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# A NEW MODEL OF INTERPRETATION OF SOME ACOUSTIC PHENOMENA

## CIRCULAR HARMONIC SYSTEM – C.HA.S.

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## ABSTRACT

This article presents the results of research into sound, specifically vibrating strings and the frequency combinations which give rise to *beats*. The main aim of the research was to define a *sound scale* that takes account of the divergent *partials* 3 and 5, and to construct a reliable model in both theoretical and practical terms. At the same time there was a need for an accurate rule able to counterbalance and manage the string inharmonicity which is partly responsible for increases in frequencies of partials, giving rise to the need to stretch the 2:1 octave interval ratio. The question was how to order a scale of frequencies in proportional terms without the 2:1 ratio.

Systematic analysis of *beats frequencies* revealed a new differences constant, that is, a 1:1 *beats* ratio on harmonic partials 3 and 4. The order of sounds described here thus constitutes a set of proportional frequencies as a function of synchronic beats: a dynamic, stable and perfectly resonant system.

## 1.0 – INTRODUCTION

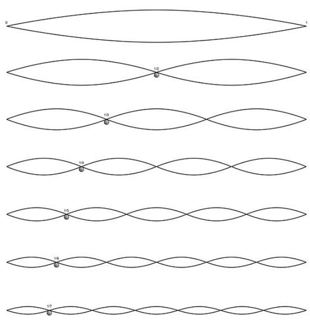
This article describes a new approach to the temperament of the chromatic scale. The linkography of selected sites[1] will enable the more demanding reader to further investigate with ease historical and theoretical aspects, partly provided here as context for presenting the study and describing the chas model.

A string, like many compounds, is made up of compact and elastic matter; *sound* can be understood as the effect of the energy passing through it. A vibrating string rearranges itself and releases superfluous energy, *sound*, which retains and conveys the imprint of the string's intrinsic structural characteristics. In this way string matter causes interference with all other vibrating matter.

The behaviour of vibrating strings and the origin of *beats* are central to the problems of ordering sounds in a scale.

## 1.1 – MODES OF VIBRATING STRINGS

A string anchored at both extremities vibrates according to its normal modes, describing first a double C, then two Ss and then increasingly complex figures, a clear natural example of auto-similar forms produced by an increasing number of *bellies* and *nodes*.



The number of *nodes* thus describes the wavelengths at which a string can vibrate.

Given that the number of nodes must be a whole number, subsequent wavelengths have ratios of 1:2, 1:3, 1:4, …1:n. This specific and natural order, the harmonic series, is infinite and was described in scientific terms in the 18th century by the physicist Sauveur[2].

Since the frequency value is inverse to wavelength, a vibrating string produces a first frequency, the fundamental, together with infinite other frequencies of ever-diminishing amplitude, known as harmonic partials, or overtones, and expressed as whole multiples of the fundamental. Thus a string producing a first fundamental frequency sound **1**, will theoretically simultaneously produce a second harmonic *partial* frequency **2**, a third *partial* frequency **3**, and so on.

The simple *combination of two sounds* thus translates into a complex juxtaposition between two fundamentals and their related partial frequencies.

## 1.2 – COMBINATIONS OF SOUNDS AND BEATS

Two sounds with *wavelengths* which relate to each other in ratios consisting of small integers, such as 2:1, 3:2 or 5:4, can in theory blend perfectly, since it is possible for the *nodes* of the respective waves to match each other.

However, two frequencies of 1.25 (5:4) and 1.5 (3:2) cannot multiply in the same scale without producing node mismatches. The series of sounds deriving from the 3:2 ratio (interval of fifth) will not match those deriving from the 5:4 ratio (interval of third) since these ratios are based on the prime numbers 2, 3, and 5, which do not have the same exponents. Three lines suffice to show this below (highlighted in red and blue).

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| FREQUENCY VALUES ***5/4** | **1** | | | | **1.25** | | | **1.5625** | | | | | **1.9531** |
| SOUNDS IN SCALE | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | | | | | THIRD | | | | | | | | |
| FREQUENCY VALUES ***2/1** | **1** | | | | | | | | | | | | **2** |
| SOUNDS IN SCALE | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| FREQUENCY VALUES *3/2: (4/3) | **1** | | **1.125** | | **1.2656** | | | **1.5** | | **1.6875** | | | |
| SOUNDS IN SCALE | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

The pure third (5:4) does not reach the octave ratio (2:1): 1*(5:4)*(5:4)*(5:4) = 1.9531

The pure fifth (3:2) and pure fourth (4:3) go beyond the third: 1*(3:2)/(4:3)*(3:2)/(4:3) = 1.265625

The *numeric differences* from the exact arithmetic ratios give rise to *beats*, perceivable as a pulse-like variation in amplitude, which have a specific frequency. When two bellies match, the two sounds add to each other (constructive interference), and when they do not, the two sounds subtract from each other (destructive interference). When two sounds are close in frequency, beat frequency will be determined by the difference between the two frequencies[3].

Generally, the greater the distance from a theoretical concurrence point of two *fundamental* or *partial* frequencies, the faster the frequency of the beat will be.

Indeed beats, which our auditory system perceives as rhythmic pulsation, faithfully reproduce the match and non-match between two bellies, and hence the *precise proportions* related to two different wavelengths.

## 1.3 – HARMONIC RATIOS – CONSONANCE AND DISSONANCE

In the west, Pythagorus is believed to have first described harmonic ratios related to the *length of a vibrating string*: 1:2 for the octave and 2:3 for the fifth.

The numbers 1 and 2, the unit and the dyad, seemed able to blend perfectly and close the interval known as *dià pason* (through all), the octave.

The pythagorean school saw these musical ratios as evidence of a universe which was fully harmonious or "consonant", and which could be interpreted through the relationships between small numbers. The octave interval, theorised as a 2:1 ratio, has since been considered the most harmonious.

The history of music has always been influenced – and there is still debate over to what extent and how – by factors of harmonic consonance and dissonance[4]. This is because various levels of consonance distinguish sound combinations in the scale and arouse differing sensations and states of mind in the listener. We could also say that the converse statement is true: that it is listeners' sensations that define the level of consonance between two sounds. In general consonance is synonymous for the listener with calm or relaxation, while dissonance is synonymous with turbulence or tension, thus producing the dichotomy between a static and a dynamic principle.

As with any debatable issue, consonance and dissonance have led to different approaches and theories relating to harmony and musical scales.

**1.4 - MUSICAL SCALES**

Two facts have considerably affected the development of a solid theory of temperament: the apparent impossibility of combining, in scale, ratios deriving from prime numbers, and the great consonance deriving from the theoretical concurrence of partials 2, 3 and 5 (and the related 2:1, 3:2 and 5:4 ratios).

The simple harmonic ratios of octave and fifth enabled the Pythagoreans to build the first diatonic scale, the succession of 7 notes on which the western system of music is built.

In the following centuries, renowned mathematicians such as Archytas, Philolaos, Didymus, and Ptolomy, contributed to the adoption of simple ratios for the other intervals. Thus the so-called natural scale[5] came into being, formalised in the 16th century by Gioseffo Zarlino[6], and built on the ratios of 2:1, 3:2, 4:3, 5:4 and 6:5, respectively for the octave, fifth, fourth, major third and minor third intervals.

As early as the 12th century, the use of poliphony and more complex combinations of sound raised the question still under debate today: how was it possible to make all chords melodious?

**1.5 - TEMPERAMENTS AND THE STATE OF THE ART**

Concepts of melody and harmony over the ages have influenced the search for new temperaments, which strove to match new styles of composition[7] with broader sound horizons. This happened as tonal music was formalised.

Moving away from a pure ratio involved a loss of consonance, while favouring such a ratio caused strong dissonance in at least one interval, known for this reason as the "wolf fifth". The only solution seemed to be to make the best of a bad job, and avoid as far as possible those scale differences responsible for beats.

Within the pure octave, the scale initially calculated as a function of a 3:2 ratio (Pythagorean scale) was reordered as a function of a 5:4 ratio (meantone temperament). A variety of irregular temperaments in the 17th century maintained the 2:1 ratio, aiming to facilitate modulations in all keys. Today more than a hundred temperaments are identifiable, for scales which contain from 12 to as many as 665 notes in an octave[8].

The current system was developed at the end of the 17th century: it maintains the pure octave and distributes the so-called commas, or differences produced by the 3:2 and 5:4 ratios, equally across 12 semitones. This equal temperament introduces a compromise: it multiplies the first frequency and subsequent frequencies by $2^{(1/12)}$ so that the 1st and 13th frequency, in the 0-12 combination, have a ratio of 1:2.

In this way, the scale of sound values is formed of natural numbers which are multiples of 2, and, for the first time, of algebraic numbers in a logarithmic progression, that suggests Jakob Bernoulli's spira mirabilis[9] and can also contain the $5^{(1/2)}$ component of the golden ratio (section 4.3) .

