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Bachelard, Enriques and Weyl: comparing some of their ideas

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Abstract. Some aspects of Federigo Enriques mathematical philosophy thought are taken as central reference points for a critical historic-epistemological comparison between it and some of the main aspects of the philosophical thought of other his contemporary thinkers like, Gaston Bachelard and Hermann Weyl. From what will be exposed, it will be also possible to make out possible educational implications of the historic-epistemological approach.

Sunto. In questo lavoro sono considerati alcuni aspetti del pensiero di Federigo Enriques riguardanti la filosofia matematica, criticamente posti a confronto storico-epistemologico con i principali aspetti del pensiero filosofico di altri studiosi suoi contemporanei, come Gaston Bachelard e Hermann Weyl. Da quanto verrà esposto, sarà anche possibile scorgere eventuali risvolti didattici dell'approccio storico-epistemologico.

1. Introduction

Even in modern textbooks and treatises on History of Philosophy and Philosophy of Science, both Italian[1] and foreign, there exist neither a whole chapter nor few sections, devoted to the fundamental epistemological work of Federigo Enriques, whose philosophical thought is dismissed in few lines amongst the subjects related to the modern Italian Philosophy between the end of the 19*th*-Century and the beginning of the 20*th*-one. An exception is made by both some prefaces to the various anastatic reprints of Federigo Enriques works and some remarkable collective and proceeding works mainly edited by the *Centro Studi Federigo Enriques* in Livorno (IT). All that is quite unfair respect to the wide cleverness and acuteness of the forerunner Enriques' thought: he has been remembered only for his high and celebrated contributions to Algebraic Geometry, and only recently a certain further attention has appeared towards this author[2].

An almost identical or similar fate has been undergone by Giovanni Vailati, almost to witness that absurd but real (and still effective) kind of reciprocal dislike that there exist, by both sides, between philosophers and scientists, which embed its historical roots into the secular dispute between *Geisteswissenschaften* on the one hand, and the *Naturwissenschaften* on the other hand[3]. Historically, many renowned scholars have tried to settle such a dispute, but with very poor results, despite of the immemorial historical course of the scientific culture. In Italy then this problematic situation has been (and still is) much more incisive than abroad, above all after the famous strong disagreement which has had as main protagonists the neo-Hegelian idealistic philosophers (amongst whom Giovanni Gentile and Benedetto Croce) against the neo-positivist ones (amongst whom Antonio Aliotta, Roberto Ardigò and Ugo Spirito).

Federigo Enriques (1871-1946) was one of the main Italian scientists, near to the exponents of the logic positivism and of the neo-rationalism, who tried (unfortunately, in vain) to overcome this useless gap between the ones and the others. Just (but not exclusively) on particular aspects of this last wholehearted attempt of reconciliation and the underlying philosophical motivations, it is based this brief note; in

[1] For instance, Ludovico Geymonat, in his celebrated treatise on history of philosophic and scientific thought – see (Geymonat, 1976) – devotes a whole chapter to the life and work of Gaston Bachelard but not to those of Federigo Enriques. This turns out to be even stranger because of the fact that the same Geymonat has been a supporter and a follower of Enriques work, as himself has affirmed. Moreover, this author devote sections of his treatise to Giovanni Vailati and even to Giovanni Gentile and Benedetto Croce (see (Geymonat, 1976, Volume VII, Chapter XI), but not to Enriques, who is mentioned, here and there in few lines, in such a chapter. Likewise for the Nicola Abbagnano (see (Abbagnano, 1993-1995)) treatise.

[2] See, above all, the works of Mario Castellana quoted in References. See also the recent interesting paper (Lolli, 2012).

[3] See (Dalla Chiara & Toraldo di Francia, 1999, Chapter 15, Section 15.1) and (von Weizsäcker, 1994, Chapter V, Section 6.B).

particular, we would like to highlight one of the main implications of Enriques' work, namely that concerning his unappreciated intuitions on the general science education side.

2. Enriques and the Italian philosophy of the time

In what follows, we mainly consider the few lines written by Ludovico Geymonat in (Geymonat, 1976, Volume VII, Chapter XI and Volume VIII, Chapter III) who was one of the main exponents of the Italian neo-positivistic current of the 20*th*-Century, on the wake left by his predecessors, amongst whom the same Enriques.

The main fact that immediately jumps out to the attention is related to the known, difficult relationships between the Italian philosophy and the scientific context of the time[4], whose main causes should be ascribed to the Croce and Gentile[5] neo-idealism. Nevertheless, it wouldn't be historically correct neglect certain other antecedent facts which will concur to exacerbate such problematic relationships. Indeed, some of the main exponents of the same Italian neo-positivism of the end of 19*th*-Century, amongst whom Roberto Ardigò, did not give the right relevance that will deserve the fundamental epistemological works made, for instance, by Vailati, Peano and Beltrami as concern the foundations of mathematics. Amongst them, above all Giovanni Vailati (1863-1909) tried to stem this incipient breaking between the Italian philosophers and the scientific thought, having as reference point the recent work and thought manifested by Federigo Enriques since the last years of the 19*th*-Century, who, after the premature death of Vailati, continued himself, in first person, to bring forward this program of reciprocal collaboration.

