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# Developing the Radian Concept Understanding and the Historical Point of View

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**Abstract.** *Making a short review on the history of the trigonometry and it's current use, we are looking for a motivating real situation example, which can help both to introduce and understand the concept of radian measure. We shall discuss how the concept of angular mil which is based on milliradian and used in indirect measurement can be used to help students to understand the concept of radian.*

**Resumé.** *Dans cet article-ci nous faisons une courte révision de l'histoire de la trigonométrie et de son utilisation contemporaine. Nous cherchons un exemple motivant, qui pourrait être utile pour l'introduction de la mesure des angles en radians et pour la compréhension de cette notion. Nous examinons l'utilisation du concept de mil angulaire. Il s'agit d'un concept qui est utilisé pour mesurer des angles de facon indirecte et qui est basé sur la notion de miliradian. Nous examinons comment ce concept-ci peut être utilisé pour aider les étudiants à comprendre la notion de radian.*

**Sommario**.
*In questo articolo viene presentata una ristretta revisione della storia della trigonometria e della sua utilizzazione contemporanea. Cerchiamo un esempio motivante che potrebbe essere utile per l'introduzione della misura degli angoli in radianti per la comprensione di questa nozione. Esaminiamo l'utilizzazione del concetto di "mil" angolare. Si tratta di un concetto che é utilizzato per misurare degli angoli in modo indiretto e che é basato sulla nozione di milliradiante. Esaminiamo come questo concetto può essere utilizzato per aiutare gli studenti a comprendere la nozine di radiante.*

## Introduction

In Slovakia, trigonometric functions are introduced in two learning cycles. In the first learning cycle, at age 14, sine, cosine, tangent and cotangent are introduced using the right triangle and measurement problems are solved. In the second learning cycle, at age 16, the emphasis is changing from the study of measurement to the study of functions. Trigonometric functions are introduced using the unit circle model. The problem, both for teachers and students, is that the unit circle model is introduced simultaneously with a new unit of the angle measure - the radian. From the didactical point of view, it's impossible to grasp two new concepts at the same time. So teachers have to sacrifice the radian concept understanding in the favor of the "sine as function" concept understanding. In fact, if they really want to follow the curriculum and at the same time support students understanding, the only manageable way of using radians is the $\pi = 180°$ equivalence. So, in students understanding, radians have nothing to do with the length of the arc on the unit circle. Which leads to the faulty interpretation of the radian concept: *"180 degrees equals π radians. The π used with trigonometric functions is just a symbol to evaluate the length of a cycle in a different way. We could use any other symbol instead of π."* To find out, how wide spread is this interpretation of radian, we made a small study amongst 44 college students, who are undertaking the mathematics teacher education at Comenius University. These students were asked to draw a graph of a sine function and answer the following question: Does the Ludolphian number $\pi$ have something common with the symbol $\pi$ used with trigonometric functions? Although all 44 students marked the x-axes with symbols $\pi/2$, $\pi$, $3\pi/2$ and $2\pi$, 30 students answered they did not see any relation between the constant 3,14 and the symbol $\pi$ used in trigonometry. Only two out of the 14 students, who claimed to see a relation, involved in their answers the concept of the arc length. One of these 14 students has seen the relation in completely misinterpreted way: the length of a half circle is $\pi$, and the graph of the sine function is a composition of 2 half circles.

When discussing this topic with high school math teachers, most of the time only those who were teaching physics too, could see the meaningfulness of the radian concept. The rest of them regarded radians as something what students probably will need in higher education – and have to manage now. For them, radians were just an inconvenient exercise of converting degrees into multiples of $\pi$**,** using the basic formula $\pi = 180°$. These teachers realized that to develop a deep understanding of trigonometric functions, a mental presentation of an angle measured in degrees is more appropriate in this phase of learning cycle. Weber (2005) describes a college trigonometry course, where the unit cycle model is used with degree measurement of an angle and the students developed a deep understanding of trigonometric functions.

What's happening when introducing radians at high school is that, after the transposition process, school loses the rationale of the knowledge that is to be taught, that is, the questions that motivated the creation of this knowledge: What were radians introduced for? Why do we need radians? In this case, we obtain what Chevallard (2004) called a 'monumentalistic' education, in which students are invited to contemplate bodies of knowledge the rationale of which have perished in time.

