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Interactive tools to support Linear Algebra students: GeoUniud

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Abstract. GeoUniud è una piattaforma intuitiva integrata attraverso tutor interattivi. Permette agli studenti di indagare su specifiche attività selezionando i loro input iniziali e risolvendo gradualmente un problema. In ogni fase viene fornito un insieme di feedback immediato. In questo articolo verranno descritti solo alcuni strumenti sulle applicazioni lineari e gli autovettori per migliorare il *sense-making* degli studenti in un ambiente di geometria dinamica nella prospettiva della teoria della mediazione semiotica.

1. Introduction

Nowadays it is an obvious remark to point out that young generation finds most of its basic notions (useful for its moral, scientific, and professional education) from the web. But the average quality of the information we can extract from the web depends a lot on our degree of awareness. This is also true in the case of scientific or educational platforms. This vicious circle can be broken and rearranged in a virtuous one by those educational centres, as School and Universities, which accept the challenge, and become able to create their own platforms. We think that a new frontier for school education will consist of the ability to create flexible school platforms suitable to arrange topics according to teachers' goals, and student needs. We think that the ready availability of interactive platforms has produced a new generation of students able to utilize computer-based learning tools with ease and comfort. University must respond to the challenges of society and the world of work, increase the quality level of graduates, accommodate an increasing number of students. It must integrate research into teaching and respond to the needs of the territory to which it belongs, keeping the spirit of international development unchanged. All these difficult tasks of the university today flow into didactic actions, which require innovation in terms of contents, methods, and tools. But integration of technology into existing pedagogy requires careful thought as to the redesign of classroom instruction and technology tools should "serve as intellectual partners during activities requiring problem solving or critical thinking" (Ertmer & Ottenbreit-Leftwich, 2013, p. 176). Effective teaching with technology demands specialist knowledge and specific learning skills that require appropriate training and professional development (Bowers & Doerr, 2001; Albano & Ferrari 2008; Thomas & Chinnappan, 2008). Artigue (2002) attributes some of the lack of success of technology innovations to the fact that "the education system does not easily recognise this fact and has little taste for dedicating the necessary time and energy to this learning" (p. 11). Thomas and Chinnappan (2008) suggest that teachers require time and assistance to develop pedagogical technology knowledge. Domains of complex knowledge such as mathematics require the acquisition of pieces of knowledge organized in the form of a system, connected to each other, even in complex ways, with any prerequisite constraints; in addition, their use depends on methods that are not attributable to mere algorithms and didactic strategies. For such cases, it is unthinkable to imagine that the achievement of knowledge is the spontaneous result of a construction process borne by the student; on the contrary, a fine planning of didactic courses that can favour such construction is necessary (Balacheff, 2000; Ferrari, 2011). In particular, the structure of learning environments, the possibilities of interaction, immediate feedback and assessment promote the development of metacognition and self-regulation, which are at the core of learning processes (Persico and Steffens, 2017). Recent research on inquiry-based experimental approach for mathematics pedagogy forefronts the use of tools, digital technology, as an epistemic medium (Baccaglini-Frank et al., 2017). Designing suitable pedagogical tasks is a key to implement inquiry-based learning and the use of tools is a major epistemic component in carrying out the systematic inquiry-based process. Student inquiry serves two primary functions: it enables students to learn new mathematics through engagement in genuine exploration and argumentation, and it serves to empower learners to see themselves as capable of reinventing important mathematical ideas.

The aim of this paper is to describe interactive tools about linear transformation and its matrix representation and about eigenvectors in order to develop students' sense-making within the perspective of semiotic mediation (for other examples see Lepellere et al., 2020a, 2020b). These tools are part of a project, GeoUniud promoted by the Department of Mathematics, Computer Science and Physics of the University of Udine with the aim to provide university teachers of Linear Algebra innovative contents. The paper is organized as follows: Section 2 presents the literature about some student difficulty in linear algebra and Section 3 provide the theoretical framework involved in this study. Section 4 furnish the description of the platform and Section 5 describe the tools. Conclusions and future works are presented in Section 6.

