"Quaderni di Ricerca in Didattica (Matematica)", n.20, 2010.
G.R.I.M. (Department of Mathematics, University of Palermo, Italy)

THE GAP BETWEEN REAL NUMBERS AND TRIGONOMETRIC RELATIONS

Nevin ORHUN
norhun@anadolu.edu.tr
Anadolu University Faculty of Science
Department of Mathematics, 26470 Eskisehir,Turkey

Abstract.
For most of the students in higher education, it is necessary the analytical part of trigonometry. In other words, the trigonometry of the numbers is more important than the other subjects in trigonometry. The purpose of this study is to investigate the relationship between real numbers and trigonometric relations. The subjects of this study were freshman calculus students from the Science Faculty at Anadolu University during the summer term of 2007. 134 students participated in this study. The data of this research was obtained from a test with 15 items in open-ended format. The questions on the test have dealt with the structural and procedural activities of trigonometric expressions in real numbers. Two levels were defined for measure of students' knowledge. First one is radian concept the second one is trigonometric function concept. It has been realized that students' skills for usage of radian concept have not been developed. The findings have shown that the students perform unsatisfactorily in the operation with real numbers of trigonometric expressions. We have pointed out a remarkable obstacle to reach a good performance in trigonometric problem solving related with radian. In the presentation of the example, the students have been seen as immediately responsive to the visual structure of algebraic rules. However, the structure of trigonometric expressions has been difficult because of the different treatment of expressions in algebra and arithmetic. The students must learn that trigonometry is neither the generalization of numerical processes nor the way of symbolization. Some conjectures are made as to why students did not understand the trigonometric concepts in real numbers. As it is generally to be in teaching of mathematics, if the development-principles of notions and methods on trigonometry would put forward, it would find the tricks which make easy to learn.

Keywords: Trigonometry, Radian, Trigonometric Functions

1. Introduction

Trigonometry is the branch of geometry which forms an important background for the solution of problems according to many disciplines. Trigonometry is frequently used in mathematical explanations and definitions of new ideas and concepts. Recently, the studies on research of some factors that explain variation in mathematics achievement have been realized (e.g., Kirshner, Awtry, 2004; Porter, Masingila, 2000; Dorier et al., 2000; Radford, 2000; Romberg, Shafer, Webb, 2000). A lot of studies have researched mathematics learning in the formal setting of school, including analyses of teacher – student interactions and instruction methods (e.g., Bowers, Doerr, 2001; Nathan, Knuth, 2003; Hacker, Tenent, 2002). There is little research on freshman calculus students' understanding of trigonometry. It is known that the students have some common misconceptions and difficulties in trigonometry. Unless the students are well informed about the reasons why the errors are made, it is unattainable to get rid of these errors. To express the reason of these errors, we must interpret the errors in terms of an appropriate learning theory. Generally, the teachers consider that the applications are important especially, on the level of high school and secondary school. They are often wary of theory. But unless there is an appropriate learning theory, solving the errors are difficult. In addition, each application and theory has motivated mathematical relationship which is a commonplace observation (Kirshner, Awtry, 2004). Avoiding from the theory by the teachers is due to the students that they are unwilling, lazy and have a weak level of understanding. A relationship between these causes and errors can not be constructed. These factors are important; if we find out the causes of student-errors, the importance of these abstract feelings will be reduced. Although it is considered that the relationship between real numbers and trigonometric concepts plays important role in the development of mathematical notion, the teaching of these concepts has not been the subject of research until the 1960's. So far we have measured angles in degrees. But degree is not only way to measure the angles. There is another way to measure an angle, which involves comparing the length of arc that the angle cuts off on a circle to the radius of the circle. This subject is very helpful in calculus. The radian measure of the angle θ at the center of the unit circle is defined to be the length of the arc spanned by the angle θ. It can be shown in Figure 1, the radian measure of the angle θ at the center of the unit circle.