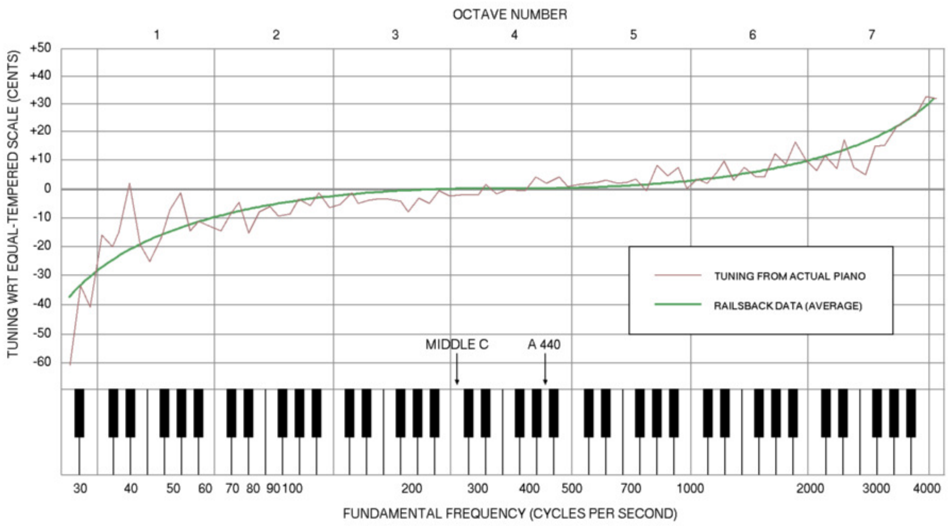
There is still debate between the supporters and detractors of this solution. There is no doubt, however, that all differences are subject to partial 2, a solution which today may appear difficult to justify.

Here it is appropriate to look at the results of more recent studies describing inharmonicity, the phenomenon correlated to string rigidity.

**1.6 – STRING INHARMONICITY**

The term inharmonicity describes the deviation of partial frequencies from the natural values of the harmonic series. String rigidity is one of the causes of this phenomenon. String length, diameter, density and tension all contribute to calculating inharmonicity. The phenomenon, discovered last century, obliges the 2:1 octave ratio to be stretched.

Railsback[10] measured average deviation from the 2:1 ratio in the pianoforte; from the lower sounds, the curve gradually flattens toward the middle sounds, where the degree of inharmonicity is slight, and again grows as the notes become higher.




RAILSBACK CURVE –

String parameters vary in all instruments, and this means that correction of any theoretical order of frequencies in scale will be required. This may explain loss of momentum in the search for a semitone temperament able to reach beyond the limitations of equal temperament in terms of both logic and efficiency.

The chas octave deviation curve is in line with the Railsback curve, as shown below (section 4.2).

**2.0 - THE CHAS MODEL APPROACH – SEMITONE TO MICROTONE**

We know that pure ratios between small whole numbers determine harmonious sounds.

However building a chromatic scale as a function of ratio 3:2 creates excessively high frequencies for the thirds and octaves, just as ratio 5:4 creates frequencies which are excessively low for fifths and octaves (section 1.2). In other words, pure fifths and thirds do not help the octave.

So it remains to be understood with what logic the pure octave could help fifths and thirds.

We also know that string rigidity causes a deviation of scale frequencies from pure ratios (section 1.6).

Thus two questions arise. The first: is it correct to theorise that the octave interval must have a 2:1 ratio? The second: which temperament model today is reliable in theoretical terms and is commonly applied in the practice of tuning?

We see that although equal temperament finds a compromise between ratios 3:2 and 5:4, it bases distribution of frequencies on the pure octave, that is the 2:1 ratio, a proportion with no logical or practical justification.

The chas model approach starts from the traditional chromatic scale, but brings innovation to the theory and practice of tuning by recognising that beats are as natural for octaves as they are for the other intervals. Octaves, too, can and must be tempered, exactly as fifths and thirds have been. Thus the need arises to combine partials 2, 3 and 5 in a new set.

It is the differences and therefore the beats that constitute the real potential of the chas model.

A set of sounds in any scale, chromatic or of any other type, can achieve extraordinary resonance by drawing on the potential of proportional beats, a resource that every element which is part of the sound set can share to the full.

Purity no longer derives from a single combination or from a pure ratio, but from a new set which is pure because it is perfectly congruent and coherent.

The sounds in the scale all give up a small part of their pure partial value for the benefit of this set which is now harmonic and dynamic since it is the result of a natural, intrinsic correlation between frequencies and beats frequencies.

In conceptual terms, the model is trans-cultural; it also responds to a new requirement on the contemporary music scene, by providing an algorithm which can give form to all kinds of microtonal sound structures. The model provides a correct logical approach to ordering a scale system and a sample of proportionality from which an infinite variety of new sound combinations can be drawn, to create new music compounds.

## 3.0 - DESCRIPTION OF THE CHAS MODEL

The chas model discards two unjustified assumptions: that the range of the scale module must be 12 semitones, and that the octave, the 12th semitone, must be double the first note.

The system develops from the following observations:

Firstly, *pure intervals are not essential*: slight frequency deviations from pure harmonic ratios do not upset the ear. In addition, string rigidity causes inharmonicity (section 1.6), making the partials higher and consequently never calculable in pure proportion.

Secondly, *differences translate into beats* (section 1.2) and these determine the harmoniousness of two or more simultaneous sounds. *Beats frequencies* produce a specific rhythm and this rhythm expresses the effect of *combinations of fundamentals and partials in time*. Hence a proportional ratio must apply for the frequencies as well as for the differences between the partials of the harmonic series and real scale frequencies.

Thirdly, a *12-semitone module is not sufficient*. This is clearly demonstrated by the fact that harmonic 3 corresponds to a sound situated beyond semitone 12, at semitone 19. Moreover, when commas are distributed only within a pure octave, the differences of intervals greater than an octave will always occur in proportion 2 (section 4.3).

Fourthly, a *double 12-semitone module is necessary*[11]. A two-octave module gives the scale set an *intermodular quality*. From the minor second degree to the nth degree, all intervals will now find their exclusive identity.

The chas scale model starts, then, from these fundamental principles:
the sound scale constitutes a system-set that is dynamic since it is defined in time by the specific rhythm of beats.
In this set the ratio must be identifiable both in the single elements forming the scale foreground, as well as in the differences arising from the infinite combinations of its elements, and forming the background.
Each frequency or element in the scale must contain and bear witness to this bi-frontal ratio, which is pure in that it is natural, exactly proportional and perfectly synchronic.
The solution to how to proceed without the pure 2:1 constant arises from analysis of ratio 3:2. This ratio marks off a span of 7 semitones in the octave module, the small set known as the dominant. It consists of 8 sounds and can create all the scale intervals: it contains ratios 6:5 and 5:4, and with 4:3, by convention, also the 2:1 ratio.
In a scale of sounds, the slightest deviation from the 3:2 ratio, which gives the first interval from which the semitone matrix can be extracted, will resound through all the other intervals, just as the slightest variation in a single lever modifies an entire system of levers. This fact suggested the existence of a correct differences constant.
Accurate synchronisation of beats, achieved through direct experimentation, led to two new coordinates: the differences produced by the two combinations 0-19 and 0-24, related to harmonics 3 and 4, now calculable in a ratio of 1:1.

### 3.1 - THE CHAS ALGORITHM

In the chromatic scale, just as 2 is the harmonic partial corresponding to semitone 12, so 3 corresponds to semitone 19, 4 to semitone 24, and 5 to semitone 28.
Where the equal temperament formula 2^(1/12) employs harmonic partial 2 and its related scale position 12, the chas algorithm employs an equation between two algebraic expressions with 2 different harmonic partials, 2 related scale positions, and 2 variables: **Δ** and **s**. The **Δ** variable stands for the differences and appears in both expressions; the scale positions 19 and 24 determine period, module size and interval size:

$$(3-\Delta)\hat{}\,(1/19) = (4+\Delta)\hat{}\,(1/24) \qquad (1)$$

### 3.2 - THE DELTA VARIABLE

In the chas algorithm, the **Δ** variable proportions the differences of two intervals, 8th+5th (12$^{th}$ degree) and 8$^{th}$+8$^{th}$ (15$^{th}$ degree), that is, combinations 0-19 and 0-24. These intervals have constant differences from their respective partials 3 and 4.

The delta variable obtains two differences, in a 1:1 ratio, equal in value and opposite in sign, **one negative and one positive** (0-19 negative and 0-24 positive).
A solution to the chas equation is:

$$\Delta = 0.0021253899646... \quad (2)$$

Substituting this value for **Δ** in (1) gives the incremental factor for frequencies in the scale:

$$(3 - 0.0021253899646)\hat{}\,(1/19) = (4 + 0.0021253899646)\hat{}\,(1/24) = 1.0594865443501... \quad (3)$$

The incremental factor is the constant ratio in the scale; delta represents the differences in constant ratio 1:1.
The 1:1 proportion of the differences related to intervals 0-19 and 0-24 is constant for all degrees 12 and 15. Their ratio, in this exponential scale, expresses a constant of linear proportionality which we find in the chromatic combinations (1-20, 1-25) – (2-21, 2-26) – (3-22, 3-27) etc.
The chas model uses the delta variable to extend the distances between the natural values of the harmonic series.
The delta variable relates to partial frequencies in such a way as to make every sound in the scale equally powerful and thus perfectly adapted for every interval, and ready for any combination. In this system every frequency becomes resonance potential.
Two homogeneous size classes are obtained: frequencies and differences. Any combination of frequencies obtains differences with just one set ratio.

