"Quaderni di Ricerca in Didattica", n. 19, 2009.
G.R.I.M. (Department of Mathematics, University of Palermo, Italy)

Insights into Game Theory: An Alternative Mathematical Experience

Ein-Ya Gura
The Center for the Study of Rationality
The Hebrew University of Jerusalem

Abstract

Few branches of mathematics have been more influential in the social sciences than game theory. In recent years, it has become an essential tool for all social scientists studying the strategic behavior of competing individuals, firms, and countries. However, the mathematical complexity of game theory is often very intimidating for students who have only a basic understanding of mathematics. *Insights into Game Theory* addresses this problem by providing students with an understanding of the key concepts and ideas of game theory without using formal mathematical notation. The authors use four different topics (college admissions, social justice and majority voting, coalitions and cooperative games, and a bankruptcy problem from the Talmud) to investigate four areas of game theory. The result is a fascinating introduction to the world of game theory and its increasingly important role in the social sciences.

This paper is a survey of the book *Insights into Game Theory: An Alternative Mathematical Experience*, by Michael Maschler and Ein-Ya Gura, published by Cambridge University Press (2008).

Introduction

Game theory is a relatively young branch of mathematics that goes back to the publication of *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern in 1944.[1]

Game theory undertakes to build mathematical models and draw conclusions from these models in connection with interactive decision-making: situations in which a group of people not necessarily sharing the same interests are required to make a decision.

The choice of the topics reflects our purpose: we wanted to present material that does not require mathematical prerequisites and yet involves deep game-theoretic ideas and some mathematical sophistication. Thus, we ruled out topics from non-cooperative game theory, which requires some knowledge of probability, matrices, and point-set topology.

Broadly speaking, the topics chosen are all related to the various meanings that can be given to the concept of "fair division." The four chapters illustrate this.

The first, "Mathematical Matching," concerns, among other things, the problem of assigning applicants to institutions of higher learning. Each applicant ranks the universities according to his scale of preferences. The universities, in turn, rank the applicants for admission according to their own scale of preferences. The question is how to effect the "matching" between the applicants and the universities. The reader will discover that this problem leads to unexpected solutions.

The second chapter, "Social Justice," concerns social decision rules. In a democratic society it is customary to make decisions by a vote. The decision supported by the majority of voters is

[1] Several "game-theoretic" topics were discussed prior to its publication, but not in any systematic way.

adopted. But the reader will discover that "majority rule" does not always yield a clear-cut solution. The attempt to find other voting rules raises unexpected difficulties.

The third chapter, "The Shapley Value in Cooperative Games," addresses, among other things, the following problem: a group of people come before an arbitrator and inform him of the expected profits of every subgroup, as well as of the whole group, if the groups operate independently. It seems that these data are sufficient for the arbitrator to decide how to divide the profits if all of the litigants operate jointly.

The fourth chapter, "Analysis of a Bankruptcy Problem from the Talmud," addresses the following problem: several creditors have claims to an estate, but the total amount of the claims exceeds the value of the estate. How should the estate be divided among the creditors? In the chapter several solutions are accepted, two of which are discussed in the Talmud.

As explained above, this book is not a textbook in game theory. Rather, it is a collection of a few topics from the theory intended to open a window onto a new and fascinating world of mathematical applications to the social sciences. Our hope is that it will motivate the reader to take a solid course in game theory.

One of the aims of the book is to acquaint the reader and the student with "a different mathematics"—a mathematics that is not buried under complicated formulas, yet contains deep mathematical thinking. Another aim is to show that mathematics can efficiently handle social issues. A third aim is to deepen the mathematical thinking of the person who studies this book.

We believe that by studying the topics of this book, the mathematical thinking of the student will be enriched.

This book selects a small number of topics and studies them in depth. It shows the student of the social sciences how a mathematical model can be constructed for real-life issues.

The chapters are independent. A teacher and a student can choose one chapter or several and cover them in any order.

In high schools, the book can be used by students on any program track or as extracurricular material. The teacher can proceed to the deeper parts of each chapter if she has a mathematically inclined class or skip some of the proofs if the class cannot handle them. The book can also be used by students who want to read independently or under the guidance of a teacher beyond what is required in school.