Nevertheless, it is historically well-known as both valuable aims, not only intentionally manifested but also put in practice with remarkable factual works[6], failed or were ignored, with the consequent eclipsing of the anti-idealistic philosophical trends and with all the consequent harmful results still today present into the Italian cultural setting. Almost like a sort of unfair retaliation of the destiny against these benevolent reconciliation and collaborative attempts, to confirmation of the Saint Bernard of Clairvaux maxim according to which *«the good intentions pave the hell's roads»*, Enriques and Vailati were almost neglected by the same Italian culture as regard their philosophical works; only abroad they received major attention, again to confirmation of another Latin maxim according to which *«nemo propheta in patria est*[7]*»*.

Nevertheless, the appreciated collaborative and mentally-open perspectives of Enriques, had remarkable parallel attempts in some foreign notable thinkers, among whom Gaston Bachelard (1884-1962) and

[4] For brief outlines concerning the relationships between Mathematics and Philosophy in the beginning of the 20*th*-Century, see also (Berzolari, 1978, Article LXI, Section 4), where, amongst other, there is a rich related literature.

[5] Just in regards to Giovanni Gentile, it is notable to recall as his son Giovannino Gentile Jr. (1906-1942) was a great physicist prematurely died. For some brief biobibliographical notes on him, see (Bernardini & Bonolis, 2002) and (Bernardini, 2007) (see also the Preface of Gilberto Bernardini to (Bernardini et al., 1947) in which there are further interesting historical remarks), from which emerges that, after all, the same Giovanni Gentile senior wasn't so adverse to the scientific knowledge as could seem at a first sight; indeed, he left full freedom to the studies chosen by his son, even eulogizing, also publicly (see (Gentile, 1941)), the natural sciences and their Galileian experimental method. Instead, it was above all Croce the main opponent of the scientific knowledge, very likely to counteract a possible advent of the neo-positivistic thought mainly headed by the so-called *Vienna Circle* (but also by the *Berliner Gesellschaft für Wissenschaftliche Philosophie* of H. Reichenbach, near to the former) to whom Enriques was into contact, strong of position conquered by him within the Italian Philosophical Society. On the other hand, after the death of his son, Gentile senior published too a book entitled *Scritti minori (di scienza, filosofia e letteratura)* which collect all the publications of his son, and from which, besides, clearly emerges an extraordinary eclecticism of Gentile junior quite similar to that of Enriques, even in the undergone fait. Indeed, the same Gentile junior was also discriminated, both by scientists and humanists, for his attempts to unify the humanistic disciplines with the scientific ones; only Giovanni Polvani and Ettore Majorana were estimators of his singular work, Ettore Majorana having also been a his strict friend (which is quite strange seen his character).

[6] In particular, the basic works *I problemi della scienza* (1906) and *Scienza e razionalismo* (1912), despite had been criticized first by Gentile then, above all, by Croce, earned to Enriques, for some years, the presidency of the same Italian Philosophical Society. The journal *Scientia – Rivista di sintesi scientifica*, founded in 1907, was the result of the great Enriques' foresight and established just as a place of meeting and cultural exchange between philosophers and scientists.

[7] And this maxim reached its highest achievement just relatively to the fate of the journal *Scientia*, whose initial programmatic manifesto was formed by the celebrated book *I problemi della scienza* (1906).

Hermann Weyl (1855-1955), of whom herein we wish to point out certain common aspects of their thought[8], putting them in critical comparison with the Enriques ones.

3. Enriques, Bachelard and Weyl: some comparative attempts

As already said above and as recalled by Geymonat in (Geymonat, 1976, Volume VIII, Chapter III), the causes of the failure of Enriques philosophical program (with the consequent neglect of the related thought) must not be imputed only to his controversy with Croce and Gentile, albeit it played a pivotal role, but also to the same mathematical community of the time. In fact, the latter was regulated by an its own 'internal behavioural deontological codex' according to which was seen with extreme diffidence every attempt turned toward historical, philosophical and foundational (above all logical) questions. It is then very strange as such a *mental crease* was long present, as a kind of *internal pure idealism*, into the mathematical sciences which were mainly understood – also nowadays – as totally detached from any type of problematic which did not be of a purely theoretical nature or, at most, technical-applicative. After an initial good consideration, Enriques was yet soon isolated both by the Italian philosophical society and (mainly for these his first interests) by the same mathematical community which was completely unrelated to these type of studies, a human fate this which was also experienced by Bachelard (see (Geymonat, 1976, Chapter X, Section II)).