### 3.3 - THE S VARIABLE

Equal temperament's geometric progression, when clear of unjustified premises, suggested infinite exponential curves related to oscillations of partial values, and identifiable through a second variable, expressing an "elastic" potential and *enabling the system to evolve*.
When we add in the **s** variable, a rational number, (**s** from the concepts of stretching, swinging and spinning), equation (1) becomes:

$$(3 - \Delta)\hat{}\,(1/19) = (4 + (\Delta * s))\hat{}\,(1/24) \quad (4)$$

The **s** variable can change the delta value and swing the logarithmic scale from ratio 3:2 to ratio 5:4, <u>including ratio 2:1 of the equal temperament scale</u>. The variable affects the distances and proportion of scale values related to partials 2, 3, 4 and 5, and obviously also the distances and logarithmic differences of all possible sound combinations.

If **s** is a fraction (s/s1), the delta values will change so that:

$$(3 - \Delta)\hat{}\,(1/19) = (4 + (\Delta * s / s1))\hat{}\,(1/24) \quad (5)$$

equals in value:

$$(3-(\Delta * s1))^\wedge(1/19) = (4+(\Delta * s))^\wedge(1/24) \qquad (6)$$

In fact, in equation (5) we can select values for s and s1, and calculate the value for delta that makes the equality true. In equation (6), we keep the same values of s and s1 and compute a different delta value that makes the equality true. The resulting incremental factor will not change.

s < 0 scale value of semitone 12 less than ratio 2:1
Incremental factor < 1.0594630943593..(equal temp. value) stretches to (5:4)^(1/4)

**s = 0** scale value of semitone 12 has ratio 2:1
Incremental factor (equal temperament) = 1.0594630943593…

0 > s < 1 scale value of semitone 12 greater than ratio 2:1

**s = 1** scale value of semitone 12 has chas ratio 2.00053127693…:1
Incremental factor (chas) = 1.0594865443501…

s > 1 scale value of semitone 12 greater than ratio 2.00053127693…:1
Incremental factor > 1.0594865443501… (chas value), stretches to (3:2)^(1/7).

The two variables **Δ** and **s** push not only the ratio for partial 2 but also for 5:4, 4:3, 3:2, 3 and 4, a mix of natural and rational numbers, thus translated into a new set of scale values.

For **s = 1** the scale values related to partials 3 and 4 add up to 7.

The *infinite scale oscillations* are all within the chas attractor which, in this version, has a period of 19*24 = 456. In a unique way, the figures 4, 5 and 6 correspond to the harmonic partials which form the first major triad.

### 3.4 - CHAS SET: THE SYMMETRY OF BEATS

Until now, the pure octave has generally consisted of a module of 13 elements, from 0 to 12.

The chas model opens up a module of 49 sound elements, in a semitonal order, from 0 to 48, whose scale ratio is:

$$\mathbf{(4+\Delta)^2}$$

With the ratios (3 – Δ) and (4 + Δ), the combinations between elements **0-19, 5-24** and **0-24** obtain constant, symmetrical beats frequencies with respect to the combinations between elements **29-48, 24-43** and **24-48**. The module appears perfectly balanced: the delta variable makes element 24 a *stable centre* in absolute terms:

$$\mathbf{0} *(3-\Delta) \rightarrow \mathbf{19} \rightarrow \mathbf{24} \rightarrow \mathbf{29} *(3-\Delta) \rightarrow \mathbf{48}$$

$$0 \rightarrow \mathbf{5} *(3-\Delta) \rightarrow \mathbf{24} *(3-\Delta) \rightarrow \mathbf{43} \rightarrow 48$$

$$\mathbf{0} *(4+\Delta) \rightarrow \mathbf{24} *(4+\Delta) \rightarrow \mathbf{48}$$

**5** → 24→ **29**
I--*(4 + Δ)--I

**12** → 24→ **36**
I--*(4 + Δ)--I

**19** → 24→ **43**
I--*(4 + Δ)—I

| **Combinations** | **0→5→12→19→24→ 29→36→43→48** | **Ratios** |
|---|---|---|
| 0 – 5, 19-24, 24-29 e 43-48 | I----I I----IOI----I I-----I | (4 + Δ)/(3 – Δ) |
| 0 –19 e 29-48 | I--------------I O I----------------I | (3 – Δ) |
| 5 – 24 e 24- 43 | I--------------------O----------------I | (3 – Δ) |
| 0 – 24 e 24- 48 | I--------------------O---------------------I | (4 + Δ) |
| 5 – 29 e 19- 43 | I---------I===O===I----------I | (4 + Δ) |
| 12 –36 | I----------O----------I | (4 + Δ) |
| **0 – 48** | **O------------------------------------------O** | **$(4 + \Delta)^2$.** |

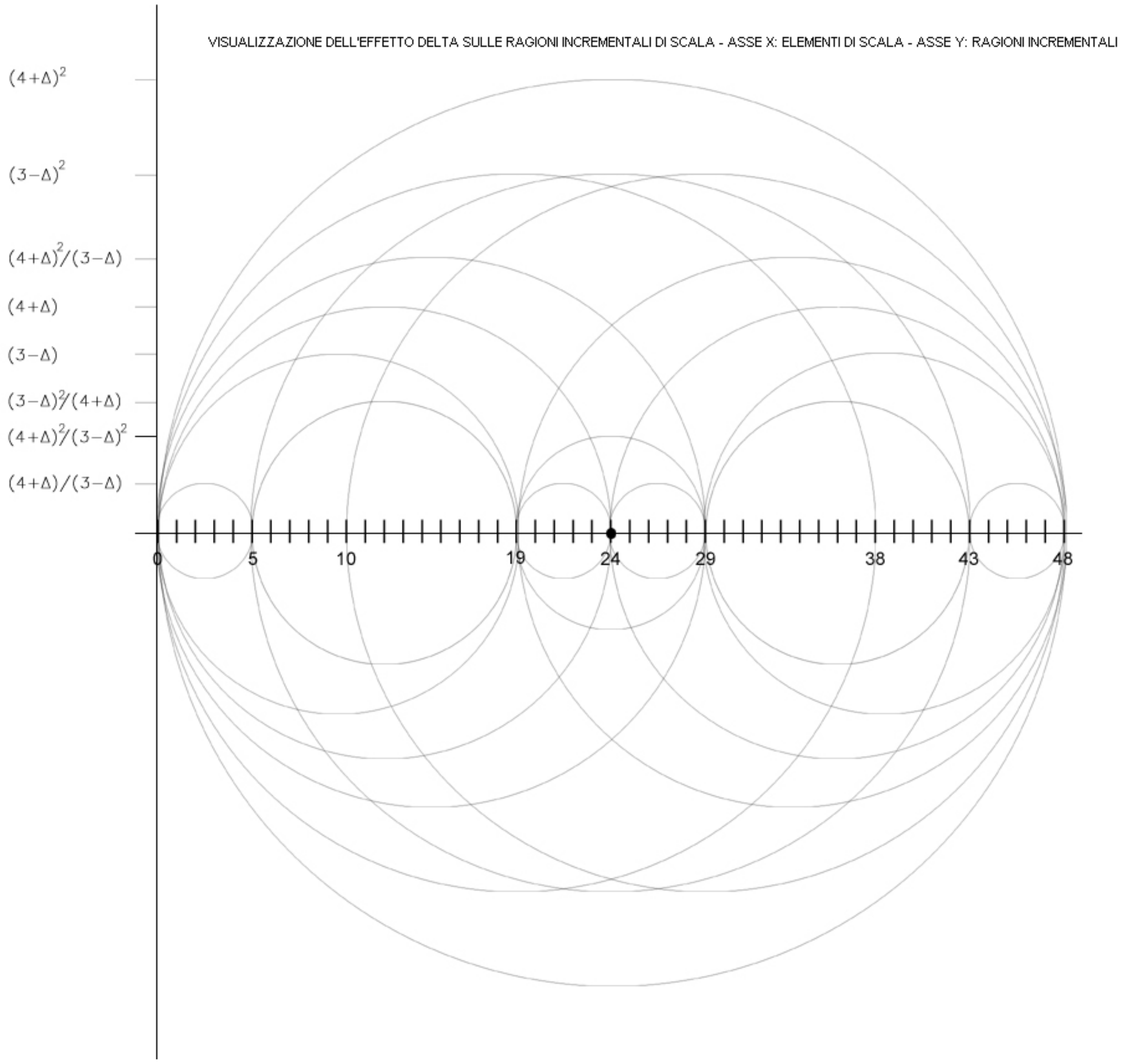
Figure showing the equilibrium and stability of the set described by the chas model [12]