The process of didactic transposition starts far away from school, in the choice of the bodies of knowledge that have to be transmitted. Then follows a clearly creative type of work — not a mere "transference", adaptation or simplification —, namely a process of de-construction and rebuilding of the different elements of the knowledge, with the aim of making it 'teachable' while keeping its power and functional character. The transpositive work is done by a plurality of agents (the 'noosphere'), including politicians, mathematicians ('scholars') and members of the teaching system (teachers in particular), and under historical and institutional conditions that are not always easy to discern. It makes teaching possible but it also imposes a lot of limitations on what can be and what cannot be done at school (Bosch & Gascon, 2006).

In our research, we are doing a transpositive work – looking for examples, which can help college students who already developed the faulty radian concept understanding to develop the correct one. We focus on the application - oriented teaching of mathematics, the transformation from a situation of the real world to a mathematical problem (Voskoglou, 2006).

Respecting the relation between the history and the psychogenesis, sometimes called the ontho-phylo parallelism (Hejný, 1988), we start with a brief insight into where the concept of radian comes from, how it was discovered and how its use in the past relates to how it is currently used.

**Radians – where do they come from – a short insight into history.**

As far as the origin of the subject is concerned, trigonometry and the development of trigonometric functions have a rich, diverse history. Trigonometry is not the work of one man or a nation. In fact, the ***ancient Egyptians and Babylonians*** had developed theorems on ratios of the sides of similar triangles (Boyer, 1991), before trigonometry was ever formalized as a subdivision of mathematics. These two groups had no clear usage of trigonometric functions but were able to use them unknowingly to their advantage. Egyptians used trigonometry to their benefit in land surveying and the building of pyramids. Babylonian astronomers related trigonometric functions to arcs of circles and the lengths of chords subtending their arcs (Gullberg, 1996). The ancestral beginnings of trigonometry are thought to be the first numerical sequences correlating shadow lengths with the time of day. The shadow tables are the ancestors of cotangent and tangent; hence, these functions were later derived from these early discoveries.

***Hipparchus*** (died about 125 B.C.), a highly credited Greek astronomer who came to be known as "the father of trigonometry," had a great influence in the developments of trigonometry and is the first person whose use of trigonometry is documented (Heath, 1981). Trigonometric tables were created for computations related to the scientific field known as astronomy. Most of what we do know comes from Claudius Ptolemy, who credits Hipparchus with a number of ideas in trigonometry and astronomy. Hipparchus' method of approaching trigonometry, as described and used by Ptolemy, is the following. The circumference of a circle is divided into 360 parts and the diameter is divided into 120 parts. Each part of the circumference and the diameter is further divided into 60 parts and each of these into 60 more, and so on according to the Babylonian system of sexagesimal fractions. Then for a given arc AB with its length expressed in circumference units, Hipparchus gives the number of units in the corresponding chord AB. (See Fig. 1).

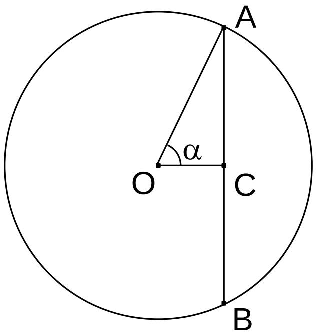



Fig.1

The number of units in the chord corresponding to our arc is equivalent to the modern sine function. If 2α is the central angle of arc AB (Fig. 1), then for a modern mathematician sin α = AC/OA. On the other hand, Hipparchus gives the length of the segment AB. For him, one unit of length is represented by one sixtieth of the length of the radius.

The advantage of choosing a large fixed radius is that fractions can be avoided because when the radius is chosen large enough, when divisions are made, these parts become whole numbers. Each side of the triangle then became a chord, defined as a straight line drawn between two points on a circle (web source [7]).

The development of Greek trigonometry and its application to astronomy culminated in the work of the Egyptian ***Claudius Ptolemy*** (d. A.D. 168). In his *Syntaxis Mathematica* (the work was referred to by the Arabs as *Almagest*), astronomy and trigonometry are commingled in thirteen books. In Chapter IX of Book I Ptolemy begins by calculating chords of arcs of a circle, thereby extending the work of Hipparchus and Menelaus. As already noted, the circumference is divided into 360 parts or units (he does not use the word "degree"). The diameter is divided into 120 units. Then he proposes, given an arc containing a given number of these (circle) units, to find the length of the chord expressed in the "diameter" units.