At universities and colleges, the book can be used in courses whose aim is to introduce general game-theoretic topics and deepen mathematical thinking.

1.0 Mathematical Matching

In 1962 a paper by David Gale and Lloyd S. Shapley[2] appeared at the RAND Corporation, whose title, "College Admissions and the Stability of Marriage," raised eyebrows. Actually, the paper dealt with a matter of some urgency.

According to Gale,[3] the paper owes its origin to an article in the *New Yorker*, dated September 10, 1960, in which the writer describes the difficulties of undergraduate admissions at Yale University. Then as now, students would apply to several universities and admissions officers had no way of telling which applicants were serious about enrolling. The students, who had every reason to manipulate, would create the impression that each university was their top choice, while the universities would enroll too many students, assuming that many of them would not attend. The

[2] Gale, D. and Shapley, L.S. 1962. "College admissions and the stability of marriage," *American Mathematical Monthly* 69: 9–15.

[3] Gale, D. 2001. "The two-sided matching problem: origin, development and current issues," *International Game Theory Review* 3: 237–52.

whole process became a guessing game. Above all, there was a feeling that actual enrollments were far from optimal.

Having read the article, Gale and Shapley collaborated. First, they defined the concept of stable matching, and then proved that stable matching between students and universities always exists. This and further developments are discussed in this chapter.

For simplicity, Gale and Shapley started with the unrealistic case in which there are exactly *n* universities and *n* applicants and each university has exactly one vacancy. A more realistic description of this case is a matching between men and women—hence the title of their paper.

THE MATCHING PROBLEM

Consider a community of men and women where the number of men equals the number of women.

Objective: Propose a good matching system for the community.[4] To be able to propose such a system, we shall need relevant data about the community. Accordingly, we shall ask every community member to rank members of the opposite sex in accordance with his or her preferences for a marriage partner. We shall assume that no men or women in the community are indifferent to a choice between two or more members of the opposite sex.[5] For example, if Al's list of preferences consists of Ann, Beth, Cher and Dot, in that order, then Al ranks Ann first, Beth second, Cher third, and Dot fourth.[6] Again, we shall assume that Al is not indifferent to a choice between two or more of the four women on his list.

Example:

The men are Al, Bob, Cal, Dan.

The women are Ann, Beth, Cher, Dot.

Their list of preferences is:

Women's Preferences

	Ann	Beth	Cher	Dot
Al	1	1	3	2
Bob	2	2	1	3
Cal	3	3	2	1
Dan	4	4	4	4

Men's Preferences

	Ann	Beth	Cher	Dot
Al	3	4	1	2
Bob	2	3	4	1
Cal	1	2	3	4
Dan	3	4	2	1

Explanation: The numbers indicate what rank a man or a woman occupies in the order of preferences. For example, according to the men's ranking of the women, Al ranks Cher first, Dot second, Ann third, and Beth last. And according to the women's ranking of the men, Cher ranks

[4] The meaning of "good" will become clear presently.

[5] This assumption is introduced to simplify our task. In the book (Section 1.10) we see how to dispense with it.

[6] If Al prefers Ann to Beth and Beth to Cher, it follows that he prefers Ann to Cher. Accordingly, we may list all his preferences in a row.

Bob first, Cal second, Al third and Dan last. Thus Al ranks Cher first, while Cher ranks Al just third. If we pair them off, the match will not work out, if the first or second candidate on Cher's preference list agrees to be paired off with her.

Given each individual's preferences, can you propose a matching system for the entire community?

A Possible Proposal:
(Al – Dot , Bob – Ann , Cal – Beth , Dan – Cher)
2 x 2 2 x 2 2 x 3 2 x 4

The numbers below each couple indicate what rank one member of a couple assigns to the other member. The number on the left indicates what rank the man assigns to the woman; the number on the right, what rank the woman assigns to the man.

Argument for the Proposal:
(1) No members of any couple rank each other first.
(2) No members of any couple rank each other 1 x 2 or 2 x 1.
(3) The members of two couples rank each other second.
(4) Cal can be paired off with Cher or Beth, but he prefers Beth.
(5) That leaves Dan and Cher, who can be paired off.