2. Literature review

There are more and more students who come to higher education institutions with a differentiated background than in the past, with different and vague visions of mathematics, its learning and role in their future careers and in their life. The study of students' math difficulties in passing from secondary school to university has been the subject of various researches (e.g. Di Martino & Gregorio, 2018; Lepellere et al., 2019) for its impacts at the individual and social level: in particular, students spend more time completing their scientific study or decide to abandon it. Students from a variety of STEM disciplines are required to take linear algebra as part of their undergraduate mathematics coursework. Difficulty in the teaching and learning of linear algebra during students' first year of undergraduate study is well documented (Hillel, 2000; Stewart et al., 2018; Stewart & Thomas, 2009; Sierpinska, 2000; Maracci, 2008). According to Wawro, "The content of linear algebra, however, can be highly abstract and formal, in stark contrast to students' previous computationally-oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly" (p. 2). The abstract concepts of linear algebra are often taught in such a way that students do not find any connections between new linear algebra topics and their previous knowledge of computational mathematics (Carlson, 1993). The unifying and generalizing nature of linear algebra has a didactic consequence: it is difficult to motivate the learning of new theory because its use will be profitable only after it may have been applied to a wide range of situations. (Dorier et al., 2000).

Linear transformations are introduced via algebraic rules and are usually associated with matrices, and matrix multiplication, as they appear in many textbooks, and after this, a number of geometric applications follow, such as reflection and rotation (Kolman & Hill, 2008). Therefore, the existing knowledge of undergraduate students concerning functions is often neglected, and linear transformations, on the one hand, are introduced as ready-made mathematics, and, on the other hand, applications of the topic are introduced in a static way. Research results from the related literature show that students are not fully aware of the mathematical relationship between the notions of function and linear transformation (Bagley et al., 2015). Research on the learning of eigenvalues, eigenvectors and eigenspaces has shown that their learning presents multiple obstacles for students, since they tend to concentrate in the procedures to handle them (Dogan, 2010). Using different representations while teaching these concepts has proved to help students to make sense of some of their properties (GolTabaghi, 2012; Stewart and Thomas, 2009) while the use of models stimulates students' understanding of these concepts (Larson et al., 2007). It can be challenging for students to coordinate algebraic interpretations with geometric ones (e.g. Stewart & Thomas, 2009; Larson & Zandieh, 2013), and students' ideas about eigenvectors are often not well-connected to other conceptual aspects of linear algebra (Lapp et al., 2010). Considering the value of these findings, researchers have developed interventions to support students in developing geometrically motivated ways of reasoning about eigenvectors and eigenvalues (Zandieh et al., 2016).

Several studies emphasized the use of dynamic geometry environment (DGE) for the visualization, especially in Geometer's Sketchpad (Gol Tabaghi 2014; Caglayan 2015) and GeoGebra (Beltrán-Meneu et al., 2016; Turgut, 2019). Cooley et al. (2014) availed themselves of the affordances of GeoGebra to aid students' visualization of the ways in which points on polygons are transformed. Turgut (2019) presents a careful analysis of students’ use of the dragging tool in GeoGebra to make sense of transformations matrix, using a lens of the theory of semiotic mediation (TMS). Gol Tabaghi (2014) highlighted the way that dragging vectors related to the understandings of eigenvectors and eigenvalues constructed by students in her study. Turgut (2019) uses TSM to describe how the tools and functions of a dynamic geometric system affect student learning. In particular, he focuses on how these tools mediated the evolution of student reasoning about linear transformations from personal meanings based on work in R^2 to new mathematical meanings in R^3 and R^n.