In the period of A.D. 200 – 1200 the Hindus made a few minor advances in trigonometry. ***Varahamihira*** (c. 500) used 120 units for the radius. Therefore Ptolemy's table of chords became for him a table of half chords, but still associated with the full arc. Later, ***Aryabhata*** (b. 476) made two changes. First he associated the half chord with half of the arc of the full chord; this Hindu notion of the sine was used by all later Hindu mathematicians. Secondly, he introduced the radius consisting of 3438 units. This number comes from assigning 360x60 units (the number of minutes) to the circumference of a circle and using $C = 2\pi r$, with $\pi$ approximated by the value 3,14. Thus in Aryabhata's scheme the sine of an arc of 30°, that is the length of the half chord corresponding to an arc of 30°, was equal to 1719 units. Since now Ptolemy's values were no longer suitable the Hindus recalculated a table of half chords. (Kline, 1972)

New work in trigonometry was done by Germans of the late fifteenth and early sixteenth century. The trigonometric work was motivated by navigation, calendar-reckoning, and astronomy, interest in the last-mentioned field having been heightened by the creation of the heliocentric theory.

The work of ***Regiomontanus*** (1436 – 1476) is based on results of Hindu and Arabian trigonometry and the book of Ptolemy. Regiomontanus's book "De triangulis omnimodis libri quinque" was the first in Europe where trigonometry was considered as a stand-alone mathematical theory. *De triangulis* is contained in five books, the first of which gives the basic definitions: quantity, ratio, equality, circles, arcs, chords, and the sine function. Then, he gives a list of the axioms he will use, followed by 56 theorems on geometry. In Book II the sine law is stated (in modern notation, not used by Regiomontanus, this is $a/\sin A = b/\sin B = c/\sin C$) and it is used to solve triangles. (Kline, 1972)

He constructed a table of sines based on a radius of 600 000 units and another based on a radius of 10 000 000 units. He also calculated a table of tangents. He gave five-place tangent tables and a decimal subdivision of the angles, a very unusual procedure for those times. (Kline, 1972)

Many scientists of the fifteenth and sixteenth centuries constructed tables, among them ***George Joachim Rhaeticus*** (1514 – 1576), ***Nicolaus Copernicus*** (1473 – 1543) and ***Bartolomaus Pitiscus*** (1561 – 1613). This work was characterized by the use of a larger and larger number of units covering the radius so that the values of trigonometric quantities could be obtained more accurately without the use of fractions or decimals. For example, Rhaeticus calculated a table of sines based on a radius of $10^{10}$ units and another based on $10^{15}$ units, and gave values for every 10 seconds of arc. Pitiscus in his *Thesaurus* (1613) corrected and published the second table of Rhaeticus. Actually, he is the first author using the word "trigonometry". (Kline, 1972)

Several mathematicians suggested using $r = 1$; this exactly produces the modern values of the trigonometric functions. Rhaeticus introduced the modern conception of trigonometric functions as ratios instead of as the lengths of certain lines (web source [3]).

A new era of trigonometry comes with the creation of calculus.

The eighteenth century mathematicians worked extensively with *trigonometric series*, especially in the astronomical theory.

Any series of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called a trigonometric series.

Since astronomical phenomena are periodic, it was useful to have trigonometric series because they are periodic functions as well. Use of trigonometric series was introduced to determine the positions of the planets and interpolation, which is a mathematical procedure that estimates the values of a function at positions between given values. In 1747, ***Euler*** applied the method of interpolation in the theory of planetary perturbations and secured a trigonometric series representation of the function (Kline, 1972).

The concept of trigonometric values as ratios and the modern notation comes from Euler’s book *Introductio in analysis infinitorium* (1748) (Struik, 1963) In this work, strict analysis of trigonometric functions was established. Sine was no longer a line segment; rather, it transformed into a number or ratio, the ordinate point on a unit circle (Boyer 443).