Enriques was just one of the few ones to try changing this unilateral perspective within the Italian context. In this framework, we want to consider what was the same situation abroad, limiting ourselves to few authors. In France, the general dislike towards the Logic found an influential supporter in Poincaré that opposed the initiative of the logician Louis Couturat to introduce the Peano and Russell ideas in his country, notwithstanding Poincaré himself was one of the greatest French scholar of Epistemology and Philosophy of Science, together to P. Duhem, the latter moreover a strenuous opponent of the Logicism. Both did themselves paladins of an *antidogmatic* conception of the science, involving a certain convenctionalism; their ideas were thereafter retaken by their successors, amongst whom L. Brunscvicg, É. Meyerson, A. Rey and A. Koyré, till to Gaston Bachelard who is considered as the most original thinker in this type of studies. For shortness, we refer also to (Abbagnano, 1995, Chapter XI, Section 799) for the exposition of the main outlines of Bachelard thought.

With a *license* in Mathematics and Philosophy, but first of all historian of science, Bachelard have drawn inspiration sources for his further epistemological reflection from his scientific researches. His conceptions are nevertheless different from the neo-positivistic ones for a major historicization of the scientific thought, this last being also seen from the various historical, technical, social, cultural and psychological standpoints in which it has evolved. He considers philosophy and science as inseparably connected among them; furthermore, according to him, it does not exist only one science but different sciences or an irreducible plurality of knowledge[9] and specific techniques, speaking of an *applied rationalism* which is very close to the Enriques *experimental* and *critical rationalisms* (see (Redondi, 1978, Chapter V, Footnote [43]) and (Castellana, 1974)). Both these philosophical trends were substantially motivated by their common interests for physical questions (and connected relationships with mathematics) which, among other things, have also been as valid and useful educational tool for exact sciences (see (Castelnuovo, 1907)), in particular for mathematics itself. Furthermore, both Enriques and Bachelard were quite adverse to the idealistic theses notwithstanding they always tried to classify and to compare their studies in the more general framework of the great philosophical systems[10].

Nevertheless, this closeness between their rationalisms does not completely extend to their respective conception and role played by the history of science, that in Enriques coincides with the history of philosophy[11] and goes on from past to future in a continuous manner (see (Enriques, 1938)), whereas in Bacherlard, though science and philosophy are inseparably connected between them (like in Enriques), nevertheless the history of science is guided only by the current rational values and only minimally is

[8] For other aspects, we refer to (Redondi, 1978).

[9] In this regards, see the epistemological and multidimensional perspectives of conceptual changes in science education context, as for instance outlined by (Treagust & Duit, 2009), which, amongst other things, just reminds the Bachelard's epistemological profile.

[10] For instance, Bachelard, in his philosophical conception of epistemology and science, outlines a sort of *philosophical topology* in which to place the various historical philosophical systems respect to which comparing the same historical evolution of science (see (Geymonat, 1976, Volume VII, Chapter X, Section V)). In this, Bachelard and Enriques are very tight.

[11] Almost to paradoxically border on the Crocian historical conception of knowledge.

influenced by the past because of 'discontinuities' due to the occurrence of certain breakings[12] (see later). Instead, a common point in their conception of the history of science methodology is findable in certain psychologistic tendencies of both authors: for instance, Enriques, in (Enriques, 1938), states that the study of the historic-phenomenological evolution of scientific ideas may turn out to be useful for understanding the *genesis* of the same scientific ideas, from which emerges the necessary inseparable copresence both of rational and empirical factor in the birth and development of it (see (Geymonat, 1976, Volume VIII, Chapter III, Section II)). This last perspective is also considered – hence, again in agreement with Enriques – both by Bachelard[13] (see (Geymonat, 1976, Volume VII, Chapter, X, Section III) and by Weyl in (Weyl, 1949, Chapter 5, Section 21) where the he argues on the formation of scientific theories, just reporting the Enriques conception of the continuous epistemic role played by the history of science in the formation of itself. A fundamental tool to pursue this common Bachelard, Enriques and Weyl standpoint, is just the history of science intended not as an erudite research but as a dynamic and active research of the scientific spirit, considered along its diachronic and synchronic development[14].

According to Bachelard, the scientific progress, instead, does not take place through a continuous and unilateral process but through *epistemological breakings* respect to the previous theoretical schemes, which, in turn, may take place only overcoming the various *epistemological obstacles*[15] that hinder the science path. Nevertheless, Bachelard inherited some of the main themes common to his predecessors, above all Poincaré and Duhem, like the aversion to the logic, the antiempiricism (differently by Enriques), the tendency to link the criticism of science with its history, the essential original creative nature of the theories, and so on. In particular, his contrariness to the logicism and formalism of Peano, Russell and Hilbert, puts him on a same level respect to Enriques who was notoriously into very cold relationships with Peano as noted by Mario Castellana in (Castellana, 1973) – who, inter alia, has also made fundamental epistemological studies just on the authors here considered: see (Castellana, 2004), (Castellana, 2005) and (Castellana, 2010), in which a deeper comparative historic-epistemological analysis of these authors is made.