### 3.5 - EFFECT OF ±DELTA ON INCREMENTAL RATIOS

x represents scale elements (spaced in cents), y represents incremental ratios of scale per degree elevation. List of scale degrees (in ascending order), of ratios, and of number of related elements:

| | | | |
|---|---|---|---|
| 4th | = (4 + Δ) / (3 – Δ) | no. elements | 6 - span from 0 to 5 |
| 7thm | = (4 + Δ)^$^2$ / (3 – Δ)^$^2$ | " " | 11 - " " 0 " 10 |
| 9th | = (3 – Δ)^$^2$ / (4 + Δ) | no. elements | 15 – span from 0 to 14 |
| 12th | = (3 – Δ) | " " | 20 - " " 0 " 19 |
| 15th | = (4 + Δ) | " " | 25 - " " 0 " 24 |
| 18th | = (4 + Δ)^$^2$ / (3 – Δ) | " " | 30 - " " 0 " 29 |
| 23rd | = (3 – Δ)^$^2$ | no. elements | 39 - span from 0 to 38 |
| 29th | = (4 + Δ)^$^2$ | " " | 49 - " " 0 to 48 |

Effect of delta on incremental scale ratios
X = scale elements
Y = incremental ratios




**4.0 - THE SEMITONE SCALE –**

**IN ALL FIGURES: s = 1**

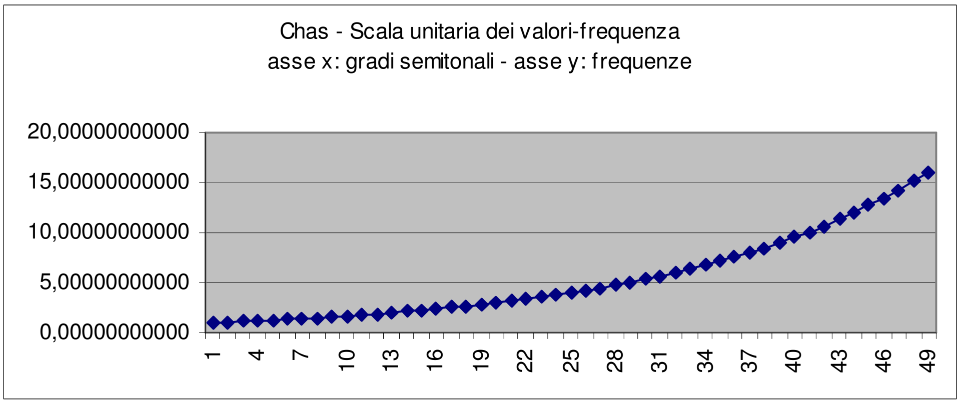
**TABLE 1**

*In the foreground*, in the values representing scale frequencies, we find the incremental logarithmic ratio: 1.0594865443501…

| | Chas scale | Scale of Frequencies | Cents | |
|---|---|---|---|---|
| Degrees | chas values | chas values | Offset in Cents | Semitone in Cents |
| I | 1.00000000000000 | 440.00000000000 | 0.00 | |
| | **1.05948654435010** | 466.17407951404 | 0.04 | 100.038318440222… |
| | 1.12251173765892 | 493.90516456992 | 0.08 | 100.038318440222… |
| IIIm | 1.18928608192467 | 523.28587604686 | 0.11 | 100.038318440222… |
| III | 1.26003260118204 | 554.41434452010 | 0.15 | 100.038318440222… |

| | | | | |
|---|---|---|---|---|
| IV | 1.33498758639483 | 587.39453801372 | 0.19 | 100.038318440222… |
| IV+ | 1.41440138465974 | 622.33660925028 | 0.23 | 100.038318440222… |
| V | 1.49853923535714 | 659.35726355714 | 0.27 | 100.038318440222… |
| | 1.58768215604158 | 698.58014865830 | 0.31 | 100.038318440222… |
| | 1.68212788103081 | 740.13626765356 | 0.34 | 100.038318440222… |
| | 1.78219185582829 | 784.16441656445 | 0.38 | 100.038318440222… |
| | 1.88820829070040 | 830.81164790818 | 0.42 | 100.038318440222… |
| VIII | **2.00053127692738** | 880.23376184805 | 0.46 | 100.038318440222… |
| | 2.11953596945608 | 932.59582656068 | 0.50 | 100.038318440222… |
| IX | 2.24561983990476 | 988.07272955810 | 0.54 | 100.038318440222… |
| | 2.37920400410472 | 1046.84976180608 | 0.57 | 100.038318440222… |
| X | 2.52073462861283 | 1109.12323658965 | 0.61 | 100.038318440222… |
| | 2.67068442089264 | 1175.10114519276 | 0.65 | 100.038318440222… |
| | 2.82955420814119 | 1245.00385158213 | 0.69 | 100.038318440222… |
| XII **(3-Δ)** | **2.99787461003480** | **1319.06482841531** | 0.73 | 100.038318440222… |
| | 3.17620781098067 | 1397.53143683150 | 0.77 | 100.038318440222… |
| | 3.36514943779371 | 1480.66575262923 | 0.80 | 100.038318440222… |
| | 3.56533054906974 | 1568.74544159069 | 0.84 | 100.038318440222… |
| | 3.77741974289974 | 1662.06468687589 | 0.88 | 100.038318440222… |
| XV **(4+Δ)** | **4.00212538996469** | **1760.93517158446** | 0.92 | 100.038318440222… |

**Graph 1** - Chas scale



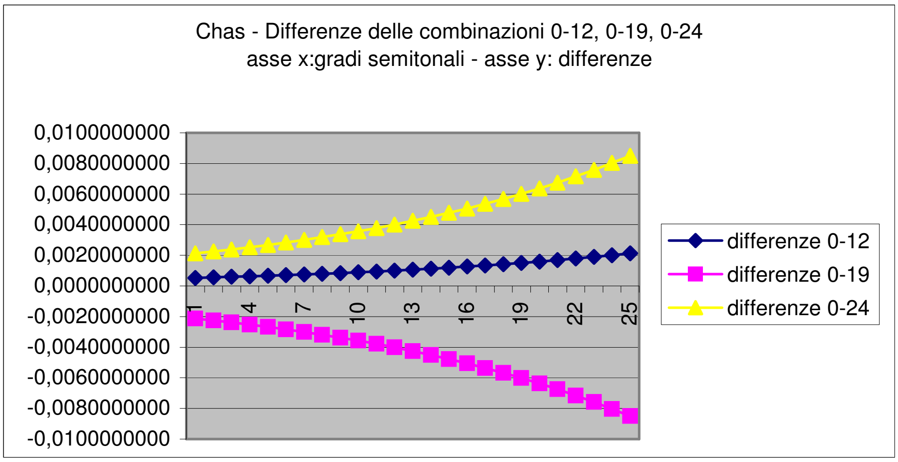

Chas scale frequency values
x = semitone degrees
y = frequencies

## 4.1 - CHAS DIFFERENCES

*In the background* we find the difference values in 1:1 proportion: ± 0.002125389965…
**TABLE 2** - Differences between chas scale values and related partials 2, 3 and 4

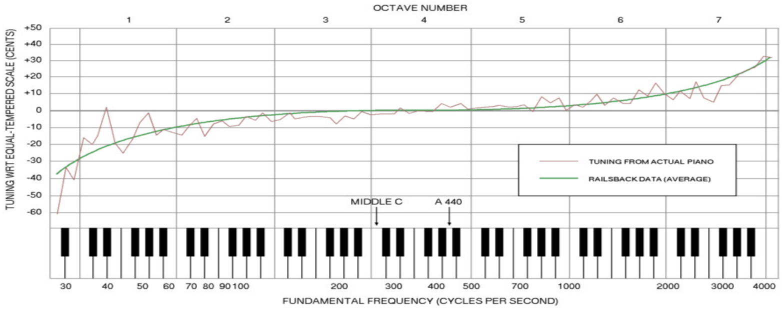
| Partial 2 differences | Partial 3 differences | Partial 4 differences |
|---|---|---|
| 0.0005312769 | **-0.002125389965** | **0.002125389965** |
| 0.0005628808 | -0.002251822070 | 0.002251822069 |
| 0.0005963646 | -0.002385775183 | 0.002385775182 |
| 0.0006318403 | -0.002527696704 | 0.002527696704 |
| 0.0006694262 | -0.002678060646 | 0.002678060646 |
| 0.0007092481 | -0.002837369220 | 0.002837369219 |
| 0.0007514388 | -0.003006154510 | 0.003006154509 |
| 0.0007961393 | -0.003184980253 | 0.003184980253 |
| 0.0008434989 | -0.003374443722 | 0.003374443722 |
| 0.0008936757 | -0.003575177719 | 0.003575177718 |
| 0.0009468374 | -0.003787852686 | 0.003787852686 |
| 0.0010031615 | -0.004013178953 | 0.004013178952 |
| 0.0010628361 | -0.004251909101 | 0.004251909100 |
| 0.0011260606 | -0.004504840480 | 0.004504840479 |
| 0.0011930460 | -0.004772817873 | 0.004772817872 |
| 0.0012640162 | -0.005056736315 | 0.005056736314 |
| 0.0013392081 | -0.005357544085 | 0.005357544083 |
| 0.0014188730 | -0.005676245868 | 0.005676245867 |
| 0.0015032769 | -0.006013906120 | 0.006013906119 |
| 0.0015927016 | -0.006371652613 | 0.006371652612 |

**Graph 2:** Chas differences curves related to partials 2, 3, and 4




Chas – Differences of combinations 0-12, 0-19, 0-24
x = semitone degrees
y = differences