The term *radian* first appeared in print on June 5, 1873 in examination questions set by ***James Thomson*** at Queen’s College, Belfast. James Thomson was a brother of Lord Kelvin. He used the term as early as 1871, while in 1869 Thomas Muir, then of St. Andrew’s University, hesitated between *rad*, *radial* and *radian*. In 1874, Muir adopted *radian* after a consultation with James Thomson. The concept of a radian measure, as opposed to the degree of an angle, should probably be credited to Roger Cotes in 1714. He had the radian in everything but name, and he recognized its naturalness as a unit of angular measure (web source [5]) .

In society today, trigonometry is used in physics, engineering and chemistry. Within mathematics it is in mainly in calculus, but also in linear algebra and statistics (web source [7]). Let’s illustrate the wide use of it by one example in the modeling process. Bachratá (2005) refers to the sampling formula **Shannon Kotelnik**, which is used for a reconstruction of the sound signal

$$f(t) = \sum_{k \in Z} f(kT)\frac{\sin \Omega(t-kT)}{\Omega(t-kT)}, \quad t \in R, \quad T = \frac{\pi}{\Omega}$$

The sample values are measured only at moments, that are integral multiples of the period *T*, and the reconstructed signal should approximate the missing values. Since the argument of the sine function is in radians, the value of

$$\frac{\sin \Omega(t-kT)}{\Omega(t-kT)}$$

is close to 1 for values of *t* which are close to *kT*. So for values of *t* which are close to *kT* the values of *f(t)* are close to *f(kT).* Therefore the reconstructed values in the close neighborhood of sample values are close enough to the sample values.

**Remarks on History**

If we examine the history of trigonometric functions, we note the following facts:

The first notion of the chord trigonometric function does not include the notion of a degree – the circle is divided into 360 units and the perimeter into 120 units. At this period, the Greeks used for π the approximate value of 3. This way, approximately same unit is measuring both the length of an arc and the length of a line segment. To get a better correspondence between the arc length and the length of the radius, the Hindu Aryabhata introduced the radius of 3438 units and assigned 360x60 units (the number of minutes) to the circumference of a circle. He was using $C = 2\pi r$, with $\pi$ approximated by 3,14. In the Renaissance, enlarging the number of radius units up to $10^{15}$ tables of greater precision were calculated. The division of the circumference does not change – 360 degrees are used, and further division into minutes and seconds is used too.

This sine function, unlike the modern one, was not a ratio but simply the length of the side opposite the angle in a right triangle of fixed hypotenuse. So the tables differ in the range of sine values. Rheticus introduced the modern conception of trigonometric functions as ratios instead of as the lengths of certain lines. This way, the values of a trigonometric function are independent on the radius, i.e. the length of the hypotenuse.

From the didactical point of view the following fact is important: the Greek and Hindu concepts of trigonometric functions reflected the demand of measuring the radius and circumference of the circle with units of same length. However, when the sin function is a ratio instead of the length of a line segment, and the argument of the sin function is still measured in degrees (that's the circumference of a circle is divided into 360 basic units), the straightforward relation between the length of the arc and the value of sine function disappears. But in calculus and its countless applications, the ancient demand of measuring the radius and circumference by identical units is very natural.

In the Lesson on *Angle Measure and Circular Functions* (web source [2]) we read: "The motivation for using radian measure in mathematics is this: The *value* of a trig function can be thought of as being *a fractional part of a circle's radius*. By having trig functions' arguments be in units of the circle's radius, a function's arguments and the function's values are in the same unit. (This is why the theorem

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

is true only when $x$ is in radians. When $x$ is in degrees, this limit is $\pi/180$.)"

In chapter on Angles from E.W. Weisstein (web source [1]) we read: "The radian is the most useful angle measure in calculus because the derivative of the trigonometric functions such as

$$\frac{d}{dx}\sin x=\cos x$$

does not require the insertion of multiplicative constants such as $\pi/180$."

There is whole chapter devoted to the aspects of (sin x)/x in the book of Eli Maor (Maor, 1998).

## Motivation and Knowledge with Understanding

When discussing the topic of radians with the future math teachers, one of their first requests is: "Can you give us some real-life examples, where trigonometric functions are used with the radian measure?" We can give them quite catchy examples from applied calculus in the signal processing.

