

## **PROBLEM SOLVING IN MATHEMATICS EDUCATION: RECENT TRENDS AND DEVELOPMENT**

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### **Abstract**

*While early work in problem solving focused mainly on describing the problem solving process, more recent investigations have focused on identifying attributes of the problem solver that contribute to the problem solving success. The aim of the present paper is to present and discuss the recent progress of the problem solving process in mathematics education.*

### **1. Introduction**

Problem – Solving (P-S) is a principal component of mathematics education from the time of its emergency as a self – sufficient science until today.

According to Schoenfeld (1983) a problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no “surprises” in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) it is an exercise.

In an earlier paper (Voskoglou, 2007a), where we have examined the role of the problem for the learning of mathematics, we have attempted a review of the evolution of the P-S from the time that Polya presented his first ideas on the subject (1945-1963) until the 1990's. Here is a rough chronology of that progress: 1970's : Emergency of mathematics education as a self – sufficient science (research methods were almost exclusively statistical) and of cognitive science (theories of learning etc).

1980's: A framework describing the P-S process and reasons for success or failure in P-S, e.g. see Schoenfeld (1980,1985a), Lester, Garofalo & Kroll (1989), etc.

1990's : Models of teaching using P-S ( e.g. constructivist teaching, see Jaworski 2006, Voskoglou 2007b, Schoenfeld 2002 , etc).

Our purpose in this paper is to present and discuss the recent progress of the problem solving process in mathematics education.

### **2. A recent trend in P-S: To focus on the problem solver**

While early work in P-S focused on describing the P-S process, more recent investigations have focused on identifying attributes of the problem solver, that contribute to P-S success.

Schoenfeld (1985a) in his book *Mathematical Problem Solving* offered a framework for analyzing how and why people are successful (or not) when they engage on P-S. He argued that the following four factors are necessary and sufficient for understanding the quality and success of the P-S attempts: ( i ) The **knowledge base**, (ii) **P-S strategies** (heuristics), (iii) **Control** : monitoring and self-regulation, or metacognition, and (iv) **Beliefs** and the practices that give rise to them .

More recent studies have cited planning and monitoring as key discriminators in P-S success and have revealed the influence of various other affective dimensions, like beliefs, attitudes and emotions (Schoenfeld, 1992, DeFranco, 1996, Carlson,1999, etc)

Lester (1994) noted a consistent finding that P-S performance appears to be a function of several independent factors, like knowledge, control, beliefs and socio-cultural contexts. He characterized “good” mathematical problem solvers as possessing more knowledge, well-connected knowledge and rich schemata. They regularly monitor and regulate their P-S efforts, Lester observes, and they tend to care about producing elegant solutions.

Today is a general agreement that problem difficulty is not so much a function of various task variables as it is the characteristics of the problem solver. Geiger & Galbraith (1998) claimed that it is the relationship between the learner and a problem that is of significance, not the perceived level of the problem as viewed within some hierarchy of abstraction. Good mathematical problem-solvers exhibit flexibility during P-S and tend to use powerful content-related processes rather, than general heuristics alone. They also appear to have a high level of self-awareness of their strengths and weaknesses and tend to focus on the underlined structure and relationships in the problem (Stillman & Galbraith, 1998).

Carlson & Bloom (2005) drawing from the large amount of literature related to P-S developed a broad taxonomy to characterize major P-S attributes that have been identifying as relevant to P-S success. The dimensions of the taxonomy are: **Resources**: The conceptual understandings, knowledge, facts and procedures used during P-S.

**Control**: This includes the selection and implementation of resources and strategies, as well as behaviors that determine the efficiency with which facts, techniques and strategies are exploited, e.g. planning, monitoring, decision making , conscious metacognitive acts etc.

**Methods**: The general strategies used when working a problem, like constructing new statements and ideas, carrying out computations, accessing resources.

**Heuristics**: More specific procedures and approaches used when working a problem, like observing symmetries, using a graph or a table, looking for counter examples, altering the given problem so that it is easier, etc..

