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A note on Oscar Chisini mean value definition

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Abstract. Mainly on the basis of some notable physical problems reported in a 1929 Oscar Chisini paper, this brief note expose further possible historic-critical remarks on the definition of statistical mean value which will lead us towards the realm of Integral Geometry, via the Felix Klein *Erlanger Programm*. Possible educational implications are also briefly discussed.

Sunto. Principalmente sulla base di alcuni notevoli problemi fisici considerati in un lavoro di Oscar Chisini del 1929, in questa breve nota si espongono ulteriori, possibili osservazioni storico-critiche in merito alla definizione di valor medio statistico che ci condurrà verso il contesto della Geometria Integrale, attraverso il *Programma di Erlangen* di Felix Klein. Infine, eventuali risvolti didattici sono pure brevemente accennati.

1. Introduction

If one identifies, from a mathematical viewpoint, the concept of *statistical variable* (of Statistic) with that of *random variable* (of Probability Theory) according to what established in (Dall'Aglio, 1987, Chapter IV, Section IV.2), then the notion of *mean value* may be included in the most general one of *expectation value* of a random variable[1], in turn included in the wider class of the *moments* of a random variable.

Following (Piccolo, 1998, Chapter 4), the *mean* concept is a primitive one for the human being, so that it is perceived with immediacy, though its measure is arbitrary since it depends on the synthesis criterion adopted. Through such a criterion, then, it will be possible to state a formal definition of mean value. The first notion of mean was due to[2] A.L. Cauchy in 1821 who simply defined it as an intermediate value between the maximum and minimum values of a given statistical variable. Such a definition, is nowadays considered as a simple range condition, called *internality Cauchy condition*. Instead, a great attention had a formal definition of mean value due to Oscar Chisini in 1929, according to whom the mean $\mathcal{M}$ of a given statistical variable X, is that value which, with respect to another given synthetic function f defined on the frequency distribution of X, leaves invariant the values of the latter, that is to say[3]

$$f(x_1, \dots, x_n) = f(\mathcal{M}, \dots, \mathcal{M}) \qquad \forall (x_1, \dots, x_n) \in dom\, f. \tag{1}$$

Following (Girone & Salvemini, 2000, Chapter 6, Section 6.1) and (Ferrauto, 1996, Chapter 4), such a mean value $\mathcal{M}$ warrants that a predetermined quantity, assumed to be invariant and formally expressed by the function f, is left unchanged. This Chisini's theoretical criterion defining a mean, is made operative by specifying the function f in dependence on the formal properties (like additivity, multiplicativity or invertibility) of the random variable X, so reaching to various possible types of means on the basis of the given f (see (Piccolo, 1998, Chapter 4, Section 4.2)). The choice of f is strictly dependent on the context of the involved problem, this being one of the central motifs of this paper.

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[1] In this regards, see also (De Finetti, 1930) and what will be said in Section 2 of the present paper.

[2] For some related historic-bibliographical notes, see (Berzolari, 1972, Article LV, Chapter II).

[3] For instance, to get the usual arithmetic mean, we choose the following weighted invariant function $f(x_1,\dots,x_n) = \sum_{i=1}^{n} p_i\, x_i$ and we impose that be $\sum_{i=1}^{n} p_i\, x_i = \sum_{i=1}^{n} \mathcal{M}\, x_i$, whence $\mathcal{M} = \sum_{i=1}^{n} p_i\, x_i / \sum_{i=1}^{n} p_i$ which is the *weighted arithmetic mean* of the variables x_i with weights p_i. Instead, the invariant function which gives rise the simple *geometrical mean* is the following $f(x_1,\dots,x_n) = \prod_{i=1}^{n} x_i$, from which, applying (1), it follows $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \mathcal{M} = \mathcal{M}^n$, whence $\mathcal{M} = \sqrt[n]{\prod_{i=1}^{n} x_i}$. Finally, for the weighted harmonic mean, we have $f(x_1,\dots,x_n) = \sum_{i=1}^{n} p_i\, / x_i$, hence $\sum_{i=1}^{n} p_i\, / x_i = \sum_{i=1}^{n} p_i\, / \mathcal{M}$ whence $\mathcal{M} = (\sum_{i=1}^{n} p_i) / (\sum_{i=1}^{n} p_i\, / x_i)$ which is the *weighted harmonic mean* with weights p_i. For further related information, see (Girone & Salvemini, 2000, Chapter 6).

