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WHAT NOTION OF REPRESENTATION IS USEFUL FOR MATHEMATICS EDUCATION?[1]

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1. INTRODUCTION

Much enlightening empirical work in mathematics education has revealed a number of results of utmost interest and of much usefulness for the educator. These include: how different representations of mathematical concepts facilitate problem solving (Elia, Panaoura, Eracleous, & Gagatsis, 2006; Elia, Gagatsis, & Demetriou, 2007), how students translate from one representation of a mathematical concept to another and the difficulties involved in these translations (Gagatsis & Shiakalli 2004, Artigue 1992, Hitt 1998), how multiple representations of the same mathematical concept is important and often essential to mathematical visualization and understanding (Duval 2002, Dufour-Janvier *et al.*1987, Greeno & Hall 1997).[2]

Many of these researches concern the understanding the concept of function and problem solving related to functions (Artigue, 1992; Hitt, 1998; Gagatsis & Shiakalli, 2004; Elia, Panaoura, Eracleous, & Gagatsis, 2006; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2008) or the limit of a function (Elia, Gagatsis, Panaoura, Zachariades, & Zoulinaki, 2009). Some others examine students' understanding of fractions (Deliyianni, & Gagatsis, 2009; Panaoura, Gagatsis, Deliyianni,& Elia, 2009). The role of representations in problem solving and in particular in additive problems or in non-routine strategy problems has been the main objective of many researches (Pantziara, Gagatsis, & Elia, 2009; Elia, Gagatsis, & Demetriou, 2007). Finally divers researches combine the affect domain in mathematics education with the role of representations and modelling in problem solving (Panaoura, Gagatsis, & Demetriou, 2009; Panaoura, Gagatsis, Deliyianni,& Elia, 2009).

Despite the usefulness of such results I do want to raise caution regarding their interpretation. In particular my concern is that our interpretations would be much more refined and accurate if an important distinction is clarified. This distinction concerns what a representation *per se* is for mathematical concepts and propositions and how a 'representation as…' of a mathematical concept is used for, among other things, conceptualizing or visualizing a mathematical problem. In this paper I try to motivate this distinction and clarify some issues regarding the notion of representation in the light of the distinction.

The first element of the distinction concerns the deductive character of mathematics in general, the reference of mathematical propositions, and the nature of mathematical concepts, and it has been an

[1] This paper is an extended version of an invited talk at the Conference of Five Cities, held in Nicosia in 2008. The Original paper was published under the title "Representation *per se*' *vs* 'Representation as', a Useful Distinction for Mathematics Education" in *Research in Mathematics Education*, Gagatsis, T. (ed.), Nicosia: University of Cyprus, 227-233.

[2] In these approaches to the notion of representation, attention is confined to the application of mathematics in modeling physical problems or problems describing possible worldly situations. The problem of representation of a mathematical calculus and of how that affects inferences by manipulating the calculus, e.g. proving a theorem by the use of some axioms or other proven theorems, is not dealt with. Although the latter is, admittedly, a more difficult task, I do think that an integral understanding of the notion of representation is one that is involved in both the application of mathematical languages to different domains and pure derivation from the calculus.

issue for debate among philosophers and logicians for more than a century. The philosophical debate derives its justification from the conviction that a mathematical representation *per se* is a canonical issue, and hence what mathematical terms represent (and the rules for making that representation) is a matter of learning for the mathematics student – in more or less the same manner as the language of mathematics is a matter of learning. In this paper, I shall spare the reader of the arguments of philosophical schools of thought on the issue, and I will restrict myself firstly in explaining what mathematical propositions are true of, hence what without qualifications they represent, and secondly to the less ambitious task of showing what a mathematical representation *per se* is not. By understanding what representation *per se* is not we can make sense of 'representation as…'. The latter concerns our understanding of mathematical problems, our understanding of the particular mathematical concepts in question, and the particular ways by which we feel most comfortable for conceptualizing or visualizing the mathematical situation at hand. These are not canonical issues, they belong, by and large, to subjective aspects of human intelligence, hence empirical research is most welcome on this issue if it can help identify the different varieties of 'representing as', if it can help categorize these varieties and if it can help improve our understanding of how these categories of varieties can function in conceptualizing and visualizing different mathematical situations. For these reasons I think the notion of 'representing as…' is most useful for mathematics education and understanding its various components seems a promising way for making progress on several issues that belong to the domain of mathematics education.