Contrarily to his teacher Brunschvicg who considers the mathematics as a simple linguistic tool, Bachelard claims as the mathematics is the *pillar of discovery* that creates the modern physical science, in opposition to the so-called 'doctrinaires of axiomatic' like Hilbert. According to Bachelard (see (Geymonat, 1976, Volume VII, Chapter X, Section III)), every formal thought is an incomplete psychological exemplification since it is a kind of never reached limit-thought, or else it is a thought around a some subject; it concerns hidden images, which will be auxiliary to build up the related formal framework. The mathematics of the new physics is fed by its own experimental applications, whereas the science, in its educational aspects (to whom Bachelard devotes much attention), cannot be exposed in its direct axiomatic form (against the later Bourbakism[16]) but it should first be exposed for being understood, upon which thereafter building up its rigorous theoretical framework. To this purpose, according to Bachelard, the mathematics should be taught with an applicative method oriented toward the sciences, like physics and chemistry, hence together these. From all that, it is evident the common points with the related Enriques thought, which was notoriously opposed to any form of strict and curt formalism, as well as favourable to this educational way of teaching.

On the other hand, the Enriques' dislike to logicism is clearly identifiable in someone of his fundamental works on Algebraic Geometry: indeed, taking into account the introduction[17] of Guido Castelnuovo – who was one of the closer collaborator of Enriques, and himself a clever mathematician – to the posthumous publication of the first 1942 edition of the basic work (Enriques, 1949), it is possible to glimpse what practical conception of the mathematics had Enriques. From that, Castelnuovo also expresses an his own worry as concerns the new course undertaken by the mathematics at the beginnings of the 20*th*-Century, which was quite different from the intuitive and imaginative one characterizing the very fruitful and advantageous 19*th*-Century mathematical thought. On the basis of Enriques work, Castelnuovo argues on the

[12] In this sense, anticipating the Thomas S. Kuhn thought about scientific revolutions.
[13] Under a certain Husserlian philosophy influence.
[14] Albeit these authors give a different weight just to these two basic aspects of the historical evolution: for instance, Enriques gives much more importance to the diachronic aspects, whereas Bachelard give more attention to the synchronic ones.
[15] Besides, to explain their occurrence, Bachelard appeal, inter alia, to the Freudian and Jungian psychoanalytic theories as well as to the Husserlian phenomenology. The epistemological obstacle theory led thereafter Bachelard to his *Philosophy of No*, which will have fruitful implications from an educational viewpoint (see (Treagust & Duit, 2008)).
[16] On interesting historic-epistemological remarks on Bourbakism in mathematics, see (Israel, 1977).
[17] See (Enriques, 1949, pp. V-VIII).

new way of doing mathematics in the first half of the 20*th*-Century, more oriented toward the technical and logical aspects rather than sight, at first the general framework of the mathematical question upon which then formally building up the theory. This is compared too with an analogous situation which was taking place in the artistic context of the time: even there, the imagination and fantasy were dismissed and pejoratively considered as arising from the romantic era, giving instead more consideration to the technical and tool aspects. From all that, it is evident why rightly Enriques did not appreciate the logicistic way of doing mathematics that has gradually taken place ever more. Enriques tried to find confirmations to his way of seeing mathematics into the philosophical context, reaching to very original and interesting, innovative explanatory modes concerning a mathematical reasoning, with possible pedagogical insights.

The intuitive and imagination view in mathematics, was that mainly adopted and thereafter carried out by the great mathematicians of the 19*th*-Century, like Gauss, Riemann, Abel, Jacoby, Poincaré and others. To clearer make the idea of that, we herein report the exact Castelnuovo textual words, which we surely may consider reflecting what Enriques himself believed in this regards[18]

> *«La fantasia, la intuizione che guidavano la ricerca di allora sono oggi guardate con sospetto per il terrore degli errori a cui possono condurre. Le teorie sorgevano per rispondere al bisogno che il matematico provava di delineare e precisare degli oggetti del pensiero che erano già, in forma vaga, presenti alla sua mente. Era l'esplorazione di un ampio territorio intravisto da una cima lontana. Si costruirono così nel secolo scorso quei gioielli che si chiamano teoria delle funzioni analitiche, delle funzioni ellittiche, abeliane, superficie ad area minima, superficie cubiche…. Oggi più che il terreno da esplorare interessa la via che vi conduce, e questa via ora vien seminata di ostacoli artificiali, ora si libra tra le nuvole».*
>
> [*«The fantasy and the intuition which driven the research of then, are nowadays seen with suspicious due to the fear to make errors to which them may lead. The theories born for answering to the mathematician's need for outlining and specify the objects of her/his thought which were already preformed in her/his mind but into a vague form. It was like the exploration of a wide land sighted from a far peak. So, those jewels named analytic function theory, elliptic and Abelian functions, cubic and minimal area surfaces, and so on, arose from this way of doing mathematics in the last century. Today, rather than the landscape to explore, there is more interest to the formal way which leads to it, and this path is either sowed by artificial obstacles or hovers around the clouds»*.]