Other possible definitions of mean have also been proposed, like that proposed by O. Wald (1950) and the one proposed by M. Nagumo, A.N. Kolmogorov and B. De Finetti (see (Piccolo, 1998, Chapter 4, Section 4.2)), which substantially make use of methods analogous to the functional one of Chisini whose essential idea is the following: through the function f, it is possible to consider the *transferability* of the initial statistical variable X amongst the unities of the statistical population in which it is defined.

In this brief note, we want above all to deal with the notion of mean value according to Chisini, on which then one of his former students, Bruno De Finetti, has mainly based his subsequent fundamental paper (De Finetti, 1930), and from which, amongst other things, it will turn out to be clear the close dependence of the notion of mean value by the related involved problematic context.

2. On Chisini’s mean definition

In the general statistical framework of a critical discussion of the mean value notion, De Finetti has centered his 1930 paper on a review of the notion of statistical mean according to Oscar Chisini with its possible features and applications. He first stated that an extension of the concept of mean to an arbitrary random variable is also possible through the Chisini definition.

Oscar Chisini (1889-1967) was a pupil of Federigo Enriques, and his main research field was in Algebraic Geometry[4]. In 1929, he incidentally had to consider some statistical questions from which derived his brief but meaningful note on the general notion of a mean value. In it, he first of all criticizes the old 1821 Cauchy definition of mean simply conceived as a certain value comprised between the minimum and maximum values of the set of values of a given variable. Indeed, it does not provide neither any synthetic information which gives a global vision of the phenomenon described by this variable nor puts into evidence the typical relative character that a mean must have. According to Chisini, these last requirements might be accomplished by means of the choice of a certain function, say f, depending on the observed quantities of this phenomenon. To this purpose he refer to some meaningful kinematical[5] and geometrical[6] problems as practical examples of this his basic point of view on what a mean should be: for instance, to point out the relative character of a mean, that is to say, its dependence on the circumstances of the involved problematic situation, he argues, inter alia, on a physical problem concerning the determination of the mean resistance of three conductors, whose result clearly depend on the geometry of the this physical problem which is related to parallel or sequential disposition of these conductors. At last, he also considers the determination of this statistical parameter – a mean value – regarding interesting physical problems concerning the oscillations of certain physical systems (like a pendulum), in which are also involved some not negligible geometrical considerations, in turn connected to mass distribution problems whose inertial momenta are but that second order statistical momenta (see (De Finetti, 1970, Volume I, Chapter II, Sections 8, 9 and 10)).

Thereafter, Chisini provides a general definition of mean of an arbitrary distribution of a quantity given in certain circumstances and situations[7], as that unique value of it which may be substituted without to have any

[4] He was one of the exponents of the so-called Italian geometric school, but also with wide interests in mathematics education (like many other members of this celebrated school of which Federigo Enriques was charismatic leader).

[5] In this regards, it is classical examples those related to the computation of the mean velocity of certain kinematical problems, the same usually reported by the common treatises and textbooks on Statistics and Probability Theory: see, for instance, besides (De Finetti, 1930), also (Girone & Salvemini, 2000, Chapter 6, Section 6.12) and (Dall’Aglio, 1987, Chapter IV, Section 2, Example IV.2.1).

[6] Above all, the examples reported at points 4. and 6. of the paper (Chisini, 1929), are very meaningful to show the dependence of some types of means by the geometrical aspects of the problem in which they are involved. In particular, the first example reported at point 6. might be extended considering, in (Chisini, 1929, formula (12)), a path integral along the distribution line of the values given by $x = x(t)$ instead of a scalar integral which, besides, depends too by the geometry of the problem, being it the area underlying the line of equation $x = x(t)$. It is likewise interesting the other following examples of the same point 6., from which it turns out to be always non-negligible the geometrical aspects of the considered problem. Finally, the argumentations carried out at the final point 7. of Chisini paper, clearly show what significant effects have a change of independent variables of the function f of (1), leading us toward the more general group theory considerations which will be given in the next Section 4. However, for a deeper discussion of these type of argumentations, see (De Finetti, 1970, Volume I, Chapter II, Sections 8, 9 and 10).