3. Theoretical framework

The theory of semiotic mediation, proposed by Bartolini Bussi and Mariotti (2008), considers that in a mathematics class, when using an artefact for accomplishing a mathematical task, students can be led to produce personal signs which can be put in relationship with mathematical signs. The construction of such relationship should be assumed as an explicit educational aim by the teacher, who can intentionally orient her/his own action towards promoting the evolution of signs expressing the relationship between the artefact and the tasks into signs expressing the relationship between the artefact and the knowledge at stake. The semiotic potential of an artefact consists of a twofold relationship “… (1) between a tool and meanings emerging in the accomplishment of the task and (2) between the tool and meanings related to specific mathematical content evoked by that use and recognizable by an expert.” (Mariotti, 2014, p.157.)”. Recent studies have developed specific theoretical constructs, including the notions of semiotic node (Radford, 2003) and of semiotic bundle (Arzarello, 2006), which can provide insight into the nature of the signs emerging during instrumented activity. About DGE, Leung et al. (2006) write: “A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities. This dynamic tool – dragging – induces potential dialectic between the conceptual realm (abstraction) of mathematical entities and the world of virtual empirical objects. Because of this possibility, dragging has been a major focus of research in DGE resulting in fruitful discussions on promising dragging modalities and strategies that seem to be conducive to knowledge construction” (p. 346). Previous research has shown that DGEs are particularly apt for triggering an inquiring approach in geometry (Arzarello et al. 2002; Sinclair & Robutti, 2013). How to design interactive tools to develop students’ sense-making regarding matrix representation of geometric transformations and eigenvectors within the perspective of semiotic mediation? The focus is on students’ reasoning on the transition from the notion of function to transformation and to matrix representation of geometric transformations in R^2. Along these lines, the theory of semiotic mediation is referred to as a theoretical framework in the design of a teaching and learning environment for the emergence of mathematical thinking. The designed material is suitable for use of the inquiry-oriented theory (Rasmussen & Kwon’s, 2007), which applies to both student activity and to instructor activity. In this approach, students learn new mathematics by engaging in cognitively demanding tasks that prompt exploration of important mathematical relationships and concepts; engaging in mathematical discussions; developing and testing conjectures; and explaining and justifying their thinking.

4. The platform

GeoUniud is organized according to a modular structure, to guarantee maximum flexibility and accessibility. It was implemented to guarantee compatibility with the University's current e-learning Moodle platform. The platform does not aim to replace but to help the teacher, assisting him/her in the preparation of lessons/content. It has the possibility to integrate the work “in vertical mode”, carried out in class through a frontal lesson, with work done independently, in “horizontal mode”, controlled indirectly by the teacher by means of adaptive self-regulation criteria inserted into the platform. GeoUniud is able to present flowcharts of mathematical task. It contains a virtually unlimited number of self-generating exercises, often accompanied with graphic displays, in a very user-friendly context. It could include a series of self-regulation criteria, which make it possible to integrate, with intelligent work controlled remotely by the teacher, the understanding acquired by the students. This structure makes also possible to implement a self-evaluating process by the student on each one of the topics treated in the platform. The content is organized in a series of HTML/JavaScript applications inserted in a web platform which also contains an extratextual outline element that describes the theme, the modalities of interaction and links to other material. The self-assessment tools will once again rotate around a series of JavaScript applications, which will offer the student exercises and questions defined by the teacher using general parameters. Moreover, the exercises can be calibrated on student's abilities; for example, a student can reduce the difficulty level (of the exercises) in the case of a full series of incorrect answers or increasing it in the opposite case. In the case of Linear Algebra there is the possibility of "manipulating" virtual representations of the elements of a vector space on the screen, with integrated DGEs. It is possible to display properties and characteristics and possibly modify them, thus observing the effects in a much more direct way than the study of books and handouts allows. As is expected nowadays when using hardware/software implementations, different kind of interaction are implemented: direct (through dragging) or indirect motions, clickable buttons and toggles, multiple-choice setups and input fields for numbers, symbols, or text.

5. Two interactive tools

The different libraries made available on GeoUniud let the expert user (an administrator or a trained teacher) build different experiences for different use-cases. We give here two examples implementing an inquiry-oriented learning experiences and the dragging element. Both examples work with the same structure and basic interaction models: one or two whiteboards with a dragging method, and some simple input fields. The whiteboards (Figure 1) in pair are linked: in the first one a direct-motion scheme is enacted and the user can drag elements, like vectors or points, and the second one will follow an indirect-motion scheme showing the transformed elements, following in real time the occurring direct manipulation. The underlying transformation can be explicit, defined by the teacher and hidden, or chosen by the student.