This is indeed a possible proposal, but it is not a good one.
Cher is displeased, because she is paired off with her last choice. She can propose to Bob, but she will be turned down because she is his last choice. She will fare no better with Cal, because she is his third choice while he is paired off with his second choice. On the other hand, if Cher proposes to Al, he will be very pleased, because she is his first choice.
The proposal is rejected because Cher and Al prefer each other to their actual mates, and one can reasonably assume that they will reject the matchmaker's proposal.

Another Possible Proposal: Let us try to pair off all the men with their first choice.
Al's first choice is Cher.
Bob's first choice is Dot.
Cal's first choice is Ann.
Dan's first choice is Dot.

We see that there is a problem: both Bob and Dan prefer Dot. We can try to pair off Dan with his second choice, Cher, but she is already paired off with Al. Will Dan's third choice work out? Dan's third choice is Ann, but she is already paired off with Cal. That leaves Dan with his last choice, Beth.
(Al – Cher, Bob – Dot, Cal – Ann, Dan – Beth)
1 x 3 1 x 3 1 x 3 4 x 4
Three of the four men are paired off with their first choice. Do you think this proposal will be accepted or rejected?

Still Another Possible Proposal: Now we shall try to pair off all the women with their first choice. Is it possible?
Ann's first choice is Al.
Beth's first choice is Al.

Cher's first choice is Bob.
Dot's first choice is Cal.

We see that if we pair off Ann with her first choice, Al, then Beth cannot be paired off with him too. We can pair off Beth with her second choice, Bob, but he is already paired off with Cher. And Beth's third choice, Cal, is already paired off with Dot. Beth is therefore left with her last choice, Dan.

The new matching system is:
(Ann – Al , Beth – Dan , Cher – Bob , Dot – Cal)
3 x 1 4 x 4 4 x 1 4 x 1

Three of the four women are paired off with their first choice. Will they accept or reject this matching system?

Beth can fight this matching. For example, she can approach Bob and suggest that they both reject this matching and form their own pair. In so doing Beth gets her second choice—better than her fourth choice—and Bob gets his third choice—better than his fourth choice. Thus, the above matching will be rejected by Beth and Bob.

Exercise: Analyze the second proposal above and see whether it can be rejected by any pair of men and women.

The first proposal was rejected, but we can turn the failed effort to our advantage. Indeed, we have learned that a matching system must satisfy the following requirement:
A matching system must be such that under it there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates.

You will find additional explanations, definitions, theorems, proofs, and examples in the book.

2.0 Social Justice

PRESENTATION OF THE PROBLEM

In democratic society, the prevalent method of decision-making is majority rule. This method attempts to aggregate many individual views and opinions into a single social decision.

Suppose there is a community of three voters who must make a decision by choosing one of three alternatives (say, disarmament, cold war, or open war). A society that behaves rationally will establish a preference order with regard to the three alternatives on the basis of voter preferences, choosing the alternative that is the first preference. If, for example, the society establishes a preference order in which the first preference is disarmament, the second preference is cold war, and the third preference is open war, the choice will be disarmament.

Majority rule is the natural way to make a social decision on the basis of voter preferences.

Consider the following example, known as the "voting paradox."

A certain amount of municipal budget is unspent and the city council must decide how to invest it. It has three options: investment in education, investment in security, investment in health. (The sum is too small to divide feasibly among the three options.)

Sitting on the city council are representatives of three parties:

Left party: 3 members
Center party: 4 members
Right party: 5 members

The parties' list of preferences is:

Center (4)	**Left (3)**	**Right (5)**
health	education	security
security	health	education
education	security	health

The preferences are listed in the columns in descending order. For example, the Right party prefers to invest the money in security. Its second preference is investment in education, while its third preference is investment in health.

It makes little sense to vote on all alternatives together in one vote. Such a vote would result in a decision to invest in security (5 to 3 or 4), whereas there is a clear majority in favor of health over security (7 to 5), because both the Left party and the Center party prefer investment in health to investment in security. Therefore, a proposal is adopted to vote on the different alternatives in pairs ("pairwise voting").