[7] About the choice of a given mean, De Finetti, in (De Finetti, Volume I, Chapter II, Section 9), speaks of the *relative* and *functional* meaning that it must be identified for answering to the purpose whose is aimed the given problem. According to the author, this problem’s purpose may be summarized by means of the German term *zweckmässig*, where

change in the above contextual problematic framework. To our purposes, we stress on this last peculiarity, that is to say, the just mentioned requirement of general invariance about the circumstantial and situational setting of the given statistical distribution. In the general case of an arbitrary random variable ξ with distribution given by the partition function $\Phi(\xi)$, then we should consider a functional of the type $F[\Phi(\xi)] = \int \psi(\xi)d\Phi(\xi)$ instead of $f(x_1, \dots, x_n)$, and request to be valid the condition $F[\Phi(\xi)] = F\xi(x)$ if x is the required mean for such a random variable, with $F\xi(x) = \delta(\xi - x)$ distribution function of the random variable ξ centered at x. Therefore, under the hypothesis of invertibility of F, we have[8] $x = F_{\xi}^{-1}(\mathcal{F}[\Phi(\xi)])$.

3. A particular case related to non-commutativity.

One of the main formal properties of a statistical mean is the commutativity one, or else its invariance under the action of permutation group. Indeed, following the seminal Steven's paper[9] (Stevens, 1946), the first measurement approach to statistical variables both qualitative and quantitative, consists in their classification according to one of the four main measure levels stated by S.S. Stevens, namely the *nominal*, *ordinal*, *interval* and *ratio scales*, of which we herein reports what the same Stevens says in (Stevens, 1946, p. 677)

> *«Paraphrasing N. R. Campbell (Final Report, p. 340), we may say that measurement, in the broadest sense, is defined as the assignment of numerals to objects or events according to rules. The fact that numerals can be assigned under different rules leads to different kinds of scales and different kinds of measurement. The problem then becomes that of making explicit (a) the various rules for the assignment of numerals, (b) the mathematical properties (or group structure) of the resulting scales, and (c) the statistical operations applicable to measurements made with each type of scale».*

Subsequently, at page 678 of (Stevens, 1946), about the description of the third column of the basic Table I (see later), Stevens states that

> *«In the column which records the group structure of each scale are listed the mathematical transformations which leave the scale-form invariant. Thus, any numeral, x, on a scale can be replaced by another numeral, x', where x' is the function of x listed in this column. Each mathematical group in the column is contained in the group immediately above it. The last column presents examples of the type of statistical operations appropriate to each scale. This column is cumulative in that all statistics listed are admissible for data scaled against a ratio scale. The criterion for the appropriateness of a statistic is invariance under the transformations in column 3».*

We herein report the Table I of (Stevens, 1946) with the additions and corrections given in (Stevens, 1958)

Measurement Scale	**Basic Empirical Operations**	**Mathematical Group Structure**	**Permissible Statistics (Invariantive)**	**Typical examples**
NOMINAL	Determination of equalities	*Permutation group x' = f(x)* with *f* bijective correspondence	Number of cases, Mode, Contingency correlation, Information measure	Numerations
ORDINAL	Determination of greater or less	*Isotonic group x' = f(x)* with *f* injective map	Median, Percentiles, Ordinary correlations	Intelligence test coarse scorings, Mineral hardness

zweck means ''purpose'' whereas *mässig* means ''suitable'', that is to say, the aim of the problem must be ''suitable to the purpose'' (zweckmässig).

[8] All the above considerations have been drew from the papers (Chisini, 1929) and (De Finetti, 1930); in this regards, see also (De Finetti, 1970).

[9] See also (Ferrauto, 1996, Chapter 1) and (Piccolo, 1998, Chapter 2, Section 2.3).