As Poincaré himself said in his celebrated work (Poincaré, 1905), a mathematical construction is necessarily composed first by an *intuitive* process, which discovers, then by a *logic* process, which proves, coherently with what Enriques says just above through the Castelnuovo report. Hence, Enriques philosophy of mathematical thought is mainly based first on intuition, and this is a common perspective to almost all the celebrated exponents of the Italian algebraic geometry school of the time (among to which E. Beltrami, L. Cremona, F. Severi, E. Castelnuovo, C. Segre, B. Segre, G. Veronese, G. Fano and others). It is also in accordance with the Poincaré thought[19] as well as with that of Riemann whose geometric standpoint was, amongst other, one of the main common point of the thought of Bachelard, Enriques and Weyl, as witnessed by (Castellana, 2004) and[20] (Redondi, 1978). Furthermore, Enriques, Bachelard and Poincaré were also joined amongst them by the common, constant doing reference to the psychological sciences, but not in a reductive way. In this regards, the work of Enriques was abundant of suggestions for the subsequent works of Jean Piaget and Pierre Gonseth[21] in epistemology, while the Poincaré philosophical legacy will be, for instance, later retaken by J. Hadamard in his celebrated work (Hadamard, 1945).

[18] The Enriques' considerations are presently very true.

[19] It is known as, after Poincaré, in France the logicism and formalism trends attained their highest height with the *Bourbakism* which has been the prevailing educational address until few years ago. In this regards, Vladimir I. Arnold, which may be considered as a great intuitive mathematician, was very critical on this, trying to reintroduce many mathematical textbooks oriented towards the intuitive and imaginary way of doing mathematics (from a PhD Seminar lesson held by Prof. Giorgio Bolondi). For a criticism against the Bourbakism trend from an educational viewpoint, see what says F.G. Tricomi in (Tricomi, 1967), which, besides, was also in a certain opposition to the Turin Peano's school. See also (Israel, 1977).

[20] To which we refer for a more careful study.

[21] See Sections 1 and 2 of the introductory survey by O. Pompeo Faracovi to the Italian edition of (Enriques, 1938). Moreover, about the relationships among Enriques, Bachelard and Gonseth, see above all (Castellana, 2005).

From that, it is easy to find interesting historical connections between the Enriques ideas and the work of another as much great mathematician, Hermann Weyl. Both authors were two among the greatest mathematicians of history whose work on pure and applied mathematics allowed them to be able to understand the various aspects of a mathematical reasoning, so that their philosophy mathematics thought should be taken into great account. As regards Weyl's thought on the nature of mathematical reasoning, it is enough to recall[22] as, according to him, in the edification of a mathematical theory, the general starting point is represented by what he calls an *operative framework* (*Operationsbereich*), formed by the choice of a number of fundamental categories of entities respect to which are given certain properties and relations, from which afterwards to go on for building up the whole theoretical system, through the creative iterative application of certain *generative* processes which include two main types, a *logic* process and a *mathematical* one. The former generate new properties and relations (said *derived*), starting from an initial stock of primitive relations and properties related to the entities of certain *initial* categories, applying the common usual elementary logical operations[23]; the latter, instead, allows to constitute new *ideal* entities from a given system of properties and relations related to certain entities already known, identifying hence a class of entities including only those having such properties. Subsequently, Weyl himself, in his celebrated work (Weyl, 1949, Chapter 1), represents this same distinction between logic and mathematical process by means of the distinction between *combinatorial* and *creative* definition, the combinatorial one being legitimated by the logic process, whereas the creative one is that legitimated by the mathematical process. The creative iteration of these two inseparable processes lead to the notions of *types* and *orders*, through the so-called *expanded* and *limited* processes.

The Weyl mathematical philosophy thought, relatively to the properly mathematical context, has evolved in time from the first work *Das Kontinuum* (1918) to the final *Philosophy of Mathematics and Natural Science* (1949) which is a revised and enlarged edition of a first German 1926 paper published in the *Handbuch der Philosophie*, and that recently it has been republished in a new 2009 edition with an introduction by the Physics Nobel laureate A.F. Wilczek regarding the parts more properly physical of this crucial Weylian work. We here do not wish to discuss the Weyl philosophical positions and their evolution, but rather point out only few of their aspects which may be quite close to the Enriques ones. First, both thinkers belonged to the very restrict class of pure scientists which could not do without to consider also the philosophical questions inherent a given mathematical or scientific problem: out of these, we remember Poincaré, Riemann, Einstein, Eddington, Mach, Russell and few other scientists of 19*th*- and 20*th*-Century; unfortunately, this is a valuable cultural tradition that will go ever more to disappear[24]. In particular, Weyl himself, in the *Preface* to (Weyl, 1949), states as it has been no possible to him leave aside from philosophical questions each time that the opportunity will arose, ever trying to put the given mathematical or physical question into comparison with the suitable known philosophical frameworks. On the other hand, it is never enough the importance given to the philosophical thought in motivating and stimulating the same mathematical or scientific production, in this case being sufficient to recall the Riemann[25] and Einstein idea history. Furthermore, from every part of his book, it gives rise the Weylian idea according to which there is almost always a prevalence of the *imaginative* components for the occurrence of a mathematical insight[26] (either it concerns a proof or the institution of a new formal object).