"Security" vs. "education"—the majority is in favor of "security" (9 to 3).
"Security" vs. "health"—the majority is in favor of "health" (7 to 5).

In other words, the majority prefers "health" to "security", and it prefers "security" to "education". It therefore seems that the city council will prefer "health" to "security" and "security" to "education."

One may therefore conclude that the social preference is:
health
security
education

However, one of the council members calls for a vote on "health" vs. "education." Remarkably, it turns out that in the majority opinion, "education" is preferred to "health" (8 to 4). In this example, decision by majority leads to the absurd:

health	security	education
security	education	health

The voting results show that health is preferred to security, security is preferred to education, and education is preferred to health. We shall formulate this using a preference symbol:

health $\succ$ security $\succ$ education $\succ$ health

This relation is a cyclic preference relation because, for any alternative, there exists another alternative that is preferred to it. That is, there is no most-preferred alternative and therefore majority rule provides no clear guidance as to how to spend the budget.

This paradox has long been known. The French mathematician and philosopher Marquis de Condorcet first noted it in 1785.

It seems that in this example majority rule, which establishes a social preference on the basis of voter preferences, does not yield rational behavior.

Let us consider another way of making a social decision. The social decision on how to spend the rest of the budget will depend on the relative power of the parties.

security 5/12

health 4/12

education 3/12

Explain why this proposal will be rejected by a majority.

Another option is that the social decision will be whatever the most powerful party dictates.

security
education
health

Does this meet your intuition of a just decision?

The question is whether there is a decision-making model for society that can aggregate known personal preferences in a way that will meet our intuitive demands for a just method of decision-making.

The American economist Kenneth Arrow[7] tried to answer this question. In this chapter we discuss the results of his research.

You will find additional definitions, explanations, theorems, and proofs in the book.

3.0 The Shapley Value in Cooperative Games

Game theory has aroused interest both for its mathematical character and for its many applications to the social sciences. Game theory arises from social phenomena as opposed to physical phenomena. People act, sometimes against each other, sometimes for each other, their interests lead them to conflict or cooperation. By contrast, atoms, molecules, and stars crystallize, collide, and explode, but they do not fight or cooperate. Thus, a mathematical theory was created whose system of concepts is drawn from the social sciences.

The word "game" has different meanings for the layman and the game theorist, but the different meanings have a common denominator: the game has players and the players must interact or make decisions. As a result of the players' actions, and perhaps also by chance, the game will yield a certain outcome that is either a punishment or a reward for each one of the players. The word "player" has an unconventional meaning in that it does not necessarily signify an individual. A player can be a team, a corporation, or a state. It is convenient to refer to a group of persons having a common identifying interest and capable of making joint decisions as a single player. One might say that a *game* is a situation involving several decision-making bodies. Each decision-making body is a *player*.

[7] Arrow, K. J. 1951. *Social Choice and Individual Values*. New York: J. Wiley.

Human interactions involve many aspects, such as the capabilities of the players, their desires, their values, the role of the environment in which they function, and so on. Game theory selects a few of these aspects and constructs *mathematical models*, usually quite abstract. These models are analyzed and game theory then attempts to provide *recommendations* for behavior and possible resolutions of conflicts. As there are various issues that may be addressed, there may be various types of recommendations. Each type is called a *solution concept*.

In this chapter we shall study one class of games, called *cooperative transferable utility games*. These are games that involve a division of money among the players, and the rules of the game allow for making *binding agreements*, namely, agreements that will be honored. We shall consider a single solution concept, called the *Shapley value*, which can be regarded as a division of money that a judge or an arbitrator is likely to recommend. This Shapley value also has other interpretations which will be discussed subsequently.

COOPERATIVE GAMES

Cooperative games are games in which players enter into mutually bindings agreements. For example, economic negotiations often conclude in a contract binding on all parties, and the parties are unlikely to break the contract owing to the penalties attaching to such a breach.

