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Computer-aided graphical solving of equations of the form $f(x) = g(x)$

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Abstract.

One important disadvantage of most of the graphical methods for solving equations by a computer is that the computer screen shows only a part of the co-ordinate plane. Therefore in the general case we obtain incomplete information about the number and the distribution of the intersection points of the graphs (of *F(x)* with the abscissa for equations of the form *F(x)=0* and of *f(x)* with *g(x)* for equations of the form *f(x)=g(x)*). Here we suggest an approach, which makes it possible to eliminate this disadvantage for a limited (but large enough for the school level) set of functions *f(x)* and *g(x)*. It provides the opportunity of creating effective didactic technologies for graphical solving and exploring of equations, including parametric ones, by dynamic geometry software (DGS) like GEONExT (version 1.71) - see [1] or GeoGebra (GeoGebra_3_0_0_0_Release_Candidate_1) [2], [3].

Example for illustration.

In order to illustrate our idea (which will be discussed in details below), we start with an example.

Problem 1. Find, with accuracy to tenths, the roots of the equation $\sqrt{4-x^2} - \frac{1}{x+1} - 2 = 0$.

Solution: This problem cannot be solved by the students analytically. On the other side, the graph of the function $y = \sqrt{4-x^2} - \frac{1}{x+1} - 2$ is unknown and its shape outside the screen cannot be predicted. Therefore we transform the equation to the form: $\sqrt{4-x^2} = \frac{1}{x+1} + 2$. Now the graphs of the functions *f(x)* = $\sqrt{4-x^2}$ and *g(x)* = $\frac{1}{x+1} + 2$ on the two sides of the latter equation are known: the graph of the first one is a semicircle of radius 2 and center at the origin and the graph of the second one is obtained by translation from the graph of ***1/x***. In **Fig. 1** the two graphs are plotted (by GEONExT) simultaneously in the same Cartesian system of coordinates. Their intersection points determine the solutions of the given equation: the picture in **Fig. 1** shows that in our case we have one intersecting point, and its abscissa is the respective solution.

The important point here is that we can be sure that this is the only solution of the equation. This is because we know well the shape of the graphs of the simple functions

$f(x) = \sqrt{4-x^2}$ and $g(x) = \dfrac{1}{x+1}+2$, and therefore we can predict their mutual disposal outside the screen.

It is easy to obtain (using the software) the solution $x=-1,8$ (with the required accuracy).

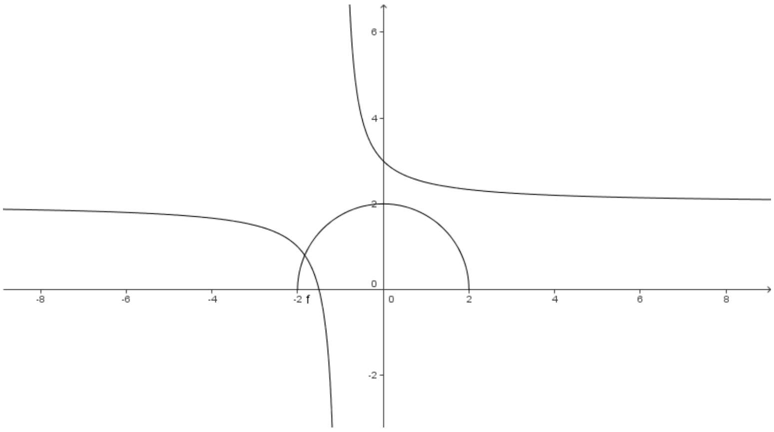

Fig. 1: Graphical representation of the equation $\sqrt{4-x^2} = \dfrac{1}{x+1}+2$ by the functions $f(x) = \sqrt{4-x^2}$ and $g(x) = \dfrac{1}{x+1}+2$.

We suggest a method, whose essential ideas can be described as follows:

To work only with equations, which can be transformed to the form $f(x) = g(x)$ ***with functions*** *f(x)* and *g(x)****, whose graphs the students know well.*** These are the functions, which the students study at school (*y=ax+b, y=* ax^2 *+bx+c, y=sinx* etc.). Some other functions that are not too complicated (e.g. $y=\sqrt{1-x^2}$, etc.) could be added gradually with suitable explanations (about their graphs).

