“Quaderni di Ricerca in Didattica (Matematica)”n.22, 2012
G.R.I.M. (Department of Mathematics, University of Palermo, Italy)

Ontogenesis and phylogenesis in the passage from arithmetic to algebraic thought

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Abstract: *With this work we retrace some of the fundamental historical and historical-epistemological steps of the evolution of algebraic thought. Given the accepted parallelism between ontogenesis and phylogenesis we believe such a research to be a useful approach to the interpretation of obstacles and difficulties that students encounter during the acquisition of algebraic thought.*

Theoretical framework

“Reading” and “interpreting” the history of mathematics from an educational viewpoint is as difficult a work as it is fascinating. Determining a biunivocal relation between history and education is a complex operation that requires the identification of its validity and applicability domain with regard to the mathematical object of research but mainly with regard to the teaching-learning phenomenon that is the aim of our experimental research.
In this respect then, the history of mathematics represents an important viewpoint of the analysis of education dynamics and thus a viewpoint of the acquisition process of mathematical contents studied by students at school.
As J. Fauvel, J. Van Maanen and some works by Y. Chevallard affirm, the research on the use of the history of mathematics in teaching/learning the discipline must be an important part of the research in Mathematics Education (Fauvel & van Maanen, 1997, p. 8; Y. Chevallard, 1989), conceiving the historical and historical-epistemological approach as possible teaching strategies and mathematics as a cultural dimension and method (Furinghetti & Somaglia, 1997, p. 43).
In our research on the problematic teaching strategies for the acquisition of algebraic thought by high school students and on the difficulties encountered by the students themselves during the transition from arithmetic to algebraic thought, the history of mathematics and the evolution of algebra in particular can serve as interesting observational lenses.
Following Piaget’s theory of convergence between historical and individual development (Garcia and Piaget 1989) it may be interesting to experimentally analyse some reasoning schemes uncovered by the students, to study the difficulties they encountered during the solution of mathematical problems, thus verifying how these may be similar to those encountered by generations of mathematicians. Moreover, on a purely educational level, already at the end of the 19th century it was suggested that mathematical education could benefit from a comparison with the development of mathematical thought throughout the centuries and that this comparison could also inspire teachers (Demattè-Furimghett, 2006).
“The birth of knowledge in the individual must follow the same course of the birth of knowledge in the race… it seems that the knowledge of the history of a science is an effective method to teach that science itself” (Cajori, 1896).
In this context, a retrospective analysis of the historical development of Algebra shows its long and difficult advancement, marked by the several debates between the mathematical cultures of

different peoples. At least in western culture[1], the historical picture shows a slow and difficult start compared to geometry and a difficult "relationship" with arithmetic, demonstrated by the constant effort to pass from computational procedures to "mathematical objects".

When do the students arrive, if they do, to algebraic thought?

What course do they take to acquire it?

What can constitute an obstacle to "thinking algebraically"?

The theoretical frameworks found in literature referring to this field of research are several and developed according to different approaches and numerous points of view. Some examples: the framework closely focused on algebraic formalisation thus on the difficulties encountered by students in the syntactic and semantic control of the employed linguistic register; the more general one centred on "thinking algebraically" and thus oriented towards the analysis of key concepts like *variable, unknown quantity, parameter*; the one specifically aimed at the analysis of the difficulties encountered at the generalisation stage etc… (Arzarello, 1994; Sfard, 1991; Bazzini, 2002; Malisani, 2006; Radford, 2000; Di Paola-Marino, 2007).

The following article crosses this field of research transversally, as a further historical-epistemological reflection on the topic. Consequently, evolving from our previous works and drawing on their experimental sections (Di Chiara-Di Paola-Marino, 2004; Di Paola-Marino, 2007), this article isn't a complete work from the historical point of view but just a further contribution to Mathematics Education and to the works previously published by GRIM. The latter focused on the analysis of epistemological and educational obstacles to algebraic language and were essentially aimed at the understanding of students' behaviour when facing problematic algebraic situations, the resolution of first and second degree equations in particular.

This article is composed of two separate but integrated and interconnected parts meant as a critical revision of some of the results achieved in our previous works. The historical-epistemological approach to the discipline is integrated with the educational one and becomes useful to the experimental part and to the analysis of modern students' types of behaviour. The following analysis represents an example of integration of the quantitative analysis discussed in previous works (Di Chiara-Di Paola-Marino, 2004; Di Paola-Marino, 2007) thus serves to a more mature observation of the obstacles encountered by the students participating in the experiment.