In contrast, *non-cooperative* games are games in which players enter into nonbinding agreements. For example, political agreements, such as those between states, are generally nonbinding, and the parties to a political agreement honor it only for as long as it suits them.

A large part of the theory of cooperative games deals with *coalition function games*,[8] whose essential features will be discussed in this section.

The mathematical model of a cooperative game is the pair $(N; v)$, where $N = \{1,2,3, \ldots, n\}$ is the set of *players*. The coalition function v will be explained shortly.

Every subset of N is called a *coalition* and is denoted by a capital letter, e.g., S. The expression "coalition S was formed" appears often in the description of games. In theory, the meaning of this expression is that all coalition members gave their consent to the formation of the coalition. In practice, this expression has various meanings. We provide some examples:

1. I went to the store and asked for a loaf of bread (and the grocer agreed to give it to me). We may say that from the moment this (binding) agreement was made a coalition was formed between the grocer and myself.
2. A group of political parties holding a majority after elections decided to form a governing coalition until the next elections. The formation of a coalition here consists of the agreement to share the burden of power.
3. A group of investors decided to found a factory. The agreement of these investors to found a factory means that a coalition was formed between them.

Of course, coalitions usually do not take place in a vacuum. Coalition building usually requires prolonged contact between the parties, intensive negotiations, and a decision-making procedure suited to the type of agreement (e.g., profit sharing). The analysis of how players decide to behave *after* the coalition is formed is the analysis of *solution concepts*. We shall study this in the sequel.

In this chapter we assume that whenever a coalition S is formed, an amount of money $v(S)$ is generated.[9] Thus, v is a function called the "coalition function," and it assigns a real number to every coalition. The number $v(S)$ is called the *coalition worth* of S.

---

[8] More precisely, *coalition function games with side payments*, because money is often distributed among the players in these games.

[9] We assume that $v(S)$ is independent of the actions taken by the players not in S.

Example: An advertising agent approaches three individuals, 1, 2, and 3, and asks them to sign an advertisement saying that they use "Sparkle" toothpaste. The agent says that he is interested in obtaining at least two signatures. If 1 and 2 sign the advertisement, the agent will pay them a total of $100. If 1 and 3 sign it, the agent will pay them a total of $100. On the other hand, if 2 and 3 sign, the agent will only pay them a total of $50. If all three agree to sign, the agent will pay them a total of $120. In this example, the formation of a coalition means the agreement of its members to sign an advertisement.

The mathematical model is:[10]

$N = \{1,2,3\}$ $v(1,2) = 100$
$v(1,3) = 100$
$v(2,3) = 50$
$v(1,2,3) = 120$
$v(1) = v(2) = v(3) = 0$
$v(\varnothing) = 0$

You will find additional explanations, definitions, theorems, proofs, and examples in the book.

4.0 Analysis of a Bankruptcy Problem from the Talmud

One often encounters a bankruptcy situation where there are claims against a given estate and the sum of the claims against the estate exceeds its actual worth. In such situations one would like to know what would be a "fair" way of dividing the estate among the claimants.

Unfortunately, there is no clear-cut answer to this question. What seems fair in one case may seem less so in another. In this chapter we shall encounter several solutions, each shedding light on the "real world" and each applicable under certain circumstances.

We start with a curious method of division that has its origin in the Talmud,[11] which represents still another fair division. It involves a man who married three women and promised them in their marriage contract the sum of 100, 200, and 300 units of money to be given to them upon his death. The man died but his estate amounted to less than 600 units. The Mishna, attributed to Rabbi Nathan (tractate Ketubot 93a), treats the cases in which the estate was worth 100, 200, and 300 units of money. The recommendation in the Mishna is given in the following table.

Claims

Estate	100	200	300
100	33⅓	33⅓	33⅓
200	50	75	75
300	50	100	150

This recommendation of Rabbi Nathan seems strange. Why equal division if the estate is small? Why proportional division if the estate is worth 300 units? Most strangely, how did Rabbi Nathan reach the division for the case in which the estate is worth 200? Above all, what should the rule be when the worth of the estate is different and there are more widows?

[10] For simplicity, we shall omit curly brackets and write, say, $v(1,2) = 100$, instead of the more precise $v(\{1,2\})=100$.