In signal processing, Fourier transformation can isolate individual components of a complex signal, concentrating them for easier detection and/or removal. Loosely speaking, the Fourier transform decomposes a function into a continuous spectrum of its frequency components, each component is a complex sinusoid and the transform variable represents angular frequency in radians per second. A large family of signal processing techniques consists of Fourier-transforming a signal (such as a clip of audio or an image), manipulating the Fourier-transformed data in a simple way, and reversing the transformation. Some examples include:

Removal of unwanted frequencies from an audio recording - used to eliminate hum from leakage of AC power into the signal, to eliminate the stereo sub carrier from FM radio recordings, or to create karaoke tracks with the vocals removed.

Image processing to remove periodic or anisotropic artifacts such as jaggies from interlaced video, stripe artifacts from strip aerial photography, or wave patterns from radio frequency interference in a digital camera (web source [8]).

An example of using radians to create karaoke tracks is quite motivating. However, the mathematical theory behind these examples is rather complicated.

So, we need an example, in applied mathematics, which can help us illustrate the basic idea of the radian: By having trig functions' arguments be in units of the circle's radius, a function's arguments and the function's values are in the same unit. Moreover, for small angles, we can replace the length of the arc by the length of the line segment.

In our opinion, to understand the idea of radian, it's meaningful to introduce the concept of milliradian – in the way it's used in indirect measurement. Indirect measurement is a topic that does not demand high mathematical background and the formulation of a problem is straightforward.

Moreover, the concept of angular mil, derived from a concept of milliradian, reflects the early division of a circle into whole number of units. Whereas one of the most difficult understandable attributes of radian is, that there is an irrational number of radians in one circle.

## Milliradians and Angular Mil in Indirect Measurement

There are more possible ways of defining a milliradian. For example: "One milliradian is the angle subtended at the center of a circle of radius 1000 by an arc of length 1". At websource [5] the practical use can be find: "The milliradian (0.001 rad, or 1 mrad) is used in gunnery and general targeting, because it corresponds to 1 m at a range of 1000 m (at such small angles, the curvature can be considered negligible) The divergence of laser beams are also usually measured in milliradians."

The following definition of the angular mil and it's use is from websource [6]: "The angular mil is commonly used by military organizations. Its relationship to the radian gives rise to the handy property that object of size $s$ that subtends an angle $\hat{I}$ angular mils is at a distance $d = 1000s/\hat{I}$. Alternatively, if the distance is known, we can determine the size of an object by $s = \hat{I}\, d/1000$. The practical form of this that is easy to remember is: 1 mil at 1 km is about 1 meter. Another example: 100 mils at 2 km is about 200 meters.

However, military mils are fixed angles not based on the above formula. All versions of the angular mil are approximately the same size as a milliradian. There are 2000 π milliradians in a circle. So a milliradian is just over 1/6283 of a circle. Each of the definitions of the angular mil is similar to that value but these values are easier to divide into many parts.

- 1/6400 of a circle in NATO countries.
- 1/6000 of a circle in the former Soviet Union and Finland .
- 1/6300 of a circle in Sweden. The Swedish term for this is *streck*, literally "line". Sweden has not been part of NATO nor the Warsaw Pact.

In the general case, where neither the distance nor the object size is known, the formulae may be of little use. In practice, sizes of observed objects are known with reasonable accuracy since they are often people, buildings and vehicles. Using the formulae, distances of the objects can be readily calculated without a calculator.

Many telescopic sights used on rifles have reticles that are marked in angular mils, and these are generally called **mil dot scopes**. The mil dots serve two purposes, range estimation and trajectory correction. By determining how many angular mils an object of known size subtends, the distance to that object can be estimated with a fair degree of accuracy."

## Lecture on Milliradian

In accordance with the ideas we have worked out above we are giving an outline of a lecture on milliradians.