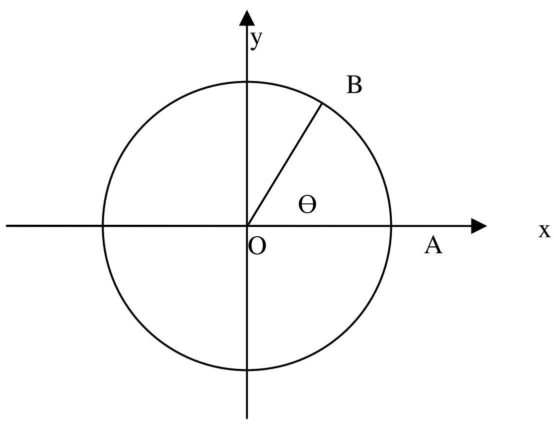

Figure 1. θ radian cuts off an arc length of θ in a unit circle

So we can assign that each real number to an angle in radian and then trigonometric functions and real numbers by cups and hold all of this while the calculate. Thus, the value of trigonometric functions when the angles are measured in radians is stated as a real number for example, cos1, cos60, cos0.4. If the angles are stated in degrees then the corresponding real numbers are written $\cos 1^0$, $\cos 60^0$, $\cos 0.4^{0.}$

2. Methodology

The goal of this study was to explore freshman calculus students' understanding the relationship between real numbers and trigonometric concepts. The present study investigates the radian concept as the length of an arc, the radian measure of an angle, the values of trigonometric functions when the angles are expressed in terms of radian, and trigonometric functions and their graphs, domain and range. The results indicated that, the existence of the gap between real numbers and trigonometric relations, the gap that can be characterized as the freshman calculus students' lack of knowledge or an earlier introduction of the topic'.

This study was conducted with 134 freshman calculus students of the Science Faculty at Anadolu University during the summer term of 2007. The instrument in the study was designed by the author. The data of this research was obtained from an exam consisting of 15 questions that measured the relationship between real numbers and trigonometric concepts. The questions on the test have dealt with defining the structural and procedural activities of trigonometric expressions. Students' responses for each of these items were analyzed qualitatively in terms of correctness and, where common mistake patterns were indicated, they were described and analyzed in terms of the probable mistakes. The students who response this exam we study on are taking the course of calculus I. This course is taken by almost all science students and some engineering students. The students had previously completed the course of calculus I, but since their marks were unsatisfactory, they were repeating this course.

3. Findings and interpretations

We have classified of students' errors observed in this study. First one is the lack of knowledge and the second one is the errors made due to the radian applied incorrectly. This classification is of considerable importance as it enables one to give appropriate remedial instruction for the types of errors.

We have pointed out that the large number of students who answered the items were not aware of several basic trigonometric concepts such as radian concept, trigonometric functions, the values of sine, cosine, and tangent of the real numbers. Firstly, we investigated students' understandings the teaching trigonometry in connection to concept of radian.

Some questions that were asked to students have been exemplified below:

Question 1 of questionnaire asked for the definition of the concept of radian.

Question was "Find the arc- length subtended by a central angle of $\frac{\pi}{2}$ radians in a unit circle".

The students gave different answers to the question. Only twenty-three out of 134 students could answer it correctly as $\frac{\pi}{2}$ (17%). Sixty-nine students answered the question as 1/4, “since $\frac{\pi}{2}$ is equal 90^0 then the arc-length is 1/4 (51%)”. Thirty-five students’ answer was 1 (26 %).

Another question was designed in order to detect possible weakness of the students in knowing π.

Question was, “represent the relation expressed by $y = \frac{\pi}{4}$ in a Cartesian plane”.

Only fourteen out of 134 students could answer it correctly (10%). The right answer has been shown in Figure 2.

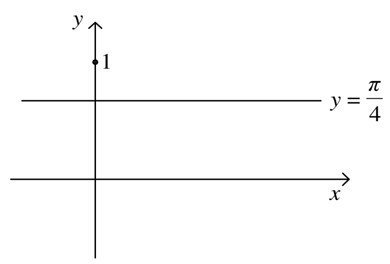

Figure 2. The graph of $y = \frac{\pi}{4}$

Seventy-seven students drew the graph incorrectly (57%). Their responses were “Since $y = \frac{\pi}{4} = 45^\circ$ the graph is y=45^0”.The graph of $y = \frac{\pi}{4}$ was drawn incorrectly by the students as shown in Figure 3.