Thinking algebraically between history and education

In order to synthetically retrace the most important steps of the evolution of algebraic thought we need to draw the historical picture of the development of Algebra and of the symbolic system used to express its concepts. A theoretical starting point can be the three phases identified by G.H.Nesselmann:

- Rhetorical phase (previous to Diofant from Alexandria, 250 a.D.): a verbal Algebra, expressed in words without symbols.
- Syncopated phase (from Diofant to the end of the 16th c.): abbreviations are introduced for unknown quantities but operations are still done in natural language.
- Symbolic phase (introduced by Viète, 1540-1603): letters are used for all quantities, unknown or not, and Algebra is not only "exploited" to find the unknown quantity as in the second phase but also to verify rules that link the different quantities, thus to express the general solutions.

If on the one hand this classification is of particular interest for our task, on the other hand it doesn't seem comprehensive, as stated by more recent studies: it is almost impossible to identify historical phases that are undoubtedly distinct and separate and that mark the development of algebraic thought. For example, the separation between rhetorical and syncopated algebra is not a clear break

[1] Different considerations should be made for Eastern cultures. For a more detailed discussion: Chemla, 2003; Spagnolo, 2005; Spagnolo, Ajello, Xiaogui,2005; Di Paola-Spagnolo, 2008.

and one did not definitely and suddenly replace the previous; it was a slow gradual change (J.Gheverghese, 2003).
Consequently, a further historical-epistemological study necessary to the above educational purposes was the analysis of the development of mathematical knowledge concerning first and second degree equations in ancient civilizations like the Babylonians, the Greeks, the Egyptians, the Indians, the Arabs and the Europeans. This was achieved by studying some of the problems written in some of the ancient books that have survived until today thanks to the work of many scholars. Based on these documents we think that western algebra was born with ancient Babylonian mathematics (Maracchia, 2001) and that it has become independent from geometry and arithmetic thanks to Viète and Bombelli.
The understanding of Babylonian mathematics is relatively recent; it is primarily due to the works on clay tablets by Otto Neugebauer (Neugebauer, 1974).
Babylonian algebra is generally rhetorical and verbal, however, according to newer studies on tablets like AO8862, one of the most ancient ones (1800/1600 b.C), and based on the analysis of the inscribed text, it is clear that Babylonians had reached a first algebraic level that traced the steps to the solution formula of second degree equations (Neugebauer, 1935). Despite the difficulties of classifying as algebraic procedures those intuitive Babylonian solving methods, geometrical and wholly verbal from statement to solution, it appears that the approach to the problem is algebraic, regardless of the type of solution and of whether there were attempts to algebraic synthesis or not. The Babylonian approach can be considered a first step towards algebra because if on the one hand it faces the need to determine one or more unknown quantities that have to verify certain conditions to achieve a predetermined result, on the other hand the several tablets-long steps that solve the problem show particular standardised operations that do not keep repeating the geometrical figure they derive from, and thus mark the first step from geometric to algebraic thought (tab. BM 345689; tab. BM 13901, ex.12; tab. BM 80209). It is indeed difficult to conceive Babylonians doing repetitive similar operations without acquiring some mechanic behaviour and thus some “solving formula” they could apply to known quantities to obtain the needed ones.
“Even Egyptians, like Babylonians, stated and solved problems in a wholly verbal manner, without explaining neither the reason why they used that particular solving method nor why it worked in that context” (Kline, 1991:25).
Egyptian mathematics was a set of simple undemonstrated rules concerning everyday problems. Their algebra and their arithmetic were thus very limited. One of the most important texts of Egyptian mathematical culture certainly is the Rhind papyrus, transcribed at the end of the 1650 a.D. and currently held at the British Museum. This papyrus deals with everyday problems, like food distribution, that can be expressed by first degree equations. The problems included in the papyrus are not of very easy solution: the unknown quantity is called *hau*, which means “heap”, addition and subtraction are depicted as a man who walks towards the number that undergoes the operation and one that walks away from it, etc…
We think that one of these problems, n.24 of the papyrus, is of particular interest for our study on epistemological obstacles in the algebraic language and for our analysis of equations in arithmetic and algebra. It asked to find the quantity of a “heap”, if a “heap” and a seventh of the “heap” equalled 19. It was asked to find the unknown quantity of the first grade equation by this formula, which we present in modern notation:

$$x + \frac{1}{7}x = 19.$$

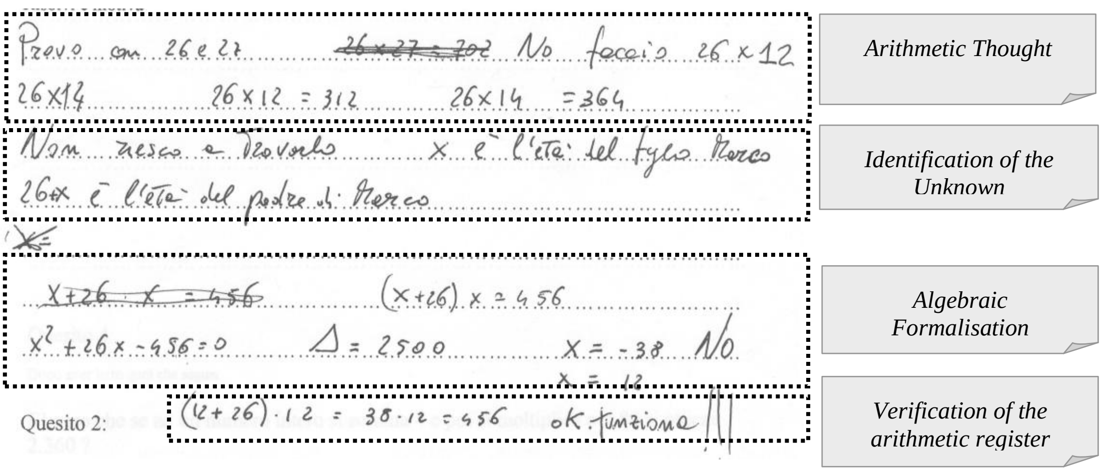
The solving method, called “*rule of false position*” by Ahmes asks to guess an unknown regardless of the accuracy of the guess and use it to perform the operations on the left side of the equal sign. After comparing the obtained result to the desired one proportions will be used to adjust the guess and find the correct answer. This method is <u>purely arithmetic</u>. With it an equation’s solution is

indeed found by adjusting the guess with subsequent approximations without making use of any abstraction.
As opposed to Babylonians and Egyptians, Greek scholars did not simply want to "employ" mathematics as a calculating method, but always tried to explain the rules used in their algebraic calculus; it is indeed during this period that the first "demonstrations" appear. It would still be wrong to define this "axiomatization of algebra" but it appears that methods are used with deeper awareness (Kline, 1991). The Greek works are witnesses of an important development of geometry, but the same cannot be said for arithmetic and algebraic calculus. In Greek equations the unknown stood for segments, rectangles, squares, cubes… in short, geometrical quantities. We could then name this kind of algebra "Geometrical Algebra".
The best part of the works of classical mathematicians has reached us through Euclid's most famous work, the Elements; thirteen books that demonstrate many calculation rules. The Second book, in which the author confirms some of the fundamental results of modern Algebra using a wholly geometrical language, is especially interesting and its sixth statement in particular. In this proposition a geometrical problem is solved, that if enunciated algebraically would take the form of a second grade equation with one unknown quantity and positive coefficients.
According to Bourbaki the geometrical methods so often used by the Greeks, and by Euclid in particular, did not further the autonomous development of Algebra in any way: "*... the path towards rigour in Euclid is met by a paralysis, and at some point even by a regression, of the development of algebraic calculus. The preponderance of geometry <u>hinders the autonomous course of algebraic notation</u>, the elements that appear in the calculi must always be represented geometrically...*" (Bourbaki, 1976:75).
Thus Algebra did not fully develop in Ancient Greece up until the late period, when Diofant of Alexandria (about 250 a.D) and Eron (100 a.D.) started to solve arithmetic and algebraic problems without "relying" on geometry, thus overcoming the geometrical conception of algebra. In order to clearly reveal Diofant's contribution to algebra's and thus algebraic thought's evolution, it would be useful to compare one of his problems to the "same" one of which we know an older solution, for example a Babylonian one, to expose the different solving strategies. The problem that we chose to analyse is the following: *"Find the dimensions of a rectangle with area 96 and semiperimeter 20"* (in modern nomenclature xy=96, x+y=20).
The Babylonian method consists of using the numbers given by the scribe's steps:

1	Halve the semiperimeter	20:2=10
2	Square it	10^2=100
3	Subtract the area, 96, from 100	100-96=4
4	Extract the square root	
5		The length is: 10+2=12; The width is: 10-2=8

What appears from the above procedure is <u>a sequence of numerical operations without any calculation on unknown quantities</u>. Diofant enunciates the above second degree Babylonian problem in a general language: "*Find two numbers whose sum and whose product are the given numbers*" (Problem n.27, Book I, *Arithmetica*). He solves it in the same way of the example: suppose the sum of the figures is 20 units, that their product is 96 units, their subtraction is 2 "aritmes" (the unknown): x+y=20; xy=96; x-y=2z. Thus, because the sum of the figures is 20 units, if we halve it we obtain two parts of 10 units each, and if we add half of the figures' difference (1 "aritimie") to one of them and we subtract it from the other, the sum of the figures is again 20 units and their difference is 2 "aritmes": x=10+z; y=10-z; xy=100-z^2.

However, the product of the figures must equal 96 units. Their product, that equals 96, is 100 units minus a square number of "aritmie"[2]. One "aritmie" then is 2 units. Thus, the biggest number will equal 12 units while the small one will be 8 units, and these figures satisfy the proportion.
Clearly, the solution depends on the auxiliary unknown z.
The appearance of such an auxiliary unknown, the so-called "aritmia", together with the symbols, constitutes a veritable conceptual change. Indeed we can affirm that Algebra was born with Diofant. Babylonian mathematics is light years behind Greek mathematics. The passage from Babylonian to Greek mathematics is sharp: the introduction of unknown and of a dedicated symbol is an innovation as important as the introduction of the zero. Diofant marks the passage from rhetorical to syncopated mathematics.
Educationally, the concepts of *variable*, of *unknown* and more generally of *parameter* are difficult concepts for students to internalize as funding elements of algebraic reasoning; consequently, in the path to algebraic thinking "*these develop slowly, moving from the initial relation between numbers to dynamic quantities related by a formula*". (Malisani, 2006:161)
The comparison between the two different above strategies (the Babylonian and Diofant's) can prove itself useful in both the experimental phase of the research on epistemological obstacles of algebraic thought and the analysis of students' behaviour during the solution of first and second grade equations in particular. As a consequence it can be useful in the analysis of their difficulties. The idea of convergence between the historical and the individual evolution appears central in this context: the two different solving strategies, the arithmetic used by the Babylonians and the pre-algebraic used by the Greeks, are actually very similar to those used by students participating in the several experiments carried out for our previous works (Di Chiara-Di Paola-Marino, 2004)[3]. An interesting example of convergence between ontogenesis and phylogenesis in relation to algebraic thought emerges from two students' attitudes towards the solution of next problem in natural language (Giuseppe - 2nd year of Liceo Scientifico, Galileo Galilei, Palermo, and Francesco - 2nd year of Liceo Classico, Garibaldi, Palermo):
"*A father was 26 when his son was born. The current age of the father multiplied by the son's age makes 456. What are the current ages of the father and son? Solve and explain.*"
In some sense, this problem can be analysed together with the Babylonian one discussed above.