**Affect**: This includes attitudes (enjoyment, motivation, interest), beliefs (self-confidence, pride, persistence, etc), emotions (joy, frustration, impatience, etc) and values/ethics (mathematical intimacy and integrity).

As their principal method of data collection Carlson & Bloom elected to investigate the behaviors of 12 experienced problem solvers, all mathematicians, while they worked through 4 mathematics problems. Their initial analysis revealed that their taxonomy was limited in its ability to characterize some of the critical behaviors being exhibited by the mathematicians in their study.

Then they reanalyzed the data using a grounded approach, employing open coding techniques (Strauss & Corbin, 1990). The resulting **Multidimensional Problem-Solving (MPS) Framework** has four phases: **Orientation, Planning, Executing** and **Checking** (these were the major phases that these mathematicians moved through when completing a problem). It has been observed that once the mathematicians oriented themselves to the problem space, the plan-execute-check cycle was then repeated through out the remainder of the solution process. Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three of the four phases, that is planning, executing and checking.

It has been also observed that, when contemplating various solution approaches during the planning phase of the P-S process, the mathematicians were at times engaged in a **conjecture-imagine-evaluate (accept/reject)** sub-cycle. This sub-cycle became evident to Carlson & Bloom as they observed the mathematicians and listened to their verbal descriptions of how they were imagining a solution, playing it out in their minds. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected with the phase of planning.

The effectiveness of the mathematicians in making intelligent decisions that led down productive paths appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics and facts, as well as their ability to manage their emotional responses. The mathematicians' well connected conceptual knowledge, in particular, appeared to be an essential attribute for effective decision making and execution throughout the P-S process.

### **3. A theory of goal-directed behavior for P-S**

As we have seen in the previous section, Schoenfeld (1985a) offered a framework for analyzing the P-S process. But this is a framework only, not a theory providing rigorous explanations of how and why things fit together, i.e. in other words why people during the P-S process made the choices they did..

In the next 20 years Schoenfeld has been working to build a theoretical approach that explains all the above and he reached to the conclusion that solving a problem , as well as other human activities like cooking, teaching a lesson and even a brain surgery(!), are all examples of goal-directed behavior (Schoenfeld, 2007).

According to his investigations the ideal domain for the development of such a theoretical approach is the process of teaching a lesson, which is a dynamic goal-oriented P-S activity: The teacher enters the classroom with certain knowledge and goals. Sometimes conducting a lesson is easy, one goes through what has been planned. But sometimes it is not and the teacher has to adapt on the spot. Indeed, so do most people jobs! They are knowledge-based and often routine, but

sometimes call for urgent on the spot decisions. According to Schoenfeld this goal-oriented “acting in the moment” can be explained and modeled by a theoretical architecture in which the following are represented : Knowledge, goals, orientations and decision-making. More specifically:

**Knowledge** is obviously the foundation of all competent behavior. The most important however is the form of knowledge organization and access. Much routine behavior is based on the individual’s possession of “knowledge packets” known as schemata (or scripts, or frames). For example, if you recognize that a mathematical problem is a max-min problem, you immediately know that you have to differentiate a function, set the derivative equal to zero, etc.

**Goals** : Much of the human behavior can be seen as goal-oriented, i.e. we act because we want to achieve something. If we are working on solving a problem the formal goal is to achieve a solution. Often we make a plan which has subgoals. We work toward the subgoals, and either achieve them (in which case we move to the next subgoal), or find alternatives. Thus progress on a problem can be seen as the establishment of and progress toward the achievement of a series of goals.

**Orientations** is a generalization of beliefs including values (e.g pure mathematics or applications?), preferences, etc. Beliefs shape behavior, for example someone who believes that mathematics word problems are merely cover stories for computational exercises will write down that the number of buses asked for in a problem is “31 remainder 12” instead of 32.

**Decision –making** : A lot of human decision making can be seen as modelable by expected-value computations, where the quantities are the subjective values assigned by the individuals. We all know for example that the decision to buy a lottery ticket is a bad decision in mathematical terms, because the expected value (which equals the probability of winning X objective value of prize - cost of the ticket) is negative. But from an average person’s subjective point of view the cost of the ticket is small and the subjective value of the prize (an easy life) is big. Thus the expected value ,which in this case is equal to the probability of winning X subjective value of prize - subjective cost of the ticket, is positive. That explains why different people will decide differently, because the subjective values they assigned are different.