## 4.2 – COMPARISON BETWEEN RAILSBACK CURVE AND CHAS OCTAVE CURVE

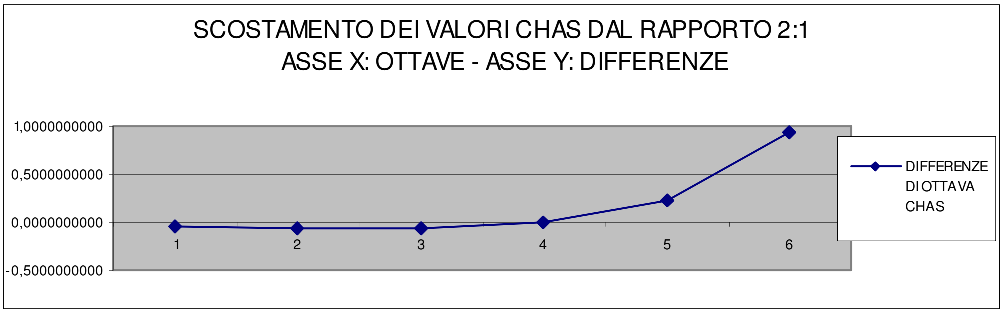



-
RAILSBACK CURVE-

**TABLE 3**

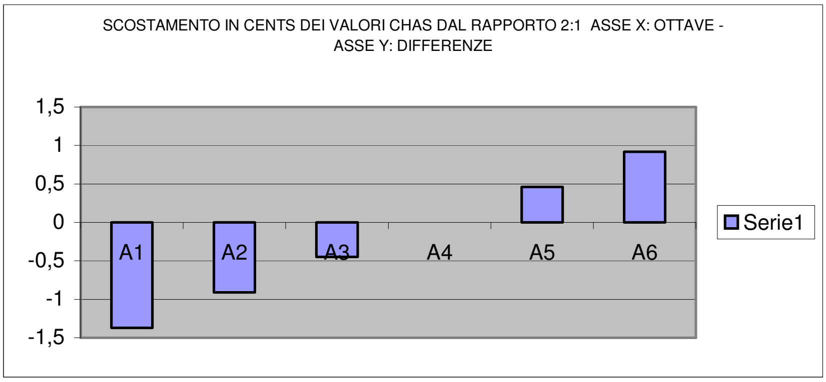
| EQUAL VALUES | CHAS VALUES | CHAS DEVIATION (Hz) | CHAS DEVIATION (CENTS) |
|---|---|---|---|
| 55.0000000 | 54.956192929 | -0.0438070708 | -1.37 |
| 110.0000000 | 109.941582816 | -0.0584171842 | -0.91 |
| 220.0000000 | 219.941575058 | -0.0584249421 | -0.45 |
| **440.0000000** | **440.000000000** | 0.0000000000 | 0.00 |
| 880.0000000 | 880.233761848 | 0.2337618481 | 0.46 |
| 1760.0000000 | 1760.935171585 | 0.9351715846 | 0.92 |

**Graph 3** – Chas deviations from ratio 2:1 (Hz)




x = octaves
y = differences

**Graph 4** – Chas deviations from ratio 2:1 (cents)



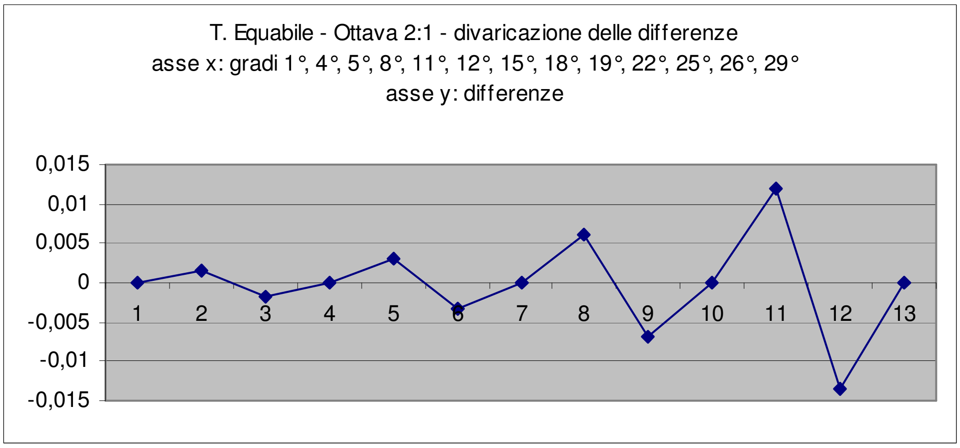

x = octaves y = differences

### 4.3 – COMPARISON BETWEEN EQUAL TEMPERAMENT AND CHAS DIFFERENCES FOR RATIOS 4:3 AND 3:2

In the equal temperament scale, based on a ratio of 2, octave intervals have zero differences. As a direct consequence, the differences for partials other than 2 have ratios which are multiples of 2.

The differences, divided by themselves, have a quotient of 2 for combinations 0-12, a quotient of 4 for combinations 0-24, and so on. With the exclusion of partial 2 and its multiples, the difference curves relating to all the other partials move away from each other exponentially in a monotone curve.

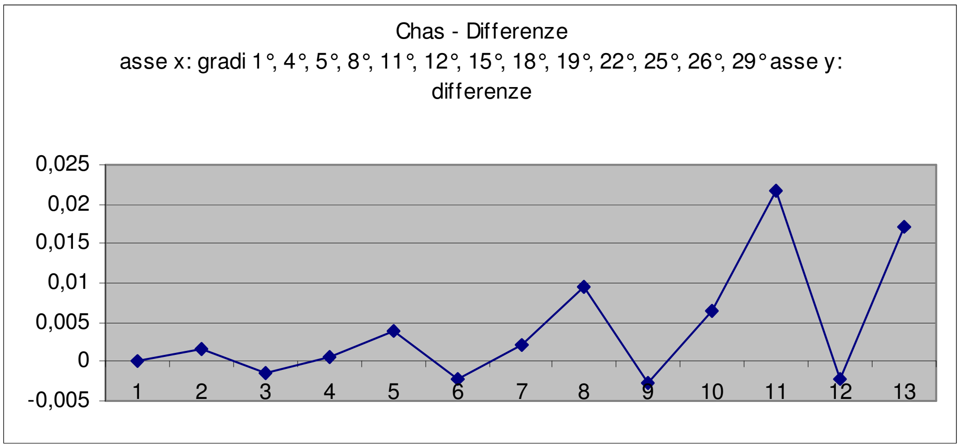
**Graph 5** - Exponential divarication of equal scale differences.




**Equal temperament - Octave 2:1** – divarication of differences
x = degrees 1, 4, 5, 8, 11, 12, 15, 18, 19, 22, 26, 29
y = differences

In the chas frequency scale the differences curves describe the exact form ordered by **the incremental ratio and by the difference ratio.**
This substantiates the optimisation of beats and the absolute coherence of the chas form.

**Graph 6 -** Exponential progression of chas differences



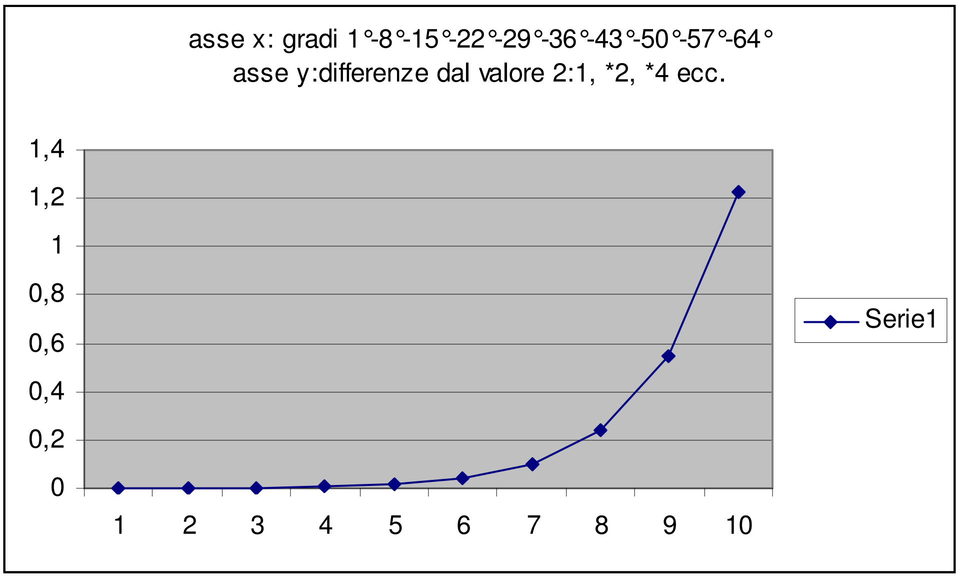

Chas differences
x = degrees 1, 4, 5, 8, 11, 12, 15, 18, 19, 22, 26, 29 y = differences