When a student is familiar with the shape of the graphs of the functions $f(x)$ and $g(x)$ in the equation $f(x) = g(x)$ and if he looks carefully at the computer screen, where these graphs are simultaneously plotted, then this student will be able to predict the behavior of the graphs out of the screen. If it is clear that these graphs could not intersect outside the screen, then:

(*) the solutions are the *x*-coordinates of the intersection points of the graphs within the screen.

Otherwise, i.e. if the graphs could intersect outside the screen, then by gradual zooming we put “within” the screen more and more of the co-ordinate plain until it is clear that the intersection points of the graphs of $y=f(x)$ and $y=g(x)$ are inside the screen. Then holds (*).

A disadvantage of the graphical solving of equations is that we find the solutions with a certain approximation. However, this is not as essential as seems at a first glance. Actually, even in the case of “exact” solving of algebraic equations for a comparison and practical use of solutions of the type $x = \sqrt{3}$ one needs its "approximate" as well. In the same time GeoGebra provides the solutions with accuracy to the 6th decimal place. In some special cases the approximations we use may significantly hamper the solving of the problem; such problems should not be given to the students. It should be noted that peculiarities of this kind appear also in the analytical "exact" form of solutions.

The solving of equations by traditional algebraic methods perplexes many students by the following reasons:

1. The algorithms for solving equations algebraically are different for different kinds of equations. We have one algorithm for linear equations, another for quadratic, third one for absolute value equations, etc.

2. When the equations are parametric, there is an additional difficulty: thorough consideration of all the possible cases.

3. The computational procedures involve risks of making errors.

4. Third- and higher-degree equations or polynomial-transcendent ones like some in [4] and [5] cannot be solved at school.

This black list could be extended. For the motivated students the overcoming of every difficulty leads to enriching in knowledge, skills and habits. For the others, which are majority, the difficulties usually are counter-indicative.

There is one and the same common algorithm for the solving of all equations of the type $f(x) = g(x)$ by graphical methods – in particular with the help of dynamic geometry software (DGS), unlike the analytical solving: plotting the graph of the function $y = f(x)$, then the graph of $y = g(x)$, finding their intersection points and then the abscissas of these points. To judge that the graphs have no common points out of the screen, or to know where in the plane to search for such points, if we suspect their existence, we should be familiar with the shapes of the graphs of $y = f(x)$ and $y = g(x)$ in the whole plain. Such knowledge the students have **only for studied functions. The teachers should explain them this peculiarity**.

The work with DGS has the following good sides:

1. The principle of using visual aids in teaching is kept. Furthermore, DGS offers opportunities that the chalk and the blackboard are not able to offer.

2. DGS clears the path for the **new technologies in math's teaching**. Students get familiar with software applications that can be used not only to do their homework but also after finishing school (**lifelong learning**). [6]

3. The learners are more motivated to work.

4. The likelihood for delusion or error during the process of problem solving is smaller.

5. The time for work becomes shorter.

In our opinion, the creation of an effective didactic technology for GSE of the form $f(x) = g(x)$ (and parametric equations $f(x,a) = g(x,a)$) for the secondary school via GEONExT (GeoGebra) should have the following features:

1) replacement of the traditional analytical solving **of a part** of the equations studied at school by graphical solving,

2) assisting, by different means, the analytical solution **of another part** of equations through visual aids,

3) a **third part** of equations should still have to be solved like before – without using graphs,

4) **extending** the set of equations studied at school by including new types of equations (such as $sin\ x = ax + b$) etc.

Revision and preliminary preparation.

To teach students GSE of the type $f(x) = g(x)$ by DGS, they should be familiar in advance with:

a) all kinds of functions $y = f(x)$ and $y = g(x)$ involved in the considered equations, and their graphs;

b) the "GSE" containing absolute value from the algebra course in 8th grade;

c) the potential of DGS for plotting graphs of functions, zooming, drawing objects in and out of the screen area etc.

For this purpose we suggest revision and preliminary preparation as follows:

1.1. Short revision of the "GSE" containing absolute value from 8th grade (for linear equations).

1.2. Brief introduction to GEONExT (GeoGebra).

Graphs of elementary functions and how to handle them in GEONExT (GeoGebra).