[11] An ancient document that forms the basis for Jewish religious, criminal, and civil law. It consists of the Mishna, which is its core, and the Gemara, which discusses the Mishna and expands on it. The Mishna was put into definitive form about 1800 years ago and the Gemara was sealed about 200 years later.

Indeed, for many years this passage was not understood, and different rules of division were adopted by different rabbinic scholars. Some thought that this division reflected special circumstances whose description was neglected. Another thought that there was a spelling mistake. The wording of the Talmud itself suggests that this recommendation was not adopted, and that a different law was applied. One important rabbinic scholar, Hai Gaon, expressed the opinion that there might be some relation between this rule and the rule for dividing a garment between two claimants (see the next section). However, Rabbi Hai Gaon did not explain the relation, and eventually retracted his opinion.

Despite myriad discussions among various scholars, no solid explanation was found until quite recently. Two game theorists, R. J. Aumann and M. Maschler, examined the rule. They decided to translate the three bankruptcy problem into game models and see if known solution concepts would yield the results stated in the Mishna. To their surprise, they found that one solution concept, called the *nucleolus*, gave precisely the numbers of the above table. It seemed that, finally, an explanation of Rabbi Nathan's recommendation had been found. There was only one "minor" problem: the nucleolus was invented by D. Schmeidler[12] in 1969. It was absolutely inconceivable that Rabbi Nathan knew what the nucleolus was.[13] There had to be another explanation for the numbers in the table. A hint was found in a paper by the game theorist A. I. Sobolev, who provided a system of axioms that characterize the nucleolus.[14] One of these axioms, called *consistency*, was the right clue.

In this chapter we explain the concept of consistency and show how it yields a reasonable explanation of Rabbi Nathan's table. Moreover, it shows clearly how similar problems with more creditors and various claims can be resolved.[15]

To understand this explanation we first have to understand another, simpler Mishna rule involving a contested garment.

THE CONTESTED GARMENT

The following Mishna appears in the Talmud (tractate Bava Metzia 2a): "Two hold a garment; both claim it all. Then the one is awarded half, the other half. Two hold a garment; one claims it all, the other claims half. Then the one is awarded 3/4, the other 1/4."

We shall now discuss the claims and the decision of this Mishna. In the first case, both sides claim the whole garment and the decision establishes that in this case each claimant gets half the length of the garment.

The second case is of much greater interest to us. The one claims the whole garment and the other claims half. In this case the decision establishes that the claimant to the whole garment receives 3/4 of it and the claimant to half the garment receives 1/4.

How was this division reached? Rabbi Shlomo Yitzhaki (Rashi) interprets the decision as follows. The claimant to the garment "concedes … that half belongs to the other, so that the dispute revolves solely around the other half. Consequently, … each of them receives half the disputed amount." Thus it is decided that the division shall be 3/4 and 1/4.

In this section we shall generalize the problem to other cases.

---

[12] Schmeidler, D. 1969. "The nucleolus of a characteristic function game," *SIAM Journal of Applied Mathematics* 17: 1163–70.

[13] A description of the nucleolus is beyond the scope of this book.

[14] Sobolev, A. I. 1975. "The characterization of optimality principles in cooperative games by functional equations," in Vorobiev, N.N. (ed.), *Matematicheskie Metody v Socialnix Naukax* 6. Academy of Sciences of the Lithuanian S.S.R., Vilnius, pp. 94–151.

[15] Aumann, R. J. and Maschler, M. 1985. "Game-theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory* 36: 195–213.

Example 1
The garment is worth 100 units of money.
One claims that his share of the garment is 50 units.
The other claims that his share of the garment is 80 units.

How should they divide it?

Solution: The claimant to 50 units of money declares in effect that he has no claim to the second 50 units, and, as far as he is concerned, the other claimant can have them. The claimant to 80 units declares that he has no claim to the remaining 20 units, and, as far as he is concerned, the first claimant can have them. Thus uncontested, 70 units of the 100 units are divided. The division therefore revolves around the remaining 30 units of money, which are to be divided equally between the two. The description of the division is as follows.