The distinction may not initially seem to be of importance to the mathematical educator. However, as I shall argue, it does present several consequences that may be of practical interest to the mathematics educator. For instance, Elia,Gagatsis, & Gras, 2005). point out that educational problems stem from the difficulties students face in interconnecting different representations of functions. They attribute these difficulties to what they call 'compartmentalization'. Compartmentalization is therein understood to be the act of splitting an idea or concept up into (often arbitrary) parts, and in an attempt to maintain simplicity the integration or the mixing together of these parts is inhibited. It is apparent that compartmentalization, in that sense, affects in a negative way the translation of one representation to another. So far as this aspect of compartmentalization is concerned, I believe the distinction between representation *per se* and 'representation as' can help overcome the problems that arise, as I shall try to explain in section 4.

2. CLARIFYING THE DISTINCTION

The distinction manifests its importance once we want to give an answer to two quite different questions: "What a mathematical concept or relation represents?" and 'How a mathematical concept or relation is –or, more precisely, can be– represented?' The first question demands an answer to what mathematical representation *per se* is and, it seems to me, it cannot be addressed by means of empirical investigations. Mathematical concepts refer to relations of numbers (or sets, or, more generally, abstract objects) hence whatever is represented by a mathematical concept belongs to the abstract realm of numbers and the relations defined upon numbers, or more generally to the realm of abstract mathematical objects and the relations defined upon those objects. The concept of 'function', for instance, represents a mapping from one set of abstract objects onto another such set, in other words what the function represents is the relation defined upon two distinct sets of abstract objects. In this sense 'representation is a form of interpretative structure (as logicians would say), that is the function is given by means of a syntax (language) which is interpreted (i.e. its semantics is supplied) by understanding its terms to refer to the mapping between the sets. More accurately, if one is to use the language of logic, a mathematical proposition is satisfied by (i.e. is true of) a mathematical structure, which is another way of saying that it is satisfied by a set of objects and a set of relations defined upon those objects. The different sets of abstract objects and relations that satisfy the given proposition is what the latter represents. Understanding our mathematical concepts and relations to represent anything other than sets of abstract objects and relations defined upon those sets, obscures the actual reference of mathematical languages and, in a sense, misguides the student of mathematics into thinking that mathematical languages are directly connected to the empirical world.[3]

The second question, i.e. "How a mathematical concept or relation can be represented?", on the other hand, invites an answer to how a mathematical concept can be 'represented as' and it concerns our attempts to visualize the concept in a particular context for a particular problem solving task. The contextualization involved in this notion of representation implies that a number of features or consequences of the mathematical situation will be abstracted (i.e. ignored) in the ostensible representation. In the process of constructing such representations the goals are simplification and resemblance. We want to simplify the complexities of the concept or of the consequences of a mathematical syllogism by abstracting some features of the concept(s) for the purposes of fitting it to a particular application, and in doing this maintain some sort of resemblance to the initial situation. Both simplification by abstraction and resemblance are key notions to 'representing as'. Abstraction (i.e. in its Aristotelian sense of subtracting some features of the actual situation at hand) is the conceptual process by which we achieve simplification without losing resemblance in relevant respects. Although it deserves an analysis of its own, I shall not herein occupy myself with the process of abstraction as it would lead me away from the central thesis of this paper. Analyzing the notion of resemblance is, however, important in order to see the philosophical underpinnings of the distinction I want to motivate.