### 4.4 – SCALE VALUES IN RATIO 1:2, 1:4, 1:8 ETC AND RELATED CHAS VALUES

**TABLE 4: Differences**

| Natural octaves 2:1 | Chas octaves | Differences | |
|---|---|---|---|
| 1 | 1 | 0 | DIFF: 00 |
| 2 | 2.000531277 | **0.000531277** | DIFF: 01 |
| 4 | 4.002125390 | **0.002125390** | DIFF: 02 |
| 8 | 8.006377018 | **0.006377018** | DIFF: 03 |
| 16 | 16.017007639 | **0.017007639** | DIFF: 04 |
| 32 | 32.042524746 | **0.042524746** | DIFF: 05 |
| 64 | 64.102072949 | **0.102072949** | DIFF: 06 |
| 128 | 128.238201856 | **0.238201856** | DIFF: 07 |
| 256 | 256.544533718 | **0.544533718** | DIFF: 08 |
| 512 | 513.225363647 | **1.225363647** | DIFF: 09 |

**Graph 7** – Differences (in bold above) of chas values, related to partials **2:1, 4:1, 8:1** etc.




x = degrees 1, 8, 15, 22, 29, 36, 43, 50, 57, 64
y = differences from value 2:1, ***2, *4** etc

**TABLE 5: Deviation of chas octave values in scale where A4 = 440.0 Hz**

| Octaves in ratio 2 | Octaves in CHAS ratio | Deviation |
|---|---|---|
| 27.5 | 27.4707991637 | -0.02920083627 |
| 55.0 | 54.9561929292 | -0.04380707077 |
| 110.0 | 109.9415828157 | -0.05841718430 |
| 220.0 | 219.9415750579 | -0.05842494211 |
| 440.0 | 440.0000000000 | 0.00000000000 |
| 880.0 | 880.2337618481 | 0.23376184808 |
| 1760.0 | 1760.9351715846 | 0.93517158460 |
| 3520.0 | 3522.8058873966 | 2.80588739660 |

**4.5 – SEQUENCE OF QUOTIENTS DERIVING FROM DIFFERENCES BETWEEN PROGRESSION IN RATIO 2 AND PROGRESSION IN CHAS RATIO 2.0005312…**

Generally if in a logarithmic scale we deviate from 2^(1/12), the combinations 0-12, 0-24, 0-36 etc. will produce differences for ratios 1:2, 1:4, 1:8, etc.

If we divide these differences by each other, we see that the quotients return values close to the *n/n+1* sequence of ratios typical of a vibrating string.

In the chas model the quotients of the differences for ratio 1:2, 1:4, 1:8 etc. return values which are very close (7th decimal point) to n/n+1 ratios. Every scale frequency value, in infinite combinations, returns these harmonic quotients for itself and for the set.

Using Table 4 (section 4.4), we divide the 01 difference by the 02 difference, the 02 difference by the 03 difference and so on, to obtain the quotients:

| | | | |
|---|---|---|---|
| DIFF. 01/ DIFF. 02 = | 0.000531277 : | 0.002125390 = $q_{00}$ | 0.249973857 |
| DIFF. 02/ DIFF. 03 = | 0.002125390 : | 0.006377018 = $q_{01}$ | 0.333289064 |
| DIFF. 03/ DIFF. 04 = | 0.006377018 : | 0.017007639 = $q_{02}$ | 0.374950195 |
| DIFF. 04/ DIFF. 05 = | 0.017007639 : | 0.042524746 = $q_{03}$ | 0.399946872 |
| DIFF. 05/ DIFF. 06 = | 0.042524746 : | 0.102072949 = $q_{04}$ | 0.416611323 |
| DIFF. 06/ DIFF. 07 = | 0.102072949 : | 0.238201856 = $q_{05}$ | 0.428514501 |
| DIFF. 07/ DIFF. 08 = | 0.238201856 : | 0.544533718 = $q_{06}$ | 0.437441884 |
| DIFF. 08/ DIFF. 09 = | 0.544533718 : | 1.225363647 = $q_{07}$ | 0.444385403 |
| DIFF. 09/ DIFF. 10 = | 1.225363647 : | 2.723392125 = $q_{08}$ | 0.449940218 |

After further dividing the quotients, we can compare the results with the n/n+1 values in table 6:

| | | |
|---|---|---|
| $q_{00} : q_{01}$ | = | 0.249973857 : 0.333289064 |
| $q_{01} : q_{02}$ | = | 0.333289064 : 0.374950195 |
| $q_{02} : q_{03}$ | = | 0.374950195 : 0.399946872 |
| $q_{03} : q_{04}$ | = | 0.399946872 : 0.416611323 |
| $q_{04} : q_{05}$ | = | 0.416611323 : 0.428514501 |
| $q_{05} : q_{06}$ | = | 0.428514501 : 0.437441884 |
| $q_{06} : q_{07}$ | = | 0.437441884 : 0.444385403 |
| $q_{07} : q_{08}$ | = | 0.444385403 : 0.449940218 |

**TABLE 6: Comparison between quotients**

| $q_n : q_{n1}$ Chas | n/n+1 – fraction | n/n+1 – decimal value |
|---|---|---|
| 0.750000004408948 | **3/4** | 0.750000000000 |
| 0.888888894114416 | **8/9** | 0.888888888889 |
| 0.937500005511253 | **15/16** | 0.937500000000 |
| 0.960000005643583 | **24/25** | 0.960000000000 |
| 0.972222227937747 | **35/36** | 0.972222222222 |
| 0.979591842493441 | **48/49** | 0.979591836735 |
| 0.984375005786949 | **63/64** | 0.984375000000 |
| 0.987654326793812 | **80/81** | 0.987654320988 |

Now we see that the logarithmic scales built on the ratios 3:2, 5:4, 3:1 and 5:1 also determine differences from ratios 2:1, 4:1, 8:1 etc. which return the same quotient sequence.

**((3/2)^(1/7)) = 1.0596340226671…**

| Differences from ratios 2:1, 4:1 etc. - | Quotients | |
|---|---|---|
| 0.0038754738 | 0.249758017 | 0.750000234 |
| 0.0155169145 | 0.333010586 | 0.888889166 |
| 0.0465958596 | 0.374636792 | 0.937500293 |
| 0.1243760907 | 0.399612453 | |
| 0.3112417786 | | |

**((5/4)^(1/4)) = 1.0573712634406…**

| Differences from ratios 2:1, 4:1 etc. – | Quotients | |
|---|---|---|
| -0.0468750000 | 0.252964427 | 0.750035151 |
| -0.1853027344 | 0.337270095 | 0.888930544 |
| -0.5494194031 | 0.379411077 | 0.937543924 |
| -1.4480847716 | 0.404686188 | |
| -3.5782905696 | | |

**3^(1/19) = 1.05952606473828…**

| Differences from ratios 2:1, 4:1 etc. - | Quotients | |
|---|---|---|
| 0.00142693 | 0.24991085 | 0.75000003 |
| 0.00570977 | 0.33321445 | 0.88888893 |
| 0.01713542 | 0.37486624 | 0.93750004 |
| 0.04571076 | 0.39985731 | |
| 0.11431768 | | |

**5^(1/28) = 1.0591640081942…**

| Differences from ratios 2:1, 4:1 etc. – | Quotients | |
|---|---|---|
| -0.00676468 | 0.25042351 | 0.75000072 |
| -0.02701298 | 0.33389769 | 0.88888974 |
| -0.08090196 | 0.37563454 | 0.93750090 |
| -0.21537411 | 0.40067646 | |
| -0.53752623 | | |

If in a semitonal logarithmic scale we wanted to favour partial 5, we would have to take value 5 and position 12+12+4 = ordinal 28, so the formula will be **5^(1/28) = 1.059164008**…In this scale, as incremental ratio of degree 9 of the scale (element 14), we find the 5^(1/2) component of the gold section. In distances of octaves, (5*2)^(1/40), (10*2)^(1/52) etc. this ratio modifies towards 2^(1/12).

If in a different logarithmic scale, we wanted to favour partial 3 we would have to take value 3 and position 12+7 = ordinal 19, so the formula will be **3^(1/19) = 1.059526065**… This ratio, too, in distances of octaves, (3*2)^(1/31), (6*2)^(1/43) etc, modifies towards 2^(1/12).

The formula 2^(1/12), at distances of octaves (position+12) does not change: 4^(1/24) = 8^(1/36) = 16^(1/48) = **1.059463094…**

The value 2 and the positional increment +12, to infinity, modify ratios 5 and 3 to make them converge on ratio 2. The two diverging series deriving from 3 and 5 find a convergence factor in 2 which in the chas model is expressed in the curves related to the difference values (section 4.3, graph 6).