These programs allow plotting graphs in Cartesian co-ordinate system. On the axes tick marks can be put up to thousandths **manually** in GEONExT and thus the roots of an equation of the type $f(x) = 0$, if they exist, can be found with accuracy to the third sign, because on the screen we can see which is the tick mark closest to the solution (maximum zoom). In GeoGebra tick marks appear **automatically** when zooming and the program has capacity to show the sixth decimal place. If we put in GEONExT the cursor on the point of intersection, its co-ordinates appear at the upper right corner of the screen, with accuracy up to the 5th decimal place at maximum zoom. But our experiments have shown that GEONExT, run on different machines, provides results, which differ with hundredths and (!) even with tenths. GeoGebra provides the same results up to the 6th decimal place precisely. GEONExT has shown further shortcomings; therefore we prefer GeoGebra.

Overview of some similar publications on computer-aided graphical solving of equations (GSE).

We do not consider sources on GSE without computer (or graphic calculator). We also omit sources, where computer is involved, but the given equations are trivial (for instance linear). We scanned many free Bulgarian and English-language Internet sites by Google and Bulgarian articles in various periodicals from the period 2000 – 2007 and below give brief comments.

On the Internet site "Trigonometry – Equations, Identities, and Modeling Lesson 3: Equations which require a Graphical Solution" [4] there are more equations of the above type, but they are solved (graphically) by graphical calculator. **The authors discuss** the problem about predictability of the graphs of functions and therefore of the possibility of existence of solutions outside the computer screen.

A graphical solution of the equation $x = 2\sin x$ by MAPLE is presented on the Internet site of the Department of Mathematics, University of South Carolina [5] in an article entitled "What is a Project Report?". It has polynomial and transcendental terms. This paper is not focused on GSE; it is only an illustration how certain empirical data can be expressed mathematically.

A didactical approach for solving parametric equations (and inequalities) by GEONExT and its Internet version is presented by Tonova, Goushev & Kopeva in [7] and [8], respectively. They are based on online dynamic Java applets. But in this didactical module GEONExT is used **only for visualization and to help the heuristics in searching for a way of analytical solving of the problems**.

In the articles [9] and [10] Lazarov & Vassileva describe graphical solutions of systems of parametric equations for university students by MATHEMATICA.

Stanilov & Slavova explain in [11] how MAPLE can be used to solve irrational equations. Obtained by the procedure "solve" and verified by "eval", the solutions are **visualized** graphically.

GSE by MATLAB is described in the cited above e-book [4] on the Internet site of the Department of Mathematics at the college of Staten Island. The method presented here prescribes to use GSE when the analytical algorithms are very complicated or if there is no such algorithm, since the graphical solving is not precise. The examples illustrating the method involve equations with both polynomial and transcendental terms. The authors sometimes transform them from the form $f(x) = g(x)$ to the form $F(x) = 0$, which causes difficulties, because the solvers do not know the shape of the graph of the function $y = F(x)$.

On the Internet site "Solve quadratic equations graphically" [12] we find a tutorial for analytical and graphical solving of quadratic equations. The GSE is presented also practically by a dynamic applet. The users can vary any of the three coefficients of the quadratic trinomial and to observe how its graph changes. If the equation has real roots, they appear on the screen with accuracy of 8 places after the decimal point.

On the Internet site "Simultaneous equations" [13] the concepts of linear equation, power of equation, system of equations are introduced formally and also graphical method for solving linear, quadratic and higher power systems of equations by dynamic Java applet and tutorial marks. Unfortunately they neither say that the graphical method is only approximate, nor provide systems with solutions, which are not integer.

There are given in [5] equations, which require graphical solution (like $x = \sin x$). The device for work is graphical calculator.

A method for graphical solving of equations and systems of equations by MAPLE with several examples is described on the Internet site "Syntax and Hints for Selected Maple Commands" [14]. The authors of [14] sometimes transform equation of the type $f(x) = g(x)$ to $F(x) = 0$, which is unsuitable, if the solvers do not know the shape of the graph outside the computer screen.

On the Internet site "Trig Solutions Review" [15] an approach is present for solving easy trigonometric equations by a handheld device. The functions are periodic and hence their graphs are predictable, as stated on the site.

For all the papers cited above except [4] the common detail is that their authors **do not require that the functions $y=F(x)$, $y = f(x)$ and $y = g(x)$ have been studied by the stu-**

dents. They do not discuss the predictability of the function graphs outside the screen. This is significant disadvantage because the students might think that they **can solve any equation graphically**. The authors do not describe didactic experiments or any impressions from classroom or results from online interviews, i.e. **most of them do not present approbated didactic technologies**.