3. WHAT REPRESENTATION *per se* IS NOT

In much of the literature on representation in mathematics the notion of 'resemblance' or 'similarity' is considered a surrogate of some sort of the notion of 'representation'. Possibly because it is a more mundane notion, 'resemblance' makes the concept of 'representation' simpler to comprehend. However, as I have claimed above resemblance is only related to 'representing as' not to representation *per se*. Because the notion of 'representation' seems to be a vital component of mathematics (and the physical sciences) its characteristics must be carefully contrasted to its ostensible synonyms, before we can jump to the conclusion that the synonymy actually holds, otherwise we are led to miscomprehensions. It is not difficult to show that representation cannot be grounded in resemblance. For the two notions to be synonymous, and thus for representation to be reducible to the concept of resemblance, the following condition must hold: "X represents Y if and only if X resembles Y". Where, X and Y are any two objects of any kind. This condition can be broken up to the conjunction of the following two: (1) "if X resembles Y then X represents Y" and (2) "if X represents Y then X resembles Y". Condition (1) expressed as a relation between X and Y states that "resemblance is a sufficient condition for representation" and condition (2) states that "resemblance is a necessary condition for representation".

Nelson Goodman (1976) has shown that to hold the view that representation is synonymous to resemblance is a naïve view of representation *per se*. His argument shows that resemblance is neither a sufficient nor a necessary condition for representation. It is not a sufficient condition because resemblance is a reflexive and symmetric notion whereas representation is neither reflexive nor symmetric. That is to say, it makes sense to claim that X resembles itself (this is the highest degree of resemblance), but it does not make sense to claim that X represents itself at least not for all X. It al-

[3] It may be the case that mathematics is not strictly speaking an *a priori* science after all, as some philosophical schools of thought would argue, but what is almost certain is that its link to the empirical world is far more complex and intricate than that implied by the naïve view that mathematical terms refer to empirical objects.

so makes sense to claim that X resembles Y implies that Y resembles X, but it does not make sense to claim that X represents Y implies that Y represents X. In other words, when we use the notion of resemblance to make claims such as, Mary resembles her sister Helen we also mean that Helen resembles Mary in much the same way. On the other hand, when we use the notion of representation to make claims such as, Picasso's Guernica represents the aftermath of the Nazi bombing of Guernica we do not mean that the aftermath of the Nazi bombing of Guernica represents Picasso's Guernica.

One could even extent Goodman's argument and claim that resemblance is transitive whereas representation is not. That is to say, if X resembles Y and Y resembles Z then it makes sense to claim that X resembles Z. On the other hand, if X represents Y and Y represents Z then it does not imply the claim that X represents Z. Think of a painting depicting the photograph of Helen, of course it represents the photograph but it does not represent Helen. In other words, representing the means of representation of a target does not imply representing the target. Since the logical properties of the two concepts are clearly different it is not logically possible (i.e. without implicitly leading to contradiction) to use resemblance in order to explicate representation. Hence the position that the concept of resemblance provides the foundation for the concept of representation is groundless.

But can we claim that resemblance is a necessary condition for representation? If yes then every representation must appreciably resemble its target. Goodman's answer is that we do not need any degree of resemblance to achieve representation. He claims, correctly I think, that almost anything can represent anything else. For instance, two stones on the ground can represent two armies ready for battle. That is to say, representation can be achieved even when the means of representing do not resemble in any way their target. I would even add that the mere concept of appreciable resemblance is context dependent, in the sense that it is relative to the domain of discourse. For instance, in the context of the Darwinian theory of Natural Selection 'man' appreciably resembles 'ape', whereas in the context of Newtonian Mechanics 'man' appreciably resembles 'table'. Thus, the claim that resemblance is a necessary condition for representation is not an assertion that admits generalization; it is dependent on the context dictated by the given discourse, and the latter's interpretation imposes psychological states from which the resemblances ensue. That is to say, because we interpret within a language that X represents Y we discern resemblance in some respects between the two objects. This conclusion is, I think, congruent with Goodman's conclusion that the core aspect of representation is *denotation* and thus it is independent of resemblance.