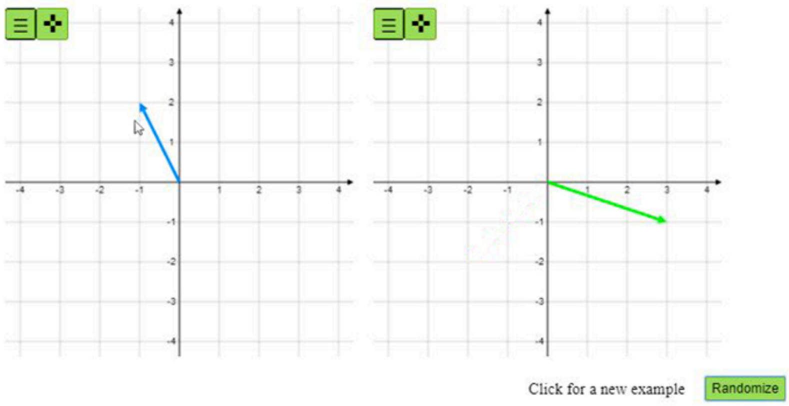

Figure 1. The whiteboards about linear functions

5.1. Example1. Linear functions.

The tool's interaction procedure, i.e. the dragging ability and the immediate visualization of vectors moving on the whiteboards, let us identify an artefact-sign of the first whiteboard: "a vector moving freely on the Cartesian plane". But the real core of the tool lies in the construction leading from the first to the second whiteboard. This construction can be hidden thus leaving the student with a sign we can describe as "two vectors moving on Cartesian planes", vectors that are known to be somehow related. This diverts attentions from the vectors themselves, which can be freely chosen, thus focusing students' attention on the notion of co-variation, leading to the mathematical concept of function. Another formulation of this tools consists of a pair of whiteboards, where a simple shape (a triangle) is shown, along with input fields to write and modify a matrix. The first whiteboard has the triangle as defined by the application; the second shows the triangle after the matrix is used as a transformation matrix.

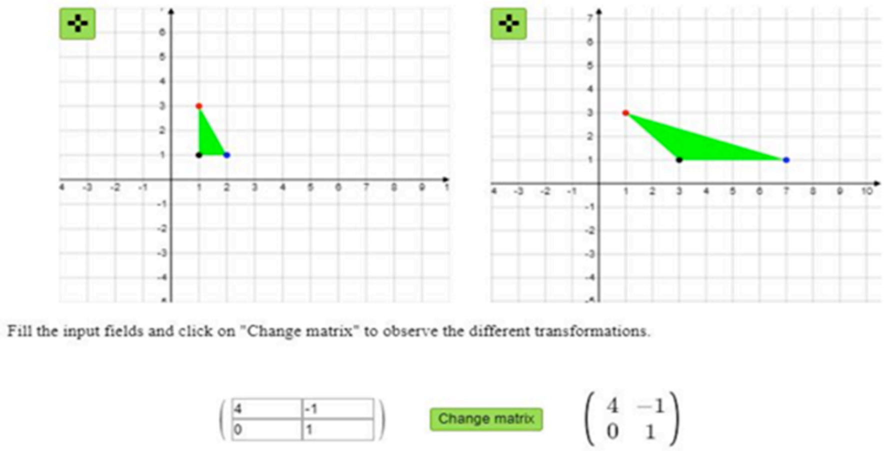

Figure 2. Transformation of a triangle through a linear function

Here (Figure 2) the student has no agency over the starting shape, which remains under the teacher's control, but he/she's free to check with the matrix: input a random one, input a matrix known to be a rotation-matrix, input a modified version of a known one. This tool is intended for students with skills in algebraic aspects, but with a still underdeveloped sense of the purely geometric aspect and is perfect to let them explore the subject without a strong teacher's leading. The starting shape is the real handle for the teacher, as it can be used to introduce symmetries and other proprieties, and then a light leading, with some hints or a list of kinds of matrices to use, will lead students to discover invariants and related matrices. The student can experience a dynamic visualization, based on the perception of variation through dragging, which can help make a conjecture on the geometric properties of the figures. In this example more conventional signs are introduced, as the matrix mimics conventions for written mathematical signs with only minor unavoidable adaptations allowing for input/output procedures. Choosing a matrix with some algebraic proprieties, like the identity matrix, the student can observe the result "the shape has not changed". The teacher can ask to find a different matrix with same results, and after some trial and error the student can produce the matrix for the symmetry with respect to one shape's axis of symmetry, given that there is one. This kind of interaction can mediate to the notion that matrices with different proprieties are linked to particular geometrical transformations.

