-test and post-test, regarding their ability to solve mathematical problems. The mean score of group was 6.13 (SD = 1.00) on the pre-test, and 7.76 (SD = 1.22) on the post-test (see Figure 2). According to the results of the paired samples t-test, a statistically significant difference was found between pre-test and post-test for the first group, regarding the ability to construct proofs t (54) = -10,181, p<0.05 (see Figure 3). The results corroborate our initial predictions that students of the first group significantly improved their ability to solve mathematical problems, when using the proposed method.

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	PRETEST	6,13	55	1,001	,135
	POSTTEST	7,76	55	1,217	,164

Table 1. Descriptive statistics of first group’s (primary school) t-test

	Paired Differences							
				95% Confidence Interval of the Difference				
	Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
PRETEST - POSTTEST	-1,636	1,192	,161	-1,959	-1,314	-10,181	54	,000

Table 2. Paired samples t-test of first group between pre-test and post-test

Regarding the second group in Gymnasium, a paired samples t-test was conducted to check for improvement of students between pre-test and post-test, regarding their ability to construct proofs. The mean score of group was 9.22 (SD =1.93) on the pre-test, and 13.46 (SD = 2.52) on the post-test (see Figure 4). According to the results of the paired samples t-test, a statistically significant difference was found between pre-test and post-test for the first group, regarding the ability to construct proofs t (66) = -13,055, p<0.05 (see Figure 5). The results corroborate our initial predictions that students of the second group significantly improved their ability to construct proofs, when using the proposed method.

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	PRETEST	9,22	67	1,929	,236
	POSTTEST	13,46	67	2,519	,308

Table 3. Descriptive statistics of second group’s (Gymnasium) t-test

	Paired Differences							
				95% Confidence Interval of the Difference				
	Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
PRETEST - POSTTEST	-4,239	2,658	,325	-4,887	-3,591	-13,055	66	,000

Table 4. Paired samples t-test of second group between pre-test and post-test

5. Discussion

According to the aforementioned results when children of both grades externalized all of their thoughts (given and blocked information) improved their performance. Pupils in primary school improve significantly their performance in solving mathematical problems. Also, students in last class of Gymnasium seemed to improve significantly their performance in constructing proofs.

In this study we offered a new and powerful approach on investigating children's construction of geometric and spatial ideas. Our sample was rather small to allow for generalizations, but this study may serve as a basis for developing hypotheses to be tested in a more extensive future project.

Also, we mention the following:

Initially, we note the following similarities that arose in both grades

1. Although pupils in primary school had to make a different work from students of secondary school, both of them did not know how to begin.
2. Also, both pupils in primary school and students of secondary school were confused i.e. they couldn't differentiate between given and derived from given, thus they couldn't understand that they could use what derived from the given.
3. Even when they drew the figure by themselves they forgot the given (because sometimes were already drawn).
4. When the figure were given ready, they only correlated the figure with the given i.e. they didn't correlate the figure with what derived from the given.
5. When they stopped, they didn't think to start the solution of the problem or the proof construction using another starting point.

Next, we note that when they used all the information they realized that:

1. They should use thoughts written both in blue and red color.
2. In order to start the solution of a problem or to construct a proof of a proposition they had to prefer the thoughts written in blue. In other words in the thoughts written in blue there was at least a concealed starting point.
3. The points, where they stuck more often during a solution or a construction of proof, were those thoughts written in red color.
4. They must write all the information of the problem or proposition.
5. They should have understood and translate all the information into mathematical relations in order to solve a problem or to construct a proof.

We must also mention that children considered the writing of their task using two colors a diverting game. Finally, Gagatsis and Elia (2004; 448) allow us to consider that blue and red written texts are representations that Gagatsis (2004) attempts to exemplify and analyze the different and conducive roles that these can play in understanding, learning and doing mathematics.