INTERVAL	Determination of equality of intervals or differences	*General linear group $x' = ax+b$*	Mean, Standard deviation, Rank-order correlation, Product-moment correlation	^{o}F and ^{o}C temperatures, Line position, Intelligence test standard scorings
RATIO	Determination of equality or ratio	*Similarity group $x' = ax$*	Coefficient of variation, Geometrical and Harmonic means	Lengths, Densities, ^{o}K temperatures, sones[10], brils[11]

According to what Stevens himself said in (Lerner, 1977, Chapter 3), the invariance is the central characteristic on which is based this classification scale. Therefore, it is possible to get an objective scientific information from a given set of data if and only if they are invariant respect to a certain group of transformations, which is the invariance group of the given scale.

The four measurement levels are cumulative and hence, in particular, the mathematical properties of one level are included into those of the higher levels[12] (see (Ferrauto, 1996, Chapter 1)), so that the commutativity, formally given by the invariance respect to the permutation group of the first measurement level, is one of the main formal properties owned by the various statistical tools therein mentioned. From this last conclusion, it is also possible to argue what follows.

Following (Bernardini, 1968, Chapter XV), (Kittel et al., 1970, Chapter 2, Section 2.6) and (Tonzig, 1981, 3), the finite angular displacements and the velocities are directional quantities which yet are not vectorial quantities because they do not verify the commutative law for the sum, so that it is not possible to consider an any their mean value in the above sense[13]. On the other hand, the non-commutativity of finite rotations[14] is due to the non-commutativity of the rotation differential operators (*generators*) L_x, L_y and L_z of the group SO(3), which, amongst other, lead to mathematics of the addition of quantum angular momenta and related selection rules. These last quantum observables cannot be summed among them with the ordinary rules of a commutative algebra but according to the irreducible representation methods of SO(3) (see (Onofri & Destri, 1996, Chapter 8, Section 8.3); in particular, it is not possible to consider, for them, the usual statistical means.

The observations made so far, above all those related to the primary above mentioned work of Stevens, clearly lead us towards a major consideration of the relationships elapsing between Group Theory and Statistic, hence between Geometry and Statistic if one takes into account the well-known 1872 Felix Klein *Erlanger Programm*, whose principle of the method sets that, roughly speaking, the main formal properties of geometrical entities are those invariant respect to the action of well-determined groups. Hence, following this pivotal Klein's idea, central concepts and tools of Geometry will be group invariance and symmetry ones. This program have had notable and fruitful features both in pure and applied mathematics, as well as in Physics: one of these, concerns that branch of Mathematics known as *Integral Geometry*, which is closely connected to the notion of *geometric probability* and related arguments.

4. Towards the Integral Geometry

[10] These are units of measurements of the volume (see (Stevens, 1958)).
[11] These are units of measurements of the brightness (see (Stevens, 1958)).
[12] As it has been already said by Stevens himself, namely when he says that «[…] *each mathematical group in the column 3 is contained in the group immediately above it*».
[13] In this regards, it is important to take into account the distinction between *polar* and *axial* vectors; the angular velocity is an axial vector. Analogously, the usual mean values, in general, cannot be applied to theoretical physics computations involving the so-called *intensive* physical quantities, like the temperatures, notwithstanding these last commute among them.
[14] But not of the infinitesimal ones.

Following[15] (Stoka, 1982, Chapter III), if G_m is an m parameter Lie group of transformations of $\mathbb{R}^n$ of the type

(2) $$y_i = \psi_i(x_1, \dots, x_n;\ a_1, \dots, a_m) = \psi_i(x; a) \qquad i = 1, \dots, n$$

depending on m parameters a_j $j = 1, \dots, m$, then a function $\Phi(x_1, \dots, x_n)$ is said to be an *integral invariant* of the group G_m if

(3) $$\int \dots \int_{\mathcal{S}} \Phi(x_1, \dots, x_n)\ dx_1 \dots dx_n = \int \dots \int_{\mathcal{S}} \Phi(y_1, \dots, y_n)\ dy_1 \dots dy_n$$

for every $\mathcal{S} \subseteq \mathbb{R}^n$ for which there exist the given integrals. On the other hand, if

$$J(x; y) \stackrel{\text{def}}{=} \frac{D(y_1, \dots, y_n)}{D(x_1, \dots, x_n)}$$

is the Jacobian determinant related to the variable change $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$ given by (2), then, from (3), it follows that

(4) $$\Phi(x_1, \dots, x_n) = J(x; y)\ \Phi(y_1, \dots, y_n) = J(x; y)\ \Phi\big(\psi_1(x; a), \dots, \psi_n\ (x; a)\big).$$

Now, the relation (1), written for Φ instead of f, is of the type (4) when $\psi_1(x; a) = \dots = \psi_n\ (x; a) = M$ and $J(x; y) = 1$ (or a non-zero constant), so that the (1) is a particular case of the more general relation (4).