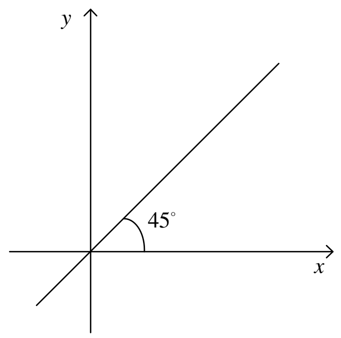

Figure 3. The graph of $y = \frac{\pi}{4}$ which was drawn incorrectly by students

$y = \frac{\pi}{4}$ is defined on real numbers but the students evaluated the value in degree. As a result, most of students could not relate π as a number. They considered π only as a equivalent to 180^0.

The questions 3 and 4 of questionnaire asked for the definition of the concept of trigonometric functions. We wanted to see whether the students know the values of trigonometric functions from real numbers to real numbers

Question 3 was, "give the sign of each of the following numbers".

Cos(-1), sin4, sin(-5), cos8.

For this question, only three students answered correctly that cos(-1)>0, sin4<0, sin(-5)>0, cos8<0. Most students (78 out of 134) answered as cos(-1)<0, sin4>0, sin(-5)<0, cos8>0 58%. The rest of students did not give any answer.

Question 4 was, "rank the following numbers in order from smallest to largest:"

sin $(2\frac{\pi}{3})$, sin3.2, sin(-3.2), sin 1.5.

Only five students answered correctly that sin 3.2 < sin(-3.2) < sin $(2\frac{\pi}{3})$ < sin 1.5 (3%).

Most students wrote the answer : sin $(2\frac{\pi}{3})$ =sin $2\frac{180^0}{3}$ = sin 120^0= $\frac{\sqrt{3}}{2}$

sin3.2= 3.2x 57^0= 182.4^0, sin (-3.2)= -182.4^0, sin 1.5= 1.5x 57^0= 85.5^0 then the rank of numbers is sin(-3.2)< sin 1.5 < sin $(2\frac{\pi}{3})$ < sin3.2.

As seen from the responses, most of students could find the values of sin function when the angle is expressed in term of π. They could convert the angle to degree measure namely,

sin $(2\frac{\pi}{3})$ =sin $2\frac{180^0}{3}$ = sin 120^0= $\frac{\sqrt{3}}{2}$

But, they could not find the values sin function when the angles are given as real numbers.

Another question was designed in order to detect of the students whether they knew the radian which was defined as an measure of angle related to the arc length of a circle.

Question was," mark the following angles given in radians on unit circle:"

1.4 , $\frac{1}{4}$, -4 , $-\frac{\pi}{4}$.

The majority of students did not give any answer (108 out of 134), 81%. The rest of students preferred in degree.

1.4x57^0= 79.8^0, $\frac{1}{4}$x57^0=16^0, (-4)x57^0= -228^0 , $-\frac{\pi}{4}$=-45^0 seen in Figure 4.

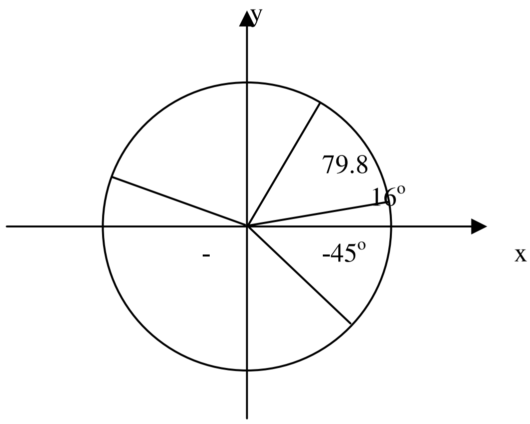

Figure 4. Angles in degree measure marked on circle.

Only 2 students give correct answer and this correct answer could be seen in Figure 5.

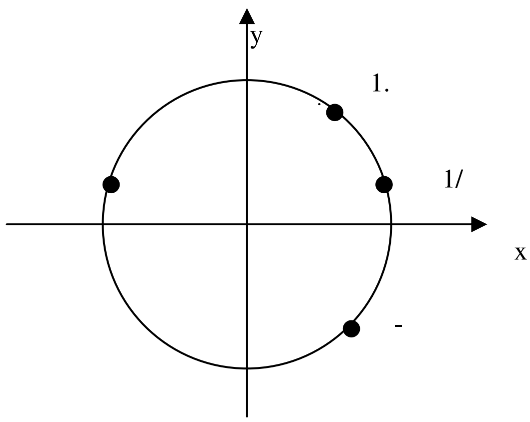

Figure 5. Angles in radian measure marked on circle.