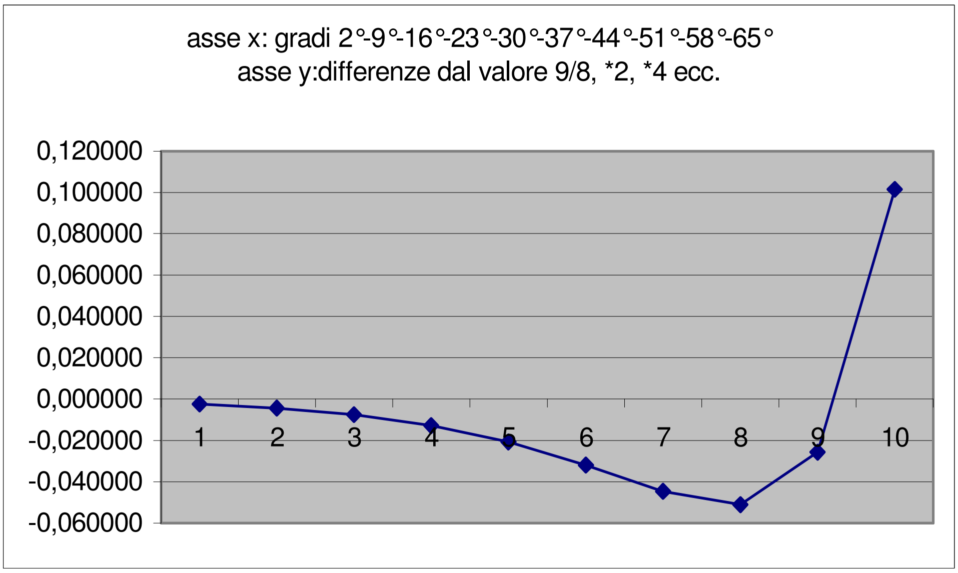
The sequence of difference quotients cannot occur in the equal temperament series calculated in a 2:1 proportion because pure octaves do not produce any difference.

### 4.6 COMPARISON BETWEEN RATIO 9:8 SCALE VALUES AND CHAS VALUES

**TABLE 7: Differences values**

| Natural ratio 9:8 | Chas ratio | Differences |
|---|---|---|
| 1.125 | 1.122511738 | **-0.002488262** |
| 2.250 | 2.245619840 | **-0.004380160** |
| 4.500 | 4.492432726 | **-0.007567274** |
| 9.000 | 8.987252178 | **-0.012747822** |
| 18.000 | 17.979279077 | **-0.020720923** |
| 36.000 | 35.968110132 | **-0.031889868** |
| 72.000 | 71.955329294 | **-0.044670706** |
| 144.000 | 143.948886799 | **-0.051113201** |
| 288.000 | 287.974250331 | **-0.025749669** |
| 576.000 | 576.101494758 | **0.101494758** |
| 1152.000 | 1152.509058989 | 0.509058989 |
| 2304.000 | 2305.630419534 | 1.630419534 |
| 4608.000 | 4612.485767480 | 4.485767480 |
| 9216.000 | 9227.422042560 | 11.422042560 |
| 18432.000 | 18459.746402221 | 27.746402221 |

**Graph 8 -** Chas differences (in bold above) for ratio **9:8**




x: degrees 2, 9, 16, 23, 30, 37, 44, 51, 58, 65
y: differences from 9/8, *2, *4 etc.

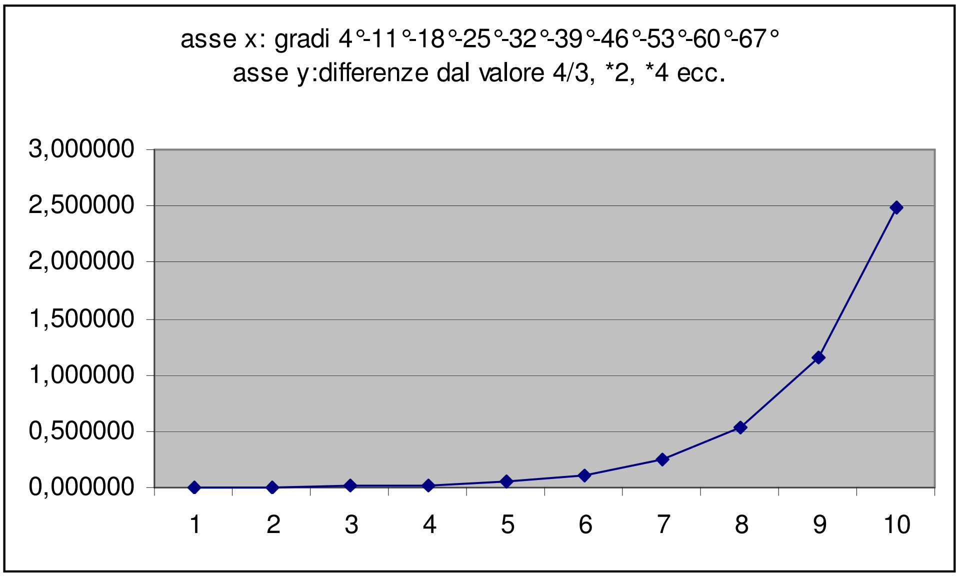
The difference curve for these intervals inverts its progression at degree 51.
**This inversion is determined by the s variable.** The same effect, we will see below, is found in degrees relating to ratios 3:2, 3:1, etc.
Thus the chas model remedies the exponential divarication of equal scale differences (section 4.3 graph 5).

## 4.7 – COMPARISON BETWEEN RATIO 4:3 SCALE VALUES AND RELATED CHAS VALUES

**TABLE 8: Differences values**

| Natural ratio 4:3 | Chas ratio | Differences |
|---|---|---|
| 1.333333333 | 1.334987586 | **0.00165425306** |
| 2.666666667 | 2.670684421 | **0.00401775432** |
| 5.333333333 | 5.342787715 | **0.00945438185** |
| 10.666666667 | 10.688413931 | **0.02174726393** |
| 21.333333333 | 21.382506370 | **0.04917303636** |
| 42.666666667 | 42.776372773 | **0.10970610655** |
| 85.333333333 | 85.575471649 | **0.24213831610** |
| 170.666666667 | 171.196407579 | **0.52974091205** |
| 341.333333333 | 342.483767871 | **1.15043453794** |
| 682.666666667 | 685.149489491 | **2.48282282462** |
| 1365.333333333 | 1370.662983148 | 5.32964981457 |
| 2730.666666667 | 2742.054168014 | 11.38750134683 |
| 5461.333333333 | 5485.565126339 | 24.23179300589 |
| 10922.666666667 | 10974.044607262 | 51.37794059540 |
| 21845.333333333 | 21953.919472021 | 108.58613868762 |

**Graph 9** - Chas difference values (in bold above) for ratio 4:3



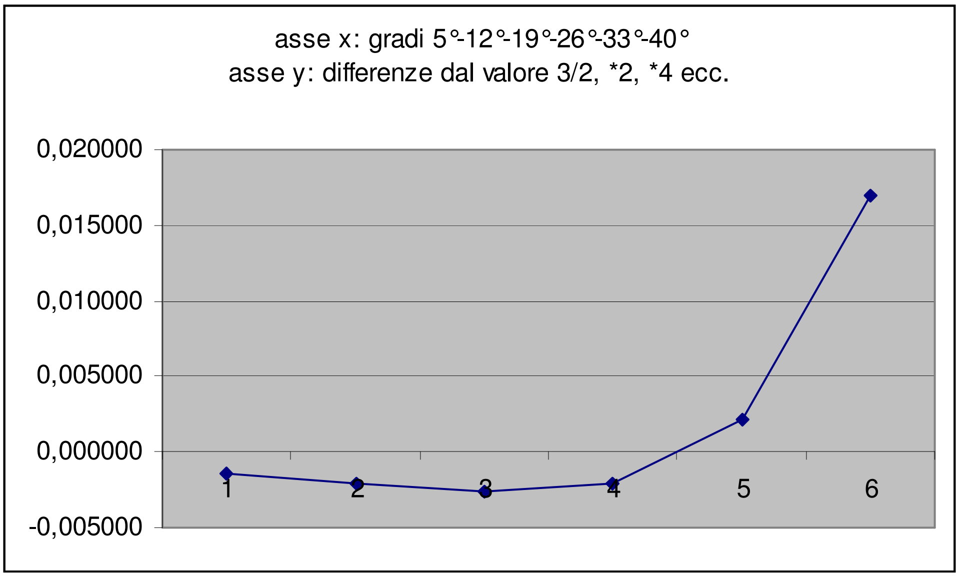

x: degrees 4, 11, 18, 25, 32, 39, 46, 53, 60, 67
y: differences from 4/3, *2, *4, etc

## 4.8 – COMPARISON BETWEEN RATIO 3:2 SCALE VALUES AND RELATED CHAS VALUES

**TABLE 9**: **Differences values**

| **Natural ratio 3:2** | **Chas ratio** | **Differences** |
|---|---|---|
| 1.50 | 1.4985392354 | **-0.0014607646** |
| 3.00 | 2.9978746101 | **-0.0021253899** |
| 6.00 | 5.9973419221 | **-0.0026580779** |
| 12.00 | 11.9978700941 | **-0.0021299059** |
| 24.00 | 24.0021143805 | **0.0021143805** |
| 48.00 | 48.0169805324 | **0.0169805324** |
| 96.00 | 96.0594713822 | 0.0594713822 |
| 192.00 | 192.1699769522 | 0.1699769522 |
| 384.00 | 384.4420493932 | 0.4420493932 |
| 768.00 | 769.0883440050 | 1.0883440050 |
| 1536.00 | 1538.5852869581 | 2.5852869581 |
| 3072.00 | 3077.9879888917 | 5.9879888917 |
| 6144.00 | 6157.6112420082 | 13.6112420082 |
| 12288.00 | 12318.4938812442 | 30.4938812442 |
| 24576.00 | 24643.5322949622 | 67.5322949622 |