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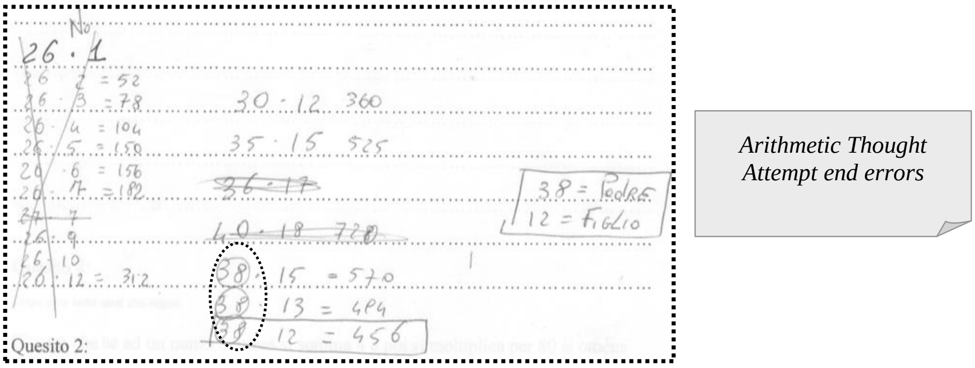
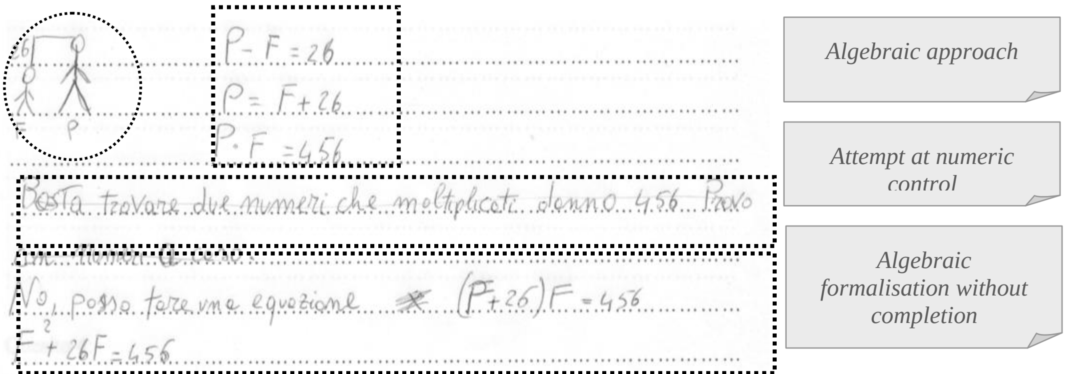
2 The formula: $(a+b)(a-b)= a^2-b^2$ is easily derived from Euclid's II,5.
3 A total of 260 high school students between 15 and 19 years old from Palermo participated to the experiments. For the complete report of the experiments conducted at school, and for the discussion on the form used during the research to whom also belongs this problem, see Di Chiara-Di Paola-Marino, 2004 and Di Paola-Marino, 2007.



From the study of both the students' works it is clear that their algebraic knowledge isn't conscious, but rather superficial. Neither of them appears used to think algebraically but if on the one hand Giuseppe's individual approach to the solution seems to follow a more mature course resembling the historical evolution of mathematics, Francesco on the other hand is crushed by his arithmetic knowledge and doesn't even attempt an algebraic systematization of the problem.
To solve the algebraic problem expressed in natural language, Giuseppe initially works like his classmates, attempting to use arithmetic methods and proceeding by trial and error with randomly chosen numbers. After some unsuccessful attempts he decides to resolve to algebraic strategies. Not all of his classmates are able to change to algebra. In most cases arithmetic is an obstacle to thinking algebraically. This is Francesco's case, who only focuses on the required product and maintains his purely arithmetic mindset, attempting to formulate numerical guesses that verify the result: 456. Giuseppe, on the contrary, formulates a possible relation between the variables but he initially suffers from mistakes in the concurrent control of the variables and the relations between them. After a first attempt he understands his mistake and correctly formalises the problem posed ((x+26)*x=456). To Giuseppe, then, the arithmetic procedure serves, by trial and error, as a model for writing the problem in algebraic language. It is not an obstacle like for Francesco and the other students (Di Paola-Marino, 2007). For Giuseppe, as observed in the history of mathematics, the determination of the unknown is a veritable conceptual change in the approach to the problem. He then shows a certain confidence in algebraic manipulation, and performs a final "test" that using the initial arithmetic register controls the correct "functioning" of the solution.
As said above, Giuseppe's knowledge is not conscious: despite possessing a series of algorithms and strategies bringing him towards thinking algebraically and helping him in solving algebraically enunciated problems, he does not acquire algebraic thought and instead stops at an intermediate condition, which has been called <u>pre-algebraic thought</u>.
The same conclusion can be drawn in Sandro's case (2nd year Liceo Classico, Garibaldi, Palermo), albeit with proper differences:



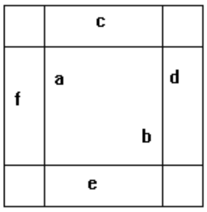
Moreover, one of the choices Francesco made in the last three attempts appears particularly interesting: he chose to keep the first number constant (n.38) and change the second one only. It seems that the student tried to narrow the range of possible cases, quickly failing. The strategy was indeed winning but it appears it was developed rather immaturely.