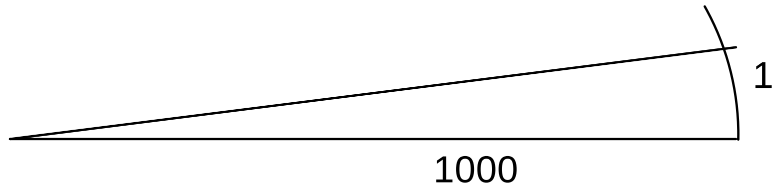
*Problem 1: A man is seen at a distance through a telescopic sight. He subtends an angle of one degree. Let us suppose that this man is about 1,80 m tall. Can you estimate the distance to this man? Can you calculate the distance to this man? Can you calculate it without a calculator?*

A typical solution may involve a sketch. Following from this sketch,

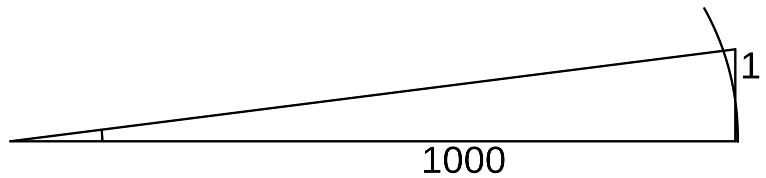
$$tg(1°) = \frac{1{,}8}{d} \qquad d = \frac{1{,}8}{tg(1°)}$$

Since usually nobody knows the value of *tg(1°)* by heart nor can estimate this value with a fair degree of accuracy, the answer is : We can't do it without a calculator.

*Problem 2: To be able to make quick estimates of distances, we are going to use a new measure of angle – the milliradian.* ***One milliradian is the angle subtended at the center of a circle of radius 1000 by an arc of length 1.***




*Let us sketch a situation, when an object of size 1 m is at distance of 1000 m. What angle, measured in milliradians does this object subtend?*




The sketch of this situation, involving the part of a large circle and the segment of the tangent line for the representation of our object, implies that the arc length and the line segment length are nearly the same. From which follows the idea already mentioned above, that's "The milliradian corresponds to 1 m at a range of 1000 m (at such small angles, the curvature can be considered negligible)".

*Problem 3: Let us sketch a situation, when we see an object of size 10 m at distance of 1000 m. Estimate what angle, measured in milliradians does this object subtend? Answer the same question for an object of size 100 m, an object of size 500 m. Just hypothetically, think about an object 5000 m high at the same distance. Estimate what angle, measured in miliradians, does this object subtend.*

Making the sketches, one can see that as the angle gets larger, the sizes of the line segment and the arc length differ more. And vice versa, as the angle gets smaller, the lengths are closer.

We can illustrate this by calculations done in Excel.

**In a circle of radius 1000**

| angle measured in milliradians | length of the tangent line segment | length of the arc |
|---|---|---|
| 1 | 1,000000333 | 1 |
| 5 | 5,000041667 | 5 |
| 10 | 10,00033335 | 10 |
| 50 | 50,04170838 | 50 |
| 100 | 100,3346721 | 100 |
| 500 | 546,3024898 | 500 |
| 1000 | 1557,407725 | 1000 |
| 1500 | 14101,41995 | 1500 |

Actually, we are just giving examples for anticipating the formula

$$\lim_{x \to o} \frac{tg(x)}{x} = 1$$

At this moment, if not earlier, one feels the need to know the correspondence between milliradians and degrees. To be able to imagine an angle 1500 mrad it is good to know the number of milliradians in one circle.

*Problem 4: What is the number of milliradians in one circle?*

It's easy to see, that the same as the number of units of a circumference of circle of radius 1000. Which is approximately 6283. However, the precise number can't be expressed with a rational number, we need the Ludolphian number.

At this stage, we anticipate a discussion triggered by students. The main idea of this discussion should be the meaningfulness of having a circle divided into an irrational number of units. At this point, the introduction of the angular mil is the natural answer.

Also, we suppose, that during the process of getting acquainted with the concept of milliradian, students start to anticipate the real meaning of marking the x-axis in a graph of a trigonometric function by the π symbol.

## Conclusion

In our future research we plan to elaborate our idea in more detail and to verify it's effectiveness in helping to develop a correct understanding of the radian measure. There are many ways of introducing the radian measure and we were looking for one, which reflects the idea of the radian in accordance with the historical point of view – the use of trigonometric functions in surveying and indirect measurement throughout the early history. The concept of a trigonometric function as a periodic one comes later, and we are aware that to grasp the radian in it's complete meaning, students need extra experience which reflect the idea of a point moving on a circle.

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