In [16] Baycheva and Kirilova present a successful didactic experiment with software for graphical solving of problems on quadratic function, quadratic equations and inequalities. The software works under DOS, its interface looks out-of-date and it deals only with quadratic trinomials.

Further examples for illustration.

Problem 2. Find the roots of the equation $\sqrt{14-x^2}-\frac{1}{x+1}-2=0$ with accuracy to tenths.

Solution: As in the solution of **Problem 1,** we transform the given equation to the form $\sqrt{14-x^2}=\frac{1}{x+1}+2$ (**Fig. 2**) and proceed in a similar way. Here $x_1=-3,3;\ x_2=-0,4;\ x_3=2,9$.

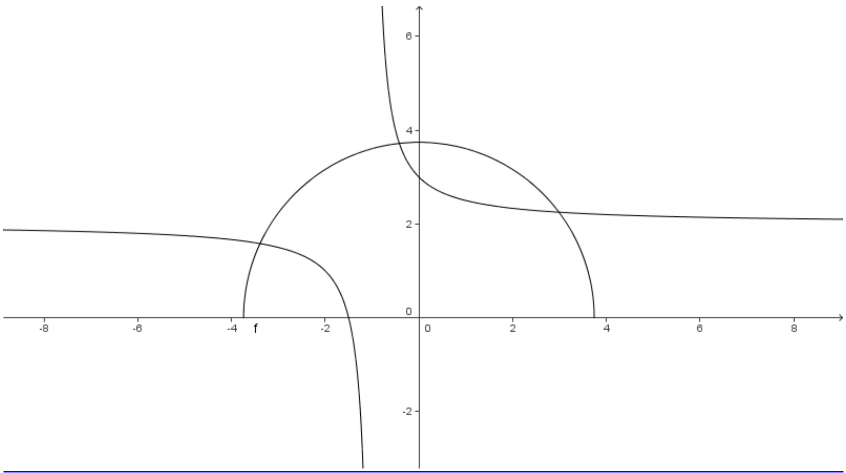

Fig. 2: Graphical representation of the equation $\sqrt{14-x^2}=\frac{1}{x+1}+2$.

Problem 3. Find, with accuracy to tenths, the roots of the equation $\sqrt{1-x^2}-\frac{1}{x+1}-2=0$.

Solution: As above, we transform the given equation to the form $\sqrt{1-x^2}=\frac{1}{x+1}+2$. Here the curves $y=\sqrt{1-x^2}$ and $y=\frac{1}{x+1}+2$ (see **Fig. 3**) have no intersection points; the problem has no solution.

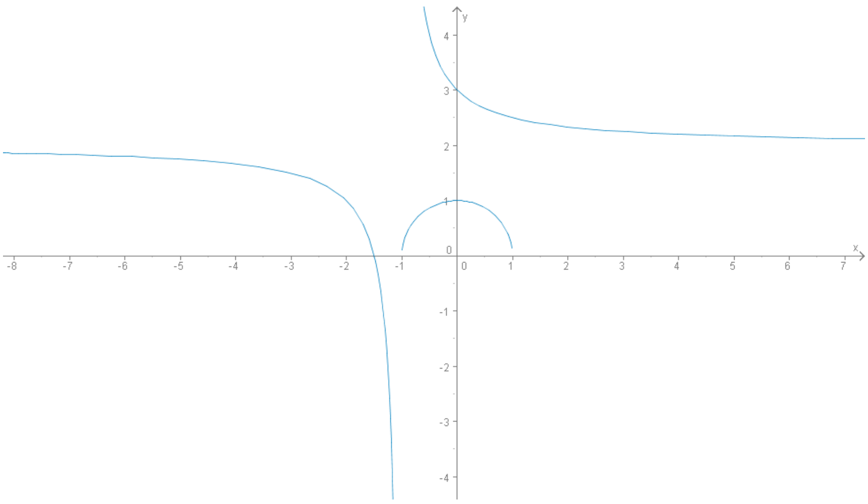

Fig. 3: Graphical representation of the equation $\sqrt{1-x^2} = \dfrac{1}{x+1} + 2$

The problems 1, 2 and 3 can serve as basis for **graphical exploration of the number of solutions of the equation**

$$\sqrt{a-x^2} - \frac{1}{x+1} - 2 = 0$$

depending on the real parameter *a*.

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