Given that Goodman's argument establishes a sharp distinction between representation and resemblance, the distinction I urge between 'representation as' and representation *per se* manifests its usefulness. An immediate implication of Goodman's argument is that representation *per se* is a product of our mathematical languages and their interpretation and as such is entirely independent from the notion of resemblance. However, it is hard to deny the usefulness of the notion of resemblance for mathematics education (and in fact any kind of education), so how can we retain it and at the same time be congruent with the above argument. The answer, I think, lies in recognizing that 'representing as' is in fact correlated to resemblance. Because representing a mathematical concept *as* something of our choice is something we do in order to visualize it, thus resemblance is imposed from our part in ways dictated by the context.

This conclusion gives a particular perspective to the use iconic (diagrammatic etc.) representations. It seems to me a truism that any picture can be described through a sufficiently rich language but that not all linguistic expressions can be represented pictorially. The well known saying that "a picture is worth a thousand words" that we all learned in reference to the economy of thought and not to the representational power of the picture, is actually reversed when the focus is mathematical representation. Representationally speaking, "every linguistic expression conveys a thousand pictures". But, as in other domains of discourse, in Mathematics no matter how many pictures we use it is impossible to represent some of the things we do represent using the linguistic expressions of our mathematical concepts.

4. SOME PRACTICAL CONSEQUENCES OF THE DISTINCTION

Mathematics education researchers are interested (and rightly so), among other things, in understanding mathematical concepts in ways that facilitate practical applications. So far the distinction I have been urging seems to be of interest only for the philosophically minded. However, two characteristics, one attached to 'representation as' and one to 'representation *per se*', are of most valuable practical interest. The characteristic implied by the practice of 'representing as' is that the latter often implies 'counterfactual' representation, i.e. X is represented as Y, could mean that Y acts *as if* it is a representation of X for a particular purpose but actually it is not, and often we know that it is not. That is, we may say that the triangle I drew in my notebook is intended to represent the concept of a mathematical isosceles triangle but it does not and, in fact, it cannot be what the concept represents. What we actually mean is that the drawn triangle acts as if it is a representation of the mathematical concept of isosceles triangle, i.e. the mathematical concept of isosceles triangle is represented as a drawn triangle. The drawn triangle acts as a representation because it resembles the mathematical concept in relevant respects and we do this because representing in diagrammatic form is one way by which we can simplify our mathematical syllogisms in particular problem solving tasks. Of course, I would be hesitant to generalize this observation, because there are cases where a particular concept is represented as something seemingly distinct, and the representation relation needs no 'as if' clause. Such is the case when we represent a mathematical function by means of a graph. The function is represented as a graph, and not as if it is a graph of two variable quantities. The reason that such cases of 'representation as' exist, and in particular graphical representation is because what the function actually represents, i.e. a mapping between two sets of objects, can also be represented by the graph. In other words, there is what looks to be a direct translation between the syntactic form of the function and the graphical form which in fact is carried out via what they actually both represent. This, however, is not always as clear in all cases of translations from one 'representation as' to another, and this brings us to the characteristic attached to representation *per se*. In many cases we must look closely in order to make sense of how a translation is carried out. Nevertheless, one thing is always clear, that a translation from one 'representation as' to another is validated if both represent the same sets of objects and relations. In other words, 'representation *per se*' always mediates in translations. I am not claiming, here, that the translating agent consciously uses the representation *per se* to guide his/her translation, but that for a translation to be valid and non-arbitrary this condition must hold.