On the other hand, quite recently, it have seen to appear some interesting researches about certain relationships between Sigmund Freud and Ludwig Wittgenstein ideas: amongst them, we remember only some papers of A.G. Gargani (see (Gargani, 2005); see also (Gargani, 1982) and (Pagnini, 2009)) in which, inter alia, the author wants bringing together the psychoanalysis with the analytical philosophy – above all in the Wittgensteinian sense – on the one hand and the methods of constructivist knowledge on the other hand, recalling into question just the mathematical constructivism of Weyl, Brouwer and of the same Wittgenstein. From here, a rather indirect link between the last Weylian mathematical philosophy thought and the psychological science, like in the Enriques work, it is possible to descry.

[22] In what follows, we refer to (Casari, 1972, Chapter XIII, Section 1).
[23] And therefore characterized by a low degree of creativity, differently from the mathematical one.
[24] Today being almost inexistent.
[25] In particular, as regards Riemann, it is enough to remember the basic notable influence exerted on his scientific production by the thought of the anti-idealist German philosopher J.F. Herbart (1776-1841) and by that of G.Th. Fechner (1801-1887) (see the *Introduction* by R. Pettoello to (Riemann, 1994)).
[26] This being in accordance with the recent research results on mathematical thought, according to which it is strictly connected with visual-spatial skills. All this is of fundamental importance from an educational viewpoint.

Moreover, taking also into account the physical science[27], in the work (Weyl, 1932) the author explains which should be the so-called *essence* of the new scientific mind turned towards the contemplation of a pluralistic and dynamic *open world* put into not aggressive but sympathetic relationships with the religious spirit. In this regards, Weyl devotes the first chapter of his work, entitled *God and the Universe*, to discuss just these last aspects, trying to justify the apparent contrasts which can arise if this argument is carried out by a mathematician. On the other hand, this last type of extreme and romantic philosophical digressions weren't estrange to the same Enriques which, in this regards, so he expresses himself at the end of his celebrated work (Enriques, 1949, Chapter XI, Section 9)

> *«A questo punto ci sia consentito fermarci un istante, come in un'ascensione alpina si ama sostare sul picco conseguito e di là contemplare lo spettacolo della Natura che si offre alla vista.*
> *Cinquant'anni or sono s'iniziava in Italia lo studio di queste teorie* [delle superfici algebriche], *appena abbozzate dal genio di un precursore (Max Noether); allora, scherzando sulle difficoltà e le eccezioni che s'incontravano da ogni parte, si soleva dire che, mentre le curve algebriche (già composte in una teoria armonica) sono create da Dio, le superficie invece sono opera del Demonio*[28].
> *Ora si palesa invece che piacque a Dio di creare per le superficie un ordine di armonie più riposte ove rifulge una meravigliosa bellezza, e ch'Ei volle in esse – diciamo col Poeta –*
>
> *del creator suo spirito,*
> *più vasta orma stampar.*
>
> *La ricchezza delle proprietà e la bellezza, lungamente nascosta, che qui si palesano, non debbono costituire ragione di vano orgoglio per la scuola geometrica italiana o per i geometri stranieri che hanno collaborato a scoprirle, ma piuttosto debbono suscitare un senso di reverenza per quell'ordine meraviglioso degli enti matematici, che il pensiero trova innanzi a sé e quasi raccoglie, al pari delle specie viventi, dalla Natura Madre; e così alimentare la fede dei giovani ricercatori che dietro alle difficoltà, alle eccezioni, alle apparenti incongruenze, c'è realmente in questo mondo di enti, una divina armonia, che gli sforzi concordi degli studiosi riusciranno sempre meglio a mettere in luce».*
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> [*«To this point, there be allowed us stop for an instant, like when in an alpine climbing, it is loved to have a break in the achieved peak and admire the Nature spectacle which is offered to our own eyes. Fifty years ago, in Italy began the study of these theories* [that is to say, those of algebraic surfaces] *just sketched by the geniality of a precursor (Max Noether); then, joking on the difficulties and the exceptions met in every its part, it was customary to say that, whilst the algebraic curves (already systemized into an harmonic theory) were made by God, the surfaces were conversely due to Devil. Now, instead, it has disclosed that pleased to God to create for surfaces an order of more secret harmonies, from which shines a wonderful beauty, and that, into them, He wanted – saying, with the poet –*
>
> *of the creator's soul,*
> *the wider trace to imprint.*
>
> *The property richness and their long hidden beauty, which here are manifested, shouldn't be reason of vain pride for the Italian geometric school or for the foreign geometers who have concurred to discover them, but rather should arouse a reverential sense for that beautiful order of the mathematical object realm that the thought finds before itself and almost accepts, like a living specie, from Nature Mother. And this, in such a way to nourish the faith of young researchers since, behind the difficulties, the exceptions and the apparent inconsistencies, in the realm of such entities really*

[27] And the constant and repeated attention to these, also provides a further common point of the Weyl and Enriques thought.
[28] In this regards, it circulates too another similar maxim but concerning the integer and complex numbers, in part included in that due to L. Kronecker according to which *«the integer numbers are due to the God's action, everything else being due to the human's one»*. The further addendum according to which yet *«the complex numbers are due to the Devil's action»*, seems instead to be anonymous.

there exists a divine harmony that the agreed attempts of the various scholars ever better will be able to put in light».]