Most students considered angle in radian only when they included π. As seen from above the responses, the majority of students could not define radian concept as the length of an arc on a circle. So, although they could convert between degree and radian angles, they could not deal with angles when given as real number without degree notation.

The questions 9 and 12 of questionnaire were designed to define trigonometric functions with domain and range consisting of real numbers.

Question 9 was, "f(x) = sin x is given, find the larger domain". The answers to this question were as follows:

Only eight students could answer it correctly as D_f=R (6%). Most students (82 out of 134) answered as $D_f=[0^\circ, 360^\circ]$, (62%). Thirty–five students answered as D_f=[-1,1], (27%). The rest of the students did not attempt to answer. Although they have learnt the concept of function in calculus, the transition from algebraic functions to trigonometric functions has not been performed well. So that we can define trigonometric functions with domain and range consisting of real numbers in radian measure.

Trigonometry is taught to the high school and secondary school students to find the unknown elements of the right triangle. At this stage, at high school and secondary school level the knowledge of trigonometry contains finding the values of sine, cosine, tangent for some angles given in degree (Cavey and Berenson, 2005). So in teaching of trigonometry, while the operations are made, generally, degree is used. Since the students easily animate "degree", the degree is preferred to radian and this teaching style is identified with trigonometry. In contrary, the radian is used in the applications of trigonometry in advance level and in calculus. Also in contrary, we can state that students' performances are good as regard to the problems dealing with degree.

Question 12 was, "if sin29=x, find x=?".

Only thirteen students answered correctly that x is $-1 \leq x \leq 0,\ x \in R$. For this question, answer of the majority was $x \cong \frac{1}{2}$. (103 out of 134, 76%).

Students are not aware of the processing load. They considered angles when given as a real number such as sin 29, in degree. The difficulty for students is that they have to know the definitions of trigonometric functions and the trigonometric functions and real numbers by cups and hold all of this while they calculate. The reason of the gap between real numbers and trigonometric concepts can be explained in terms of cognitive load. The trigonometry knowledge develops from procedural activity to structural activity. Namely, using the knowledge of trigonometry on the other subjects of calculus, it must be purified from rules. Similarly to teaching other mathematical subjects, the teaching of trigonometry must be concretized by using examples from daily living. For example, you walk 29 km around a circular area with radius 1 km. What are distances of the projections for final position on both axes to the center of this circle. The solution of this problem gives the values of cos 29 and sin 29. But, it seems obvious that the calculation of sin 29 is a procedural activity, because the operations are carried out on number to yield number. Finally, in trigonometry instruction, the structural descriptions and transformational data must be conflated.

Another question was asked the articulation graphic representation in plane.

Question was "sketch the graph of the set $\{(\cos t, \sin t) \mid t \in R\}$ in plane".

The majority of the students did not give any answer. Usually, the students were not accustomed to verify their knowledge of sketching graphs. Especially this question has showed that sketching graph of the set given as $\{(x, y) \mid x, y \in R\}$ is very hard.

Although many students drew the graph of trigonometric functions using computer, they knew neither the domain nor the range.

All of these results demonstrate how the conceptualization of subject specific knowledge for teaching.

The variation of well developed knowledge is very hard. For this reason, the students could not easily learn new and right knowledge. For example, they could not easily pass from the concept of function and the manipulations of functions to the concept of trigonometric function, because the trigonometry is neither the generalization of numerical processes nor the way of symbolization. Besides, although the students know usually the function concept, they are forced on the interpretation of trigonometric function. But, while they make operations with trigonometric functions, they regard them as polynomial functions.

If we want to dissolve the errors for students such as systematic errors from constructivist perspective, analyzing the procedures and their prerequisites is not sufficient. We must analyze whether the knowledge previously learnt influences a new concept or not.

4. Discussion

The purpose of this study was to identify some common student errors in trigonometry.