Both of these observations are, in my view, of practical interest to the mathematics educator. The practical significance stems from the synthesis of the following two things. Firstly that we know that a translation is valid when both representational systems refer to the same things (i.e. the same sets of objects and relations) and secondly we can empirically support the claim that understanding mathematical ideas entails: "(1) the ability to recognize an idea, which is embedded in a variety of qualitatively different representational systems, (2) the ability to manipulate the idea flexibly within given representational systems, and (3) the ability to translate the idea from one system to another accurately" [Gagatsis & Shiakalli 2004, pp. 645-646]. The first and second claims above, i.e. how an idea is embedded in a variety of qualitatively different representational systems, and manipulating the idea flexibly within given representational systems, are abilities that are fully acquired when the student recognizes that often we represent mathematical concepts *as if* they refer to things that actually they do not.

The above observations also can be used to tackle the phenomenon of compartmentalization. As mentioned in the introduction compartmentalization is understood to be the act of splitting an idea or concept up into parts, and in the attempt to maintain simplicity the integration or the mixing together of these parts is inhibited. Three types of knowledge compartmentalization are distinguished in (Mandl, Gruber, & Renkl, 1993), that differ with regard to their consequences concerning further learning and knowledge application, namely:

- *the compartmentalization of correct and incorrect concepts*

In this case instruction does not replace the misconceptions by correct ones, but just provides additional knowledge. Correct and incorrect knowledge stand side-by-side. The major problem with this kind of knowledge compartmentalization is that in situations where knowledge should be applied, the problem solver often relies on the old deficient misconceptions and not on the newly acquired scientific concepts which would be more adequate.

- *the compartmentalization of several correct concepts*

Different concepts that are closely interconnected are acquired as separate knowledge units and stored in different compartments. This causes inadequate oversimplifications on the application of these knowledge structures. Thus this kind of compartmentalization can yield two consequences: limited understanding and gross oversimplification in knowledge application..

- *the compartmentalization of symbol systems and real world entities*

It concerns the lacking of mapping between symbol systems and real world entities. For instance, in mathematics learning this kind of knowledge compartmentalization causes students to perform meaningless symbol manipulations without understanding the relevance for their everyday life. This leads to the situation that on the one hand real world knowledge is not used in solving arithmetical problems in school, and on the other hand the kind of mathematics taught in schools is not used in everyday activities.

Concerning the notion of representation, the phenomenon of compartmentalization reveals the cognitive difficulty that arises from the need to accomplish flexible and competent conversions back and forth between different kinds of mathematical representations. These cognitive difficulties reveal deficiencies in representational flexibility, which are indicative of fragmentary mathematical understanding. Thus, learning could be accomplished through “de-compartmentalization” (Duval, 2002). My claim is that de-compartmentalization for representation can be achieved by learning what the representation *per se* is, thus learning the mediator in translating between different kinds of ‘representation as’. This can be achieved by instruction.

5. CONCLUSION

The philosophical underpinnings of the distinction between what mathematical concepts and propositions represent (representation *per se*) and how mathematical concepts can be represented (representation as), has been demonstrated. The distinction, as such, can be of usefulness to the Mathematics Educator, as the kinds of problems that arise in the learning of mathematics can be categorized in the light of this distinction, and treated appropriately. Clearly, the ordinary notion of ‘representing as’ is embedded in everyday thinking, and does not require mathematical maturity in order to comprehend and handle. Being embedded in everyday thinking, however, makes it subject to the cognitive evolutionary-history of the agent, thus early training in ‘representing as’ is vital in shaping student's understanding. The crucial question for the mathematics educator is, therefore, what particular ways of 'representing as' facilitate the understanding of particular mathematical concepts. On the other hand, in order to recognize that it is one thing to 'represent as' for the purposes of practical problems and it is another different thing for the abstract concepts of Mathematics to represent something also abstract, requires mathematical maturity. Despite the high level of abstraction required to make sense of a representation *per* se its practical usefulness is clear. Since, the ability to identify and represent the same concept in different representations, and the flexibility in moving from one representation to another, are considered crucial in mathematics learning, instruction of the representation *per se* of a concept seems a vital component both for understanding the concept and for translating between different kinds of 'representation as' of the concept.

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