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Appendix A

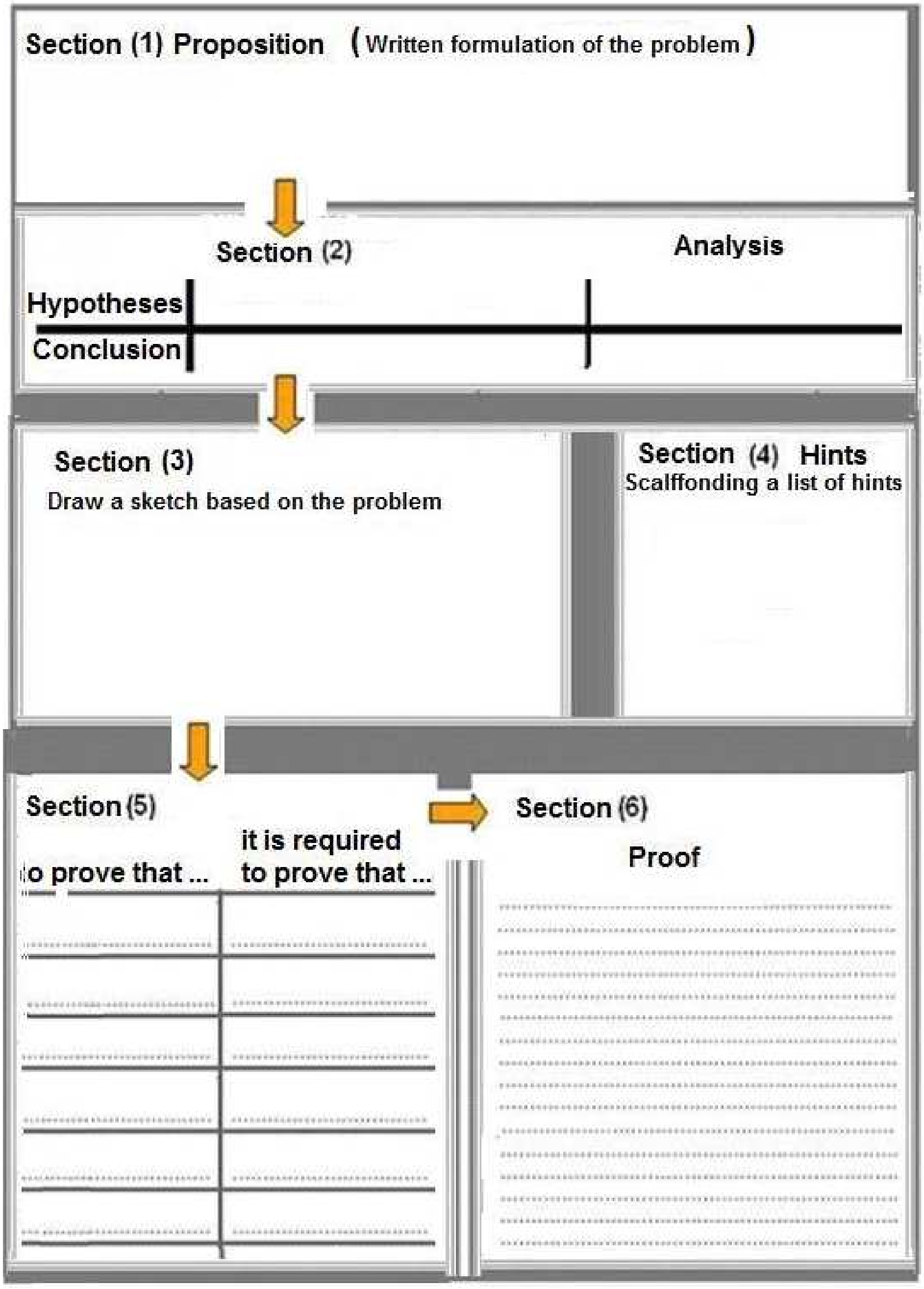
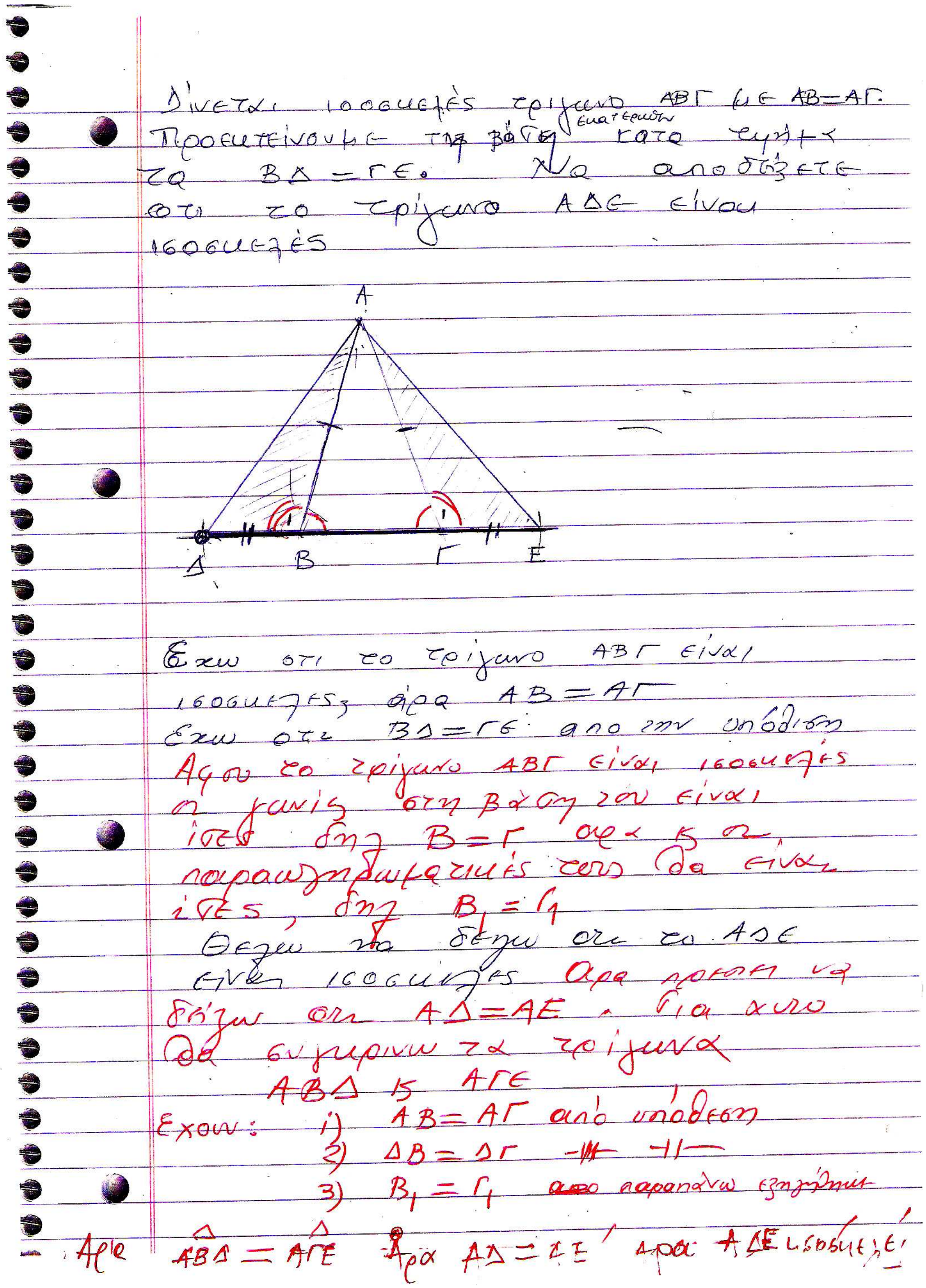