If $\xi_j(x_1, \dots, x_n)$ $j = 1, \dots, m$ are the infinitesimal generators of G_m, then a *theorem of R. Deltheil* (see (Stoka, 1982, Chapter III, Section 3.1)) states that for $\Phi(x_1, \dots, x_n)$ be an integral invariant of G_m it is necessary and sufficient that Φ be solution of the following system of first order partial differential equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \big(\xi_{ij}(x)\Phi(x)\big) = 0, \qquad j = 1, \dots, m,$$

whence it follows a close relationship between the group structure of G_m and its integral invariant functions Φ. The group G_m is said to be *measurable* if it admits an unique integral invariant function Φ, at most, up to a multiplicative constant.

Let $\mathfrak{F}_p$ be a family of p (≥ 1) dimensional and q parametric manifolds V_p of $\mathbb{R}^n$ each of which is given by the system of (parametric) equations

$$F^j\big(x_1, \dots, x_n;\ \alpha_1, \dots, \alpha_q\big) = 0 \qquad j = 1, \dots, n - p$$

with any F^j analytic and $\alpha_1, \dots, \alpha_q$ arbitrary parameters, the variability of this family being given only by the variability of these parameters α_r and not by the functions F^j. Let G be a group acting on $\mathfrak{F}_p$, that is to say, such that T: $\mathfrak{F}_p \to \mathfrak{F}_p$ for every $T \in G$, and let $\mathfrak{I}_G = \bigoplus_{V_p \in \mathfrak{F}_p} \mathfrak{I}_{V_p}$ be the internal direct product of the isotropy groups $\mathfrak{I}_{V_p} = \{T;\ T \in G,\ T\big(V_p\big) = V_p\}$, each of which is a normal subgroup of G. Hence, let $\mathfrak{G}_G = G/\mathfrak{I}_G$ be the related quotient group which has the property of leaving globally invariant the family $\mathfrak{F}_p$ without containing any transformation (different from the identity) which leaves invariant every manifold V_p of $\mathfrak{F}_p$; such a group will be said the *maximal invariance group* of $\mathfrak{F}_p$.

If $\mathfrak{G}_G$ is a Lie group of transformations of $\mathbb{R}^n$ of the type (2), said $\alpha_1, \dots, \alpha_q$ the parameters of a manifold V_p, then the parameters $\beta_1, \dots, \beta_q$ of the manifold $V'_p = T(V_p)$ will be such that

$$F^j\big(x_1, \dots, x_n; \beta_1, \dots, \beta_q\big) = F^j\big(\psi_1(x; a), \dots, \psi_n(x; a);\ \alpha_1, \dots, \alpha_q\big) \qquad j = 1, \dots, n - p$$

[15] For a more complete reference, see (Stoka, 1968).

where

$$\beta_k = \vartheta_k(\alpha_1, \dots, \alpha_q;\ a_1, \dots, a_r) \qquad k = 1, \dots, q \tag{5}$$

for certain functions ϑ_k. Therefore, if $\mathfrak{D}_q \subseteq \mathbb{R}^q$ is the space of the parameters $\alpha_1, \dots, \alpha_q$ of the family $\mathfrak{F}_p$, then to the maximal invariance group $\mathfrak{G}_G$, whose elements are of the type (2), it is possible to associate, relatively to the space $\mathfrak{D}_q$, the family of transformations (5) which form a group isomorphic to $\mathfrak{G}_G$ and that will be denoted by $\mathfrak{H}_r(\alpha)$. Hence $\mathfrak{H}_r(\alpha) \cong \mathfrak{G}_G$, the first group being also said that *associated* to $\mathfrak{G}_G$ respect to the family $\mathfrak{F}_p$. Thus, if $\mathfrak{H}_r(\alpha)$ is a measurable group with invariant integral function $\Phi(\alpha_1, \dots, \alpha_q)$, then we can define a measure on $\mathfrak{F}_p$ as follows. Said $\mathcal{A}$ a subset of $\mathfrak{F}_p$, we put