Value of the garment	100	
The two claims	80	50
Uncontested division	50	20
Equal division of remainder	15	15
	65	35

The claimant to 50 units gets 35 and the claimant to 80 units gets 65.

You will find additional explanations, definitions, theorems, proofs, and examples in the book.

Conclusion

Broadly speaking, all the chapters in this book represent attempts at reaching a decision in a conflict situation and in each of them we show the difficulties in trying to define a "superior" solution. The first chapter on matching presents a "weak" condition of stability, which nevertheless yields many matchings. One of them is best for the men and another is best for the women. The second chapter tries to reach a decision by voting and we saw that a fair voting rule is not always possible. The third chapter has probably the most successful solution. It provides a solution for an unbiased arbitrator, by supplying axioms that seem fair. However, somewhat different axioms, not covered in this book, yield different solutions. Finally, the fourth chapter, which considers the case of bankruptcy conflicts, shows that even in this simple case a superior solution cannot be defined.

In conclusion, we see that various solutions are well tailored to many real situations, but there is no single solution that fits all situations. Each solution sheds some light on the reality.

Afterword

This book owes its origin to the Ph.D. thesis of co-author Ein-Ya Gura. Her dissertation "Teaching Game Theory in High School" originated with the recognition of the narrow conception of mathematics prevalent among the general public. Mathematics was perceived as mainly technical, computational, meaningless, and therefore not intellectually challenging. An important aim of the project was to contribute to the enrichment of the mathematical world of both students and intellectuals.

A mathematical world-view is formed like that of any other scientific discipline, through personal experience. The broader and richer an individual’s experience, the more enriched and profound will be the pertinent world-view. Mathematics is no different: a rich mathematical world-view will be the result of a diversified encounter with a variety of mathematical subject-matters presented in as many ways as possible.

Schools act as a crucible in which a student’s world-view of scientific disciplines is formed. Schools are responsible for the present impoverished mathematical world-view common among students and graduates. And yet schools are capable of providing the impetus for enriching and deepening the student’s scientific world-view, by means of new curricula and new approaches to instruction. It was in this spirit that game theory was offered as an elective course in Israeli high schools. Game theory exemplified a branch of mathematics that could contribute to a change in attitudes and approaches to the discipline.

Following the results and conclusions of years of research, it may be stated that my hopes for the course were realized. The course in game theory was found to contribute to the enrichment of the students’ mathematical world-view. The high school students who took the elective course in game theory exhibited an increasingly open-minded attitude towards mathematics; that is, they related to mathematics not only as a technical and computational discipline, but also as a developing and expanding world of its own. These students discovered that the world of mathematics was much richer than they had previously thought. Indeed, it appears that their encounter with a new sphere of mathematics created a greater proclivity in them to assimilate new concepts and values.

The results and the conclusions of this project inspired us to write a high school and college textbook on game theory, which was published by the Center for Science Teaching at the Hebrew University in1996(in Hebrew) . *Insights into Game Theory: An Alternative Mathematical Experience*, published by Cambridge University Press(2008) and described above, is a combination of Gura’s dissertation, the 1996 book, and revised notes in light of relevant new research.

Bibliography

Arrow, K. J. 1951. *Social choice and Individual values.* New York: J. Wiley

Aumann, R. J. and Maschler, M. 1985. “Game-theoretic analysis of a bankruptcy problem from the Talmud,” *Journal of Economic Theory* 36: 195-213

Gale, D.2001. “The two-sided matching problem: origin, development and current issues,” *International Game Theory Review* 3: 237-52

Gale, D. and Shapley, L. S. 1962. “College admissions and the stability of marriage,” *American Mathematical Monthly* 69: 9-15

Schmeidler, D. 1969. “The nucleolus of a characteristic function game,” SIAM *Journal of Applied Mathematics* 17: 1163-70

Sobolev, A. I. 1975. “The characterization of optimality principles in cooperative games by functional equations,” in Vorobiev, N. N. (ed.), *Matematcheskie Metody v Socialnix* 6. Academy of Science of the Lithuanian S. S. R. Vilnius, pp.94-151.