Figure A1.

Appendix B



Δίνεται ισοσκελές τρίγωνο ΑΒΓ με ΑΒ=ΑΓ. Προεκτείνουμε τη βάση εκατέρωθεν κατά τμήματα ΒΔ = ΓΕ. Να αποδείξετε ότι το τρίγωνο ΑΔΕ είναι ισοσκελές

Έχω ότι το τρίγωνο ΑΒΓ είναι ισοσκελές, άρα ΑΒ = ΑΓ
Έχω ότι ΒΔ = ΓΕ από την υπόθεση
Άρα το τρίγωνο ΑΒΓ είναι ισοσκελές οι γωνίες στη βάση του είναι ίσες δηλ Β = Γ άρα και οι παραπληρωματικές τους θα είναι ίσες, δηλ $Β_1 = Γ_1$
Θέλω να δείξω ότι το ΑΔΕ είναι ισοσκελές άρα πρέπει να δείξω ότι ΑΔ=ΑΕ. Για αυτό θα συγκρίνω τα τρίγωνα ΑΒΔ κ ΑΓΕ
Έχουν: i) ΑΒ = ΑΓ από υπόθεση
2) ΔΒ = ΔΓ
3) $Β_1 = Γ_1$ από παραπάνω εξήγηση
Άρα $\hat{ΑΒΔ} = \hat{ΑΓΕ}$ Άρα ΑΔ = ΑΕ άρα ΑΔΕ ισοσκελές

Figure 10.

Below, we have translated 'Appendix A' in English.

The proposition is the following:

Extend the base BC of an isosceles triangle ABC from both sides. Take on extensions two points such BD = CE. Prove that the triangle ADE is an isosceles one.

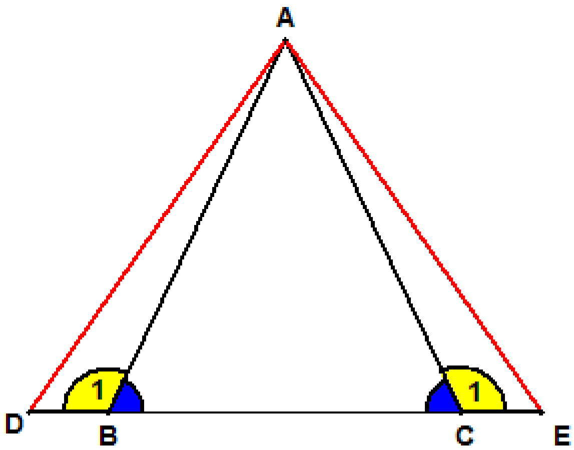


Figure B2.

The triangle ABC is an isosceles triangle, thus AB=AC. BD=CE, it is given

The triangle ABC is an isosceles one thus the angles on the base are equal, i.e. B=C. Therefore the supplement angles are equal i.e. $B_1 = C_1$.

It is required to prove that the triangle ADE is an isosceles.

In order to prove that triangle ADE is an isosceles triangle, it is required to prove that AD = AE. Thus, I compare triangles ABD and ACE. We have:

1) AB = AC, given

2) BD= CE, given

3) $B_1 = C_1$ I have already explained it above. Then, triangles ABD and ACE are equal, thus AD=AE, Therefore ADE is an isosceles triangle.