The treatise (Enriques, 1949), that Castelnuovo himself remembers to be one of the most important work of Enriques devoted to Algebraic Geometry, was prepared just after he was graduated from Scuola Normale Superiore of Pisa, and subsequently underwent to continuous remaking and revision till to the last years of his life. However, beyond the remarkable geometrical insights, one of the main features of the whole book is just the intuitive and imaginative method with which are treated the geometrical questions therein introduced, though these led himself to undergo various critical essays as regard the proof correctness of certain theorems, notwithstanding the importance of the achieved results[29]. This peculiar way of doing mathematics is characteristic of that unique kind of *scientific-humanistic* trend which Enriques wanted to pursue and that was partially retaken and kept alive by very few of some of his pupils, amongst whom Geymonat himself, A. Frajese (1902-1986), L. Campedelli (1903-1978) and L. Lombardo-Radice (1916-1982). The latter, in his preface to the anastatic reprint of (Enriques, 1938), remembers some of these distinguishing Enriques features, first of all his attempts to overcome the reductive *barrier*, or *fence*, between the *Geisteswissenschaften* and *Naturwissenschaften*, gap, this, which was inexistent in him since the beginning of his juvenile studies. The link between philosophical and exact sciences was of an indissoluble and mutual character in Enriques training, who was hostile to any form of extreme specialization, nourishing a sense of *circular unity* of knowledge, and mastering a great quantity of cognitions in many fields of knowledge but without never becoming a specialist (with an exception for Algebraic Geometry). Maybe, just due to this, he undergone the unhappy fate of the beaten and lonely scientist, like Bachelard, even if such a condition did not weight on his spiritual serenity that characterized almost the whole of his life.

In short, there have been notable scientists, like Weyl and Enriques (and, in part, also Bachelard if one takes into account his curriculum vitæ and studiorum), whose scientific work couldn't be disjoined by the philosophical speculation: for them, it is valid what Weyl himself says, namely that there exist men, like artists, scientists, technologists or politicians, which devote themselves to the construction, whereas others devote themselves to the reflection and to the philosophical speculation. These two types of attitudes should actively integrate among them, otherwise the creativity loses itself into the mechanicalness of pure routine, while the reflection becomes abstract and void natter. Another educational-methodological lesson!

4. Conclusions

The intuitive and imaginative manner to approach, in a first phase, an arbitrary mathematical question, as understood above all by Enriques and Weyl, but also by Bachelard on the wake left by Poincaré via Brunschvicg (see (Geymonat, 1976, Volume VII, Chapter X, Section III)), might have non-negligible implications from an educational perspectives if one considers the mathematics like an *immanent order of the Nature* or an *intelligible reality* external to our mind (like Plato), so re-evoking the medieval controversy between realists and nominalists. In this regards, according to Enriques and Castelnuovo (see his basic, but little known, paper (Castelnuovo, 1907)), the methodology of Physics might play a fundamental educational role also from a mathematical viewpoint, above all in Geometry[30].

They surely are *visual* mathematicians rather than *abstract* ones, above all Enriques that applied this mathematical philosophy to the active geometrical research field of the time, reaching to unique and valuable results: the work (Enriques, 1949) is considered as a valuable source of mathematical ideas as concerns the algebraic geometry of surfaces, although it were found some proofs little correct from a pure formal viewpoint and that the same author tried to remedy with a continuous revision of his work, but without substantial changes in its remarkable content of ideas. Castelnuovo himself, in the introduction to (Enriques, 1949), affirms that Enriques was forced to improve his work because of certain critical essays moved by formalists to the proofs of some his theorems. On the other hand, it was well-known, and Castelnuovo and

[29] Nevertheless, only recently he has been, in a certain sense, 'rehabilitated' from these last criticisms, because it has been ascertained, a posteriori, the formal correctness of his proofs.

[30] As regards the experimental character of mathematics, see also the brief but important note of Jean Leray in (Hamburger, 1986).

Enriques themselves were aware of this, in what state was the theory of algebraic surface at that time, hoping in a future improvement of it, from a formal viewpoint[31].