5.2. Example 2. Eigenvectors.

As a second example, a tool with only one whiteboard is presented (Figure 3: the picture shows two different states of the same tool), with two vectors of different colours. The student is able to drag the dark coloured vector, "picking it up" by the arrow point and moving it directly, and the light coloured one will follow along in real time, with indirect movement, being the first vector's image through a linear application. The associated matrix can be known to the student, and even the subject of a lesson on eigenvalues and eigenvectors to be held in advance. The starting vector is not relevant to the problem and can be randomly generated with a simple click on the button under the whiteboard.

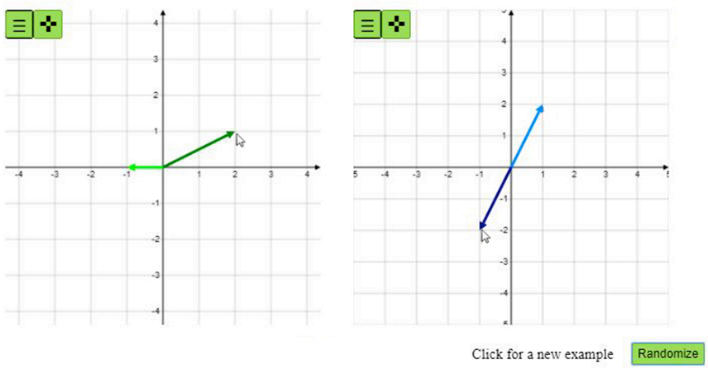

Figure 3. Eigenvectors

In this case students are invited to drag the vector around searching for an eigenvector, which could have been presented as "a vector that lies on the same line as its image". Since this observation can elude a student too keen on dragging here and there, this particular tool's instance was set up to change the colour schema when an eigenvector is found: from green (dark-green for the manipulable one, lime for its image) to blue (dark-blue/cyan), so semiotic potential of using colours can also contribute to understanding. In this setup, the choice to let the student know the linear function or the matrix in advance is up to the teacher. It's possible to set some requirements, e.g. only integer coefficients are viable, and to avoid little uncertainties due to the eyeballing the simple whiteboard when asking the student to find the eigenvectors, the dragging can be set to snap at regular intervals.

6. Conclusion

The project started in September 2018 as GeoUniud a user-friendly platform, where lessons and exercises are stored and organized with a careful use of randomized controlled contents as exercises, geometrical pictures, and abstract reasoning. The lessons are augmented by a virtually infinite collection of examples, and by interactive representations of concepts. The training pages offer randomly generated exercises, along with a system in place to verify the student's answers, flagging errors and giving contextual feedback. In this article we provide some hints in which two tools about linear transformation and eigenvectors can be integrate in a linear algebra lesson in order to improve students' sense-making in a dynamic geometry environment (DGE) within the perspective of semiotic mediation. A TSM perspective having two interrelated components is considered. On the one hand, the TSM is specific to the integration of digital tools in the teaching and learning of mathematics. On the other hand, an elaboration of the semiotic potential of an artefact guides an instructor by providing a possible learning route regarding didactic goals. The creation of a specific DGE environment provides a context for students' sense-making on matrix representation of geometric transformation and change of basis. However, this designed context can be considered as a heuristic tool for the preparation of students to enter abstract vector spaces. Moreover, as emphasized in the TSM, classroom discussion dynamics play an effective role in the transformation of personal meanings into mathematical meanings. The project will continue with teaching experiments and case studies to analyse, through their use, the semiotic potential of the proposed tools. Moreover, we are currently working on a wider platform, MatUniud, capable of providing innovative tools for all the basic mathematics courses of the STEM Area.

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