The conducted study has shown that the freshman calculus students perform unsatisfactorily in the operation of trigonometric expressions. Our investigation of student responses suggests that having to view radian as angle students' difficulties in checking trigonometric functions. The findings have shown that there are two different types of error that the freshman calculus students make. The first one is the lack of knowledge and the second one is the manipulative. For example, the first one that the radian angle is not expressed as a real number and trigonometric functions could not considered with domain and range from real numbers to real numbers. The second one that they could not used the unit circle to find the values of trigonometric functions such as sin 29. This classification is of considerable importance as it enables one to give appropriate remedial instruction for the two types of errors. If we want to account for students' systematic errors from a constructivist perspective, analyzing the procedures is not sufficient. We must analyze students' current schemas and how they interact with each other according to instruction and experience (Olivier, 1989).

As it is known, the university blames the high school for poor teaching, and the high school blames the secondary school. Some of the misconceptions in this study have showed that what the students have learnt previously is correct but this correct learning becomes the source of later misconceptions. To maintain a permanent correction, the things previously learnt must be changed so students' knowledge will not have to be changed later (Olivier, 1989). For example: the students know well enough that,

$$\cos 60^0 = ½$$

But what is the value of cos60?

Most probably, the answer is same, because in high school what is taught tends to concentrate on doing things in degrees. Students have stronger degree images used in trigonometry. Therefore, the trigonometry has been identified with degree. Thus, it is difficult to understand cos17, the domain of the function f(x) = sin x and the line of $y = \frac{\pi}{4}$ radian. They could not defined trigonometric functions from real numbers to real numbers. Many students have shown hesitancy while they are going to solve the exercises dealing with radian. Moreover, it is important to underline that the students' performances are clearly better on degree. But, the reality is that almost all of the calculus and advanced mathematics is taught within radians and the students should be ready to deal with all radians in a calculus class. According to this application, much more time should be spent in teaching the notion of radian. We assign each real number to an angle in radian and then trigonometric functions must be defined with domain and range consist of real numbers using in radian. Besides, the relation between the trigonometric functions and the real numbers must be developed, so in trigonometry the instruction of structural descriptions and transformational data must be conflated. In addition, the trigonometric functions must be described by a graph, pairs or a written statement and also the transitions must be given between each of these representations. By implementing such an instructive method, the difficulties in trigonometry can be diminished and the permanent learning can be sustained. I believe that, if the trigonometric concepts take place in the instruction of some calculus concepts, the errors in trigonometric processes will reduce.

References

Bowers, J.& Doerr, H.M. (2001). An analysis of prospective teachers’ dual roles in understanding the mathematics of change: Eliciting growth with technology. Journal of Mathematics Teacher Education, 4, 115-137.
Cavey. O. L. & Berenson. B. S. (2005). Learning to teach high school mathematics: Patterns of growth in understanding right triangle trigonometry during lesson plan study. Journal of Mathematical Behavior, 24, 171-190.
Dorier, J.,Robert,A., Robinet, J.& Rogalski, M. (2000).On a research programme concerning the teaching and learning of linear algebra in the first year of a French Science University. 31, 1, 27-35.
Hacker, D. J. & Tenent, A. (2002). Implementing reciprocal teaching in the classroom: Overcoming obstacles and making modifications. Journal of Educational Psychology, 94, 699-718.
Kirshner, D. & Awtry, T. (2004). Visual Salience of Algebraic Transformations. Journal for Research in Mathematics Education, 35,4,224-257.
Nathan, M. J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. Cognition & Instruction, 21, 175-207.
Olivier,A. (1989). Handling Pupils’ Misconceptions. Thirteenth National Convention on Mathematics, Physical Science and Biology Education, Pretoria, 3-7.
Porter, M., K. & Masingila, J. O. (2000). Examining the effects of writing on conceptual and procedural knowledge in calculus. Educational Studies in Mathematics, 42, 2, 165-177.
Radford, L. (2000). Signs and meanings in students’ emergent algebraic thinking: A semiotic analysis. Educational Studies in Mathematics, 42, 3, 237-268.
Romberg, T. & Shafer, M., Webb, N. (2000). Study of the impact of mathematics in context on student achievement. Madison, WI: Wisconsin Center for Education Research, University of Wisconsin-Madison.