In short, from what has been said so far, it clearly emerges that almost every creative mathematical process necessarily, at first, should take place by means of an intuitive and imaginative approach which could thereafter be corrected or improved by a subsequent formal or abstract revision phase which, in turn, might to provide further results susceptible of possible physical interpretations (like the discovery of antimatter expected by the physical interpretation of the eigenvalue problem solutions of electron Dirac's relativistic equation deduced from a formal relativistic extension of the Schrödinger equation). In particular, the main theses on the real nature of mathematics by Enriques, come just from the geometric context, that is to say, the Geometry, as say, is one of the main *epistemological paradigms* of a neo-Platonic conception of mathematics. Among the contemporary thinkers who agree with such an Enriques' view, we recall H. Freudenthal – that, among other things, has tried to apply this program to the educational context[32] – and R. Thom (who acquired the related Poincaré's thought legacy). All this is coherent (and prodromal) with the modern cognitive science researches according to which, as already said, at the basis of the mathematical thought there are above all visual-spatial skills. In any way, nowadays it is almost inexistent this type of reciprocal useful and fruitful relationships between Mathematics, History and Philosophy which might also turn to be useful from an educational standpoint[33].

In conclusion, from Poincaré, on the French side, and from Riemann, Klein and von Helmholtz, on the German side, Enriques pick up these influences, coeval respectively to the Bachelard and Weyl ideas, to originally develop his thought towards an intuitive view of mathematics – and applying it to his pioneering geometrical researches. His work and programmes will reach the highest values with the L.E.J. Brouwer *intuitionism*[34], does not never neglect the related philosophical counterpart. And this has been just the *leitmotiv* that has led to the drawing up this brief note, ever bearing in mind the claim according to which, in our simple opinion, the historical considerations (as those so far done) might have some educational implications both for natural sciences and mathematics[35].

Remarks. In this paper, we have limited ourselves to point out those points of Bachelard, Enriques and Weyl thought which overall lead, amongst other, to the revaluation of that line of thought referring to the visual and intuitive conception of mathematics dating back to Plato and Socrates. Between Bachelard and Enriques,

[31] Which, besides, couldn't take place without these initial results. However, this intuitive way of working was common among the above mentioned exponents of the so-called Italian geometric school (in part, following that of the German tradition dating back to Riemann, Klein and von Helmholtz), which yet attained to original and remarkable results in the geometric field. Only subsequently many other mathematicians, above all not Italians, improved their results from a formal and abstract viewpoint, often arguing (although unjustly) against such a School, in particular toward Enriques and Severi. Among them, it is no possible to omit the name of Oscar Zariski, who scientifically growth just within such a celebrated school and into its stimulating context.

[32] Furthermore, the names of H. von Helmholtz and H. Freudenthal are also historically related to some important problems concerning the axiomatic characterization of the so-called *Physical Geometry*, an important field of studies linking together basic physical questions (also correlated to General Relativity) and formal geometrical arguments (like the *Riemann-Helmholtz-Lie* and *Yamabe* problems), which has a main study subject the so-called *problem of the space*. It derives from some of those multiple intersections between Physics and Geometry, whose program is builds up along the lines traced by the works made by Riemann, Poincaré, Helmholtz, Klein (in this regards, of this author see above all (Klein, 1926-27)) and Enriques, on these arguments (for more information, see (Schmidt, 1979), (Freudenthal, 1965) and (Moore, 1919)). On the other hand, Weyl himself has had also to do with such questions, as proves his fundamental work (Weyl, 1923), so that, via Chapter IV (related to Geometry) of the 1906 Enriques work *I problemi della Scienza*, it is possible *en passant* to identify another common point which goes from Helmholtz, Riemann and Lie till to Weyl, in considering and treating this ''problem of the space'' (that, besides, will deserve a more historical attention).

[33] Bruno D'Amore, in (D'Amore, 2001, pp.75-76), highlight just these possible perspectives as arising from a constructive cooperation between historical questions and educational programs, but remembering too as so far nobody has put into practice this program, despite of distinguished historical attempts dating back just to Enriques, Campedelli and Lombardo-Radice. For instance, that valuable tendency to insert historical notes at the end of the various chapters of scientific textbooks and treatises, it's losing by this time. Nevertheless, on the epistemological side, a modern exception is given by the notable work of the theoretical physicists and science philosopher Carl Friedrich von Weizsäcker (1912-2007), among whom related works we mention, for our purposes, only (von Weizsäcker, 1994).

[34] In this regards, see also (Fieschi, 1976, Volume II, Appendix B, Section III.D).

[35] For instance, the Bachelard's epistemological profile is considered in the conceptual change theory of science education: in this regards, see (Treagust & Duit, 2008, pp. 312-313).

via Poincaré, we have tried to identify some common points more oriented toward the philosophy and history of science than toward the relationships between mathematics and physics, which besides are also present. Instead, as regard Enriques and Weyl, we have put more attention to these last types of basic relationships, as well as the relationships between philosophy and science, even if the historical questions are strongly present more in the Enriques thought than in the Weyl one. For other as much interesting common points among the mathematical philosophy thought of these authors, we refer to (Redondi, 1978).

In any way, as repeatedly said, we want to stress out what fundamental educational role may play the philosophical and epistemological thought in science, the works of Enriques and Weyl, as well as the Bachelard one, being enough to prove this. Besides, all this has already been largely witnessed by some recent science education researches (see (Treagust & Duit, 2008)).

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[36] See also the new 2009 edition still published by Princeton University Press, with the contributions of F.A. Wilczek.