Returning to the analysis of the historical development of algebra, in particular of Indian algebra we agree with Malisani (Malisani, 1996) that even though rudimental, Indian symbolism is certainly sufficient to classify Indian algebra as "almost symbolic", undoubtedly more so than Diofan't syncopated algebra.
Indian algebra indeed seems more than just verbal, phrased and without the use of symbolism, which were characteristics of the rhetoric phase (Nesselmann's phase I), but it cannot be considered fully belonging to Nesselmann's phase III, the symbolic phase, either because "*despite having developed perfect procedures and having great ability and technique this people did not motivate or demonstrate the steps in their procedures.*" (Malisani, 1996:34).
Indian mathematics did not greatly influence European mathematics, but it is almost certain that the Arabs indirectly studied Indian arithmetic and algebra thanks to the representatives of Brahmin's science welcomed by caliphs in the 9^{th} and 10^{th} centuries. During this historical period many classical works were translated into Arabic from Sanskrit and Greek and studied by Islamic scholars, as well as works of literature and science.
This prosperous cultural and scientific environment reached its peak during Al-Mamum's caliphate (809-833) with the foundation of the "House of Knowledge". Among the members of the House was the mathematician and astronomer Mohammed Ibn Musa al-Khuwarizmi, a scholar who wrote more than a dozen works of astronomy and mathematics, the oldest of which were probably derived from Indian works. The title of his most important work, "Al-jabr w'al muqabala", is the origin of our modern term "Algebra".
If, on the one hand, Arabic algebra seems to having marked an important step in the evolution of algebraic thought, on the other hand this people have taken a big step back regarding both algebra and arithmetic (indeed, despite having encountered negative numbers in their contacts with Indian scholars, they didn't adopt them). "*Arabs did not use neither abbreviations…only names to denominate the unknown quantity and its powers…thus a step back from both Indian algebra and Diofant's one*" (Malisani 1996:34). Moreover, influenced by Greeks, Arabs felt the need to explain and justify the procedures used to solve algebraic problems but to do so they only exploited geometry. Like the Greeks, they needed to justify everything geometrically. In this case, geometry's predominance seems to prevail over the autonomous development of algebraic notation. To support this we analysed some of the problems among the ones found in the Arabic texts, highlighting the solving algebraic/geometric procedures. Let's consider, for example, the following problem: "*A square number and ten roots are equal to 39 units*". Analysing the solving procedures given by Al-Khuwarizmi we can find two different techniques that we will discuss in order to demonstrate that the first one (1) was considered inferior to the other one (2) even though logically correct. The preferred one, n.2, is the geometrical solution and today we would consider it superfluous:

1a	Halve the roots	$\frac{10}{2}=5$
1b	Multiply them by themselves	$5\cdot 5 = 25$
1c	Add 39	$25+39 = 64$
1d	Extract the square root	$\sqrt{64} = 8$
1e	Subtract the half of the roots	$8-5 = 3$
		x=3 is the desired root

This is the modern notation: $x^2 +10x =39$; $=5$; $x^2 +10x+25=39+25$; $(x+5)^2 =64$; $x+5=8$; $x=3$

2a. Draw a square *ab* that represents x^2 and draw the four rectangles c,d,e,f on the four sides. Each of these rectangles is 2*½units large. To complete the bigger square add four small squares to the sides, each of which has an area of 6*¼ units (thus, add four 6*¼ units, that is 25 units, to complete the square), in order to obtain a square whose area is 39+25 units.