$$\mu_{\mathfrak{G}_G}(\mathcal{A}) \overset{\text{def}}{=} \int \dots \int_{\mathcal{A}_\alpha} |\Phi(\alpha_1, \dots, \alpha_q)|\ d\alpha_1 \dots d\alpha_q \tag{6}$$

where $\mathcal{A}_\alpha$ is the bounded set of the parameter space $\mathfrak{D}_q$, corresponding to $\mathcal{A}$ through the (5). Evidently, such a definition depends on the basic isomorphism $\mathfrak{H}_r(\alpha) \cong \mathfrak{G}_G$. Thus, we can now define a *geometric probability* as follows: if $\tilde{\mathcal{A}} \subseteq \mathcal{A}$, then the (*geometric*) *probability* for a manifold $V_p \in \mathcal{A}$ belongs to $\tilde{\mathcal{A}}$, is given by

$$P_{\mathfrak{G}_G}(\tilde{\mathcal{A}}) \overset{\text{def}}{=} \frac{\mu_{\mathfrak{G}_G}(\tilde{\mathcal{A}})}{\mu_{\mathfrak{G}_G}(\mathcal{A})}.$$

Moreover, if ξ is an arbitrary random variable associated to the set $\mathcal{A} \subseteq \mathfrak{F}_p$, then the *h-th geometric moment* of ξ is defined by

$$\mu^h_{\mathfrak{G}_G}(\xi) \overset{\text{def}}{=} \frac{\int \dots \int_{\mathcal{A}_\alpha} \xi^h(\alpha_1, \dots, \alpha_q) |\Phi(\alpha_1, \dots, \alpha_q)|\ d\alpha_1 \dots d\alpha_q}{\int \dots \int_{\mathcal{A}_\alpha} |\Phi(\alpha_1, \dots, \alpha_q)|\ d\alpha_1 \dots d\alpha_q}$$

which, as it is well-known[16], generalize the various notions of mean value (like the arithmetic, harmonic and geometric ones) of the discrete case. From here, it is possible descry a certain geometric background in Statistic, passing through the Integral Geometry and the Klein's Erlangen program.

6. Conclusions

From what has been said above, the various notions so far introduced are strictly depend on the Lie group of transformation $\mathfrak{G}_G$ of the type (2), of which we have considered a possible isomorphic image, namely $\mathfrak{H}_r(\alpha)$. Furthermore, in these discussions, it has also been possible to verify as the basic Chisini invariant relation (1) may be considered as a particular case of the more general invariant relation (4), upon which have been centred the various argumentations that followed. In turn, the latter are all closely related to the action of the given Lie group of transformations $\mathfrak{G}_G$ and its invariants (like (4)), so that, in Statistics and Probability Theory, a more properly geometric framework might also make its appearance via the general philosophy of

[16] See, for instance, the notion of *power mean value* of *index h* for the discrete case in (Girone & Salvemini, 2000, Chapter 6, Section 6.11) which, inter alia, contain, as particular cases, the notions of arithmetic, harmonic and geometric mean. In turn, this power mean value is a particular case (related to the discrete one) of the more general notion of *h*-th *moment* of an arbitrary random variable (see (Dall'Aglio, 1987, Chapter IV, Section IV.3)).

the above mentioned Felix Klein *Erlanger Programm*, if one considers the geometric probability theory as a particular chapter of the wider Integral Geometry context[17].

Finally, from an educational viewpoint, the aim of this brief note might also be interpreted as oriented to develop a more critical sense along the approach and the knowledge analysis of an arbitrary problem or question: for instance, we here have treated a possible case study of this kind, namely a critical essay of the notion of mean value, from an historic-epistemological perspective.

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[17] For brief historical outlines of this fundamental mathematical branch, with related possible applications, see, for instance, (Stoka, 1982) and references therein.