Once you understand an individual’s orientations , Schoenfeld argues, you can see how the individual prioritizes goals and outcomes and therefore you can model the possible courses of his action. Thus the importance of this theoretical approach for P-S is that an understanding of “how things work” can help to improve practice. In fact, when you understand how something skillful is done, you can help the others to do it successfully.

#### **4. Discussion and conclusions**

As we have seen the broad taxonomy for P-S developed by Carlson & Bloom was proved to be not enough to characterize in detail the behaviors of the problem-solvers (mathematicians) in their study. For this purpose, based on the reactions of

the 12 mathematicians during the P-S process of the 4 given problems, they created the MPS Framework that reveals a cyclic nature of the P-S process.

However it is not sure at all that the corresponding framework for inexperienced solvers (novices) would have the same flow. In fact, although many studies have investigated and compared the characteristics of novice and expert problem solvers (e.g. Lesh & Akerstrom, 1982, Schoenfeld 1985b, 1989, Geiger and Galbraith, 1998, Stillman and Galbraith, 1998 etc) many aspects of the problem-solving process still do not appear to be understood. For example, while the literature supports that control and metacognition are important for problem-solving success, more information is needed to understand how these behaviors are manifested during P-S and how they interact with other P-S attributes reported to influence the problem-solving process (e.g. resources, heuristics, affect, etc).

Another interesting thing is to attempt a comparison of the expert performance model of Schoenfeld (1980) with the MPS Framework. There are indeed many similarities among the five stages of Schoenfeld’s model and the four phases of MPS Framework. Thus the stage of the analysis of the problem corresponds to the phase of orientation, the stages of exploration and of design correspond to the phase of planning, the stage of the implementation of the solution corresponds to the phase of executing and finally the stage of verification corresponds to the phase of checking. In my opinion however Schoenfeld’s model has the advantage of giving, for each stage, a list of the possible heuristics that could be used in order to get through and therefore it looks to be more useful in practice.

We shall close with some comments for the theory of goal-oriented behaviour for P-S. We indubitably agree that through this theory one gets a better understanding of “how things work” for P-S. However the teacher, in order to use this theory to improve practice, has first to understand the orientations of his students and then to try to change those that prevent efficiency in P-S by engaging the suitable for each case activities. For example, if a student believes that the important thing for P-S is to memorize formulae or techniques, given a problem he will try to solve it by using the most recent technique that he has learnt. Therefore in this case the teacher must give problems that they need some extra “movements” in order to be solved. Nevertheless my strong belief is that the understanding of the students’ orientations is a very difficult task that, apart of the teacher’s great experience, demands a comfort in time, a fact that does not happen very often in practice (the teacher has 30 or more students to deal with). Moreover, since the orientations of the students are usually different, the suitable activities to be engaged are also different for each case and this imposes an extra difficulty to the teacher.

Conclusively, although the theory of goal-oriented behavior for P-S could possibly proved to be a useful tool for the researcher of mathematics education, it looks very difficult to be used by the mathematics teachers for practical applications..

Schoenfeld (2007) admits that, although his theory can help to improve practice, it does not guarantee (because of so many other factors) that there will be any improvements. Moreover he believes that, although the 40 or so years since both the cognitive sciences and mathematics education began to coalesce we have made a spectacular progress, more work needs to be done and he speaks about a

“hundred year plan”. The mind is more complex than the body, so on comparing with the evolution of medicinal practice we should expect progress in mathematical education to take as long a time

**Note:** For contributions on P-S conducted in Italy the reader can see the works of Kleinmuntz (ed.), 1966, Malara , 1990, Zan , 1991,1992,2000, Bozzolo & Ferrari, 1995, Grugnetti & Jaquet, 2000, etc. About the thematic related to “affective and P-S” it could be interesting to consider the article of De Bellis & Goldin (1997)

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