The side of the bigger square must then measure 8 units, from which two 2*½ units (5 units) must be subtracted, to find that x=3 is the solution to the equation.
This is the demonstration that the algebraic solution was indeed correct.
With time, but slowly and with some difficulties, the superior Arabic culture seeped in Europe. Spanish Arabs and Near Eastern Arabs have the merit of having refuelled European culture with life during the Dark Ages, a period between 400 and 1100 a.D. when it had lost its energy: "*Europe was not interested in the development of Mathematics, there was no progress, neither there were serious attempts at generating new knowledge. All problems were solved by means of the four operations between whole numbers… calculations were made with the abacus… the Roman numerical system was used and the zero was avoided because it was not understood. Fractions were rare and irrational numbers never appeared."* (Malisani 1996:48).
One of the important scholars of this period certainly was Leonardo Pisano, also knows as Fibonacci, who is said to have introduced the term "equation", drawing it from the Latin *aequatio*. Based on historical data we can affirm that the most important mathematical discoveries made from the 13th c. onwards were mainly made in arithmetic and algebra: indeed many scholars show an increasing interest in these disciplines, trying to deepen their studies. In these studies, the so called Abacus Treaties, algebra had an arithmetic rather than geometric base and many discussions were solved with equations.
One of the classic abacus treaties is the "Trattato d'Algibra", written at the end of the 14th c. by an anonymous writer from Florence. It is a complex text that deals with many commercial topics typical of this type of works and which are dealt with not only for material purposes but in a more abstract and complex way. Two modern scholars, Franci and Pancati think that this work is one of the best medieval and renaissance abacus treaties they have studied (Franci, Pancati, 1988). In particular, they affirm that "*the final chapters about Algebra… are fundamental for the historical reconstruction of this discipline in the 13th and 14th centuries…"* (Franci, Pancati, 1998:6).
However, the algebra developed until the 16th century was slow and imprecise: the symbols most commonly used in algebraic texts were the result of abbreviations of common words, of "pieces" of terms that named the unknown quantity of the problem. The increasing demands of science, however, brought scholars to introduce many variations; mathematicians were motivated to use a symbolic notation that allowed for faster and more precise procedures and started to understand the "power" of symbolism in Algebra. "*The most significant change during the development of algebraic language was introduced by Viète's symbolism"* (Malisani ,1996:59). Similarly to Bombelli, he did not employ symbols to stand for unknown quantities, but he used letters to visualise generic terms that needed to be expressed.
Their "new" language was not created to solve problems with algebraic calculations only but was employed to prove many general rules. Algebra thus entered its symbolic phase.

In the study of algebra’s development and symbolic language there is an important educational aspect that we think is useful to the interpretation of obstacles and difficulties in learning algebraic thought as well as the latter’s interactions with arithmetic and geometrical thinking: the constant tendency to generalise the problems that are dealt with.
Comparing Pisano’s “Liber Quadratorum”, the “Trattato d’Algibra” by the unknown author from Florence and Bombelli’s “L’algebra” we find an increasing consciousness of the necessity to generalise all solving procedures to employ them in a variety of specific cases.
Algebra is then conceived as study of general types of problems: anything that can be applied to the general case can be employed in infinite particular cases.
If on the one hand algebra is considered “universal arithmetic”, that is a discipline able to express all arithmetic rules in a more general way, on the other it was considered limited by such “link” and underwent a process of liberation. According to this vision, the variable could not be considered a generalised number, it had to be “emptied” of any meaning to become “superior”, a parameter for the solution of classes of problems. As for the rest of its history, this phase also was slow and difficult (Arzarello, Bazzini, Chiappini 1994:21).
This is also the course that with no little difficulty we try to educationally encourage in students. Indeed, this is one of the reflections that have brought us to research the idea of parameter and its conscious employment by conducting experiments with both high school students and first year university students (Di Paola-Marino, 2007). The results indicate that to almost all students participating in the experimentation the concept of parameter is an obstacle. We hypothesize that this can represent an epistemological obstacle to the full acquisition of algebraic thought.[4]

Conclusions

The historical epistemological approach to a mathematical concept is useful to the understanding of the processes that have brought to its definition and to the identification of conceptual obstacles appeared in history. As the experiments have shown, although the study of protocols is only the first step of the research, the historical epistemological approach can be employed as a valuable reference for finding new and more effective educational approaches. Based on the negative INVALSI-PISA results of the Italian students we believe that today’s research in Mathematics Education should increasingly adjust its educational approaches following the historical processes that took place in our culture and our thought.

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[4] The hypothesis should be differently redefined and criticised by treating cultures other than the European ones, like the Chinese one. Recent studies on Chinese language seem to uncover a parametric type structure that is then acquired unconsciously by students already in elementary school, supporting in the following years a more mature algebraic thought and more confidence in the solution of algebraic and pre-algebraic problems. For more on this aspect of parameters and on its role in Chinese culture please refer to Spagnolo-Di Paola, 2008; Di Paola Spagnolo, 2008; Needham 1998.

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