**Graph 10** - Chas differences (above in bold) for ratio 3:2. The difference curve for ratio 3:2 inverts its progression at degree 19.




x: degrees 5, 12, 19, 26, 33, 40
y: differences from 3:2, *2, *4 etc

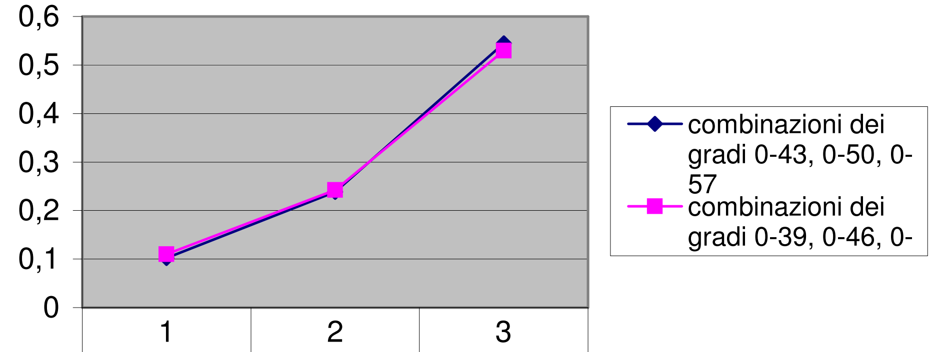
## 4.9 THE TORSION OF THE CHAS FORM

**TABLE 10: Differences related to natural harmonic values**

| 2:1 | 9:8 | 5:4 | 4:3 | 3:2 | 5:3 | 15:8 |
|---|---|---|---|---|---|---|
| 0 | -0.002488 | 0.010033 | 0.001654 | -0.001461 | 0.015461 | 0.013208 |
| 0.000531 | -0.004380 | 0.020735 | 0.004018 | -0.002125 | 0.031816 | 0.027420 |
| 0.002125 | -0.007567 | 0.042808 | 0.009454 | -0.002658 | 0.065420 | 0.056846 |
| 0.006377 | -0.012748 | 0.088296 | 0.021747 | -0.002130 | 0.134417 | 0.117707 |
| 0.017008 | -0.020721 | 0.181952 | 0.049173 | 0.002114 | 0.275988 | 0.243447 |
| 0.042525 | -0.031890 | 0.374626 | 0.109706 | 0.016981 | 0.566291 | 0.502961 |
| 0.102073 | -0.044671 | 0.770702 | 0.242138 | 0.059471 | 1.161217 | 1.038066 |
| 0.238202 | -0.051113 | 1.584315 | 0.529741 | 0.169977 | 2.379721 | 2.140436 |
| 0.544534 | -0.025750 | 3.254476 | 1.150435 | 0.442049 | 4.874046 | 4.409516 |
| 1.225364 | 0.101495 | 6.680690 | 2.482823 | 1.088344 | 9.977360 | 9.076387 |
| 2.723392 | 0.509059 | 13.704946 | 5.329650 | 2.585287 | 20.413377 | 18.667621 |
| 5.992259 | 1.630420 | 28.097209 | 11.387501 | 5.987989 | 41.744313 | 38.365212 |
| 13.075756 | 4.485767 | 57.569414 | 24.231793 | 13.611242 | 85.324228 | 78.790910 |
| 28.334570 | 11.422043 | 117.889551 | 51.377941 | 30.493881 | 174.320639 | 161.703888 |
| 61.036415 | 27.746402 | 241.282011 | 108.586139 | 67.532295 | 355.987592 | 331.654100 |

As well as the differences produced by the combination of two sounds, we can now picture also a flow of differences, or flows of synchronic beats, deriving from infinite and contemporaneous combinations of sounds. The fundamentals, with the differences related to harmonic partials 3 and 4, determine helixes of differences on a third plane. *These helixes cause the torsion in this kind of set*. The equal proportion 2:1, with its monotone differences curves, blocks this phenomenon.

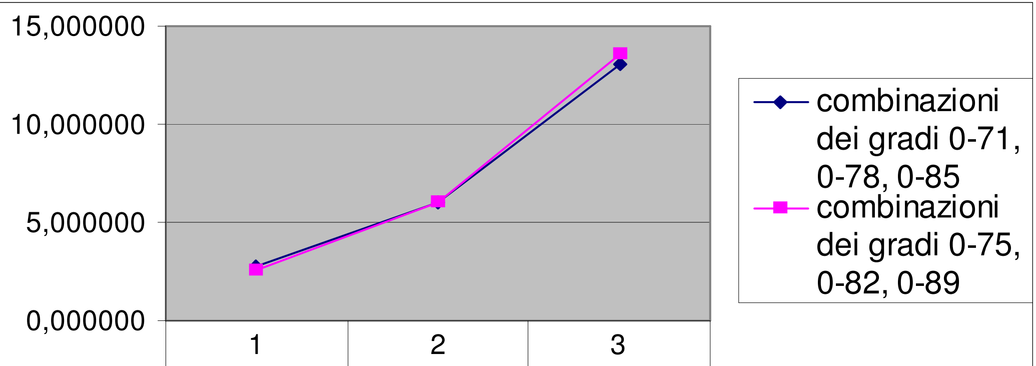
**Graph 11** – The octave difference from 2:1, at degree 57 exceeds the fourth, 4:3, at degree 53 (highlighted in green).




| | 1 | 2 | 3 |
|---|---|---|---|
| combinazioni dei gradi 0-43, 0-50, 0-57 | 0,10207295 | 0,23820186 | 0,54453372 |
| combinazioni dei gradi 0-39, 0-46, 0-53 | 0,109706 | 0,242138 | 0,529741 |

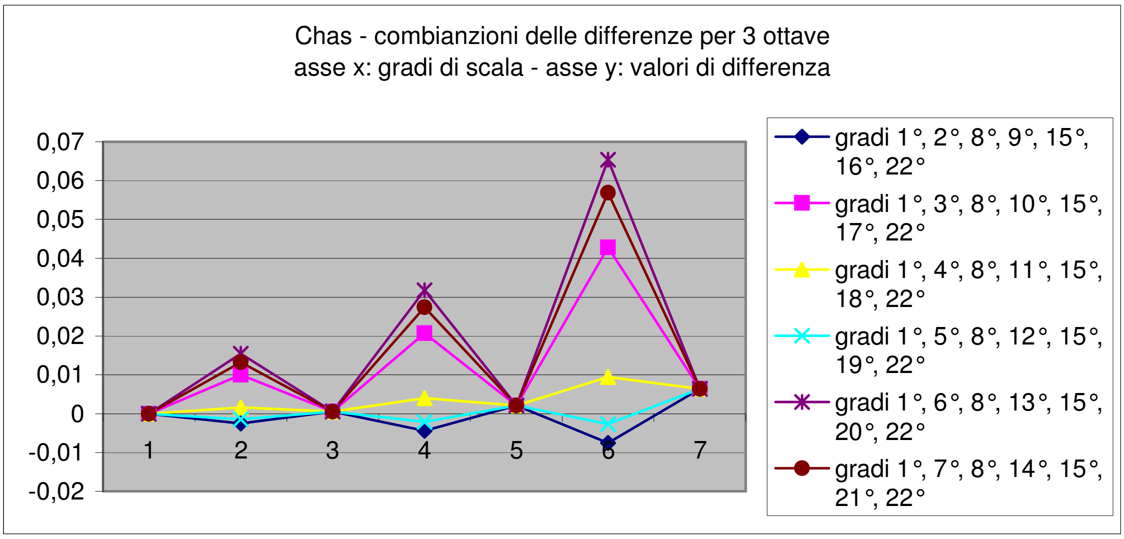
Degree combinations…

**Graph 12** – the difference of fifth, 3:2, at degree 89 exceeds the difference of octave, 2:1, at degree 85 (highlighted in yellow).




| | 1 | 2 | 3 |
|---|---|---|---|
| combinazioni dei gradi 0-71, 0-78, 0-85 | 2,723392 | 5,992259 | 13,075756 |
| combinazioni dei gradi 0-75, 0-82, 0-89 | 2,585287 | 5,987989 | 13,611242 |

**Graph 13** - Combinations of differences values for various scale degrees showing torsion of set described by chas model.




Chas – combinations of differences for 3 octaves
x: scale degrees
y: differences values

## CONCLUSIONS

The chas model highlights the fundamental relationship between frequencies and harmonic partial differences. In the chas frequency scale, the deviation of octaves from partial 2 (section 4.2) draws a curve similar to the average inharmonicity curve observed in fixed tuning instruments. This suggests that for the octave interval and more generally for partial 2 it is finally possible to adopt a natural *standard curve* of reference.

This system sheds light on a harmonic sequence, the series of n/n+1 values (section 4.5) which the harmonic partial values 3 and 5 also converge towards in their respective logarithmic scales. The n/n+1 sequence, like a spine, supports a network or rather a flow of differences, waves of synchronic beats. These differences describe a specific form which might aptly be called a "chorale", with respect to the overall effect of partial sounds.

A "chorale" draws flows of beats, the effect of the infinite combinations between fundamental and partial frequencies in vibrating strings: fundamentals and partials which are manifest in nature with precise proportional ratios and always in the same order. This suggests that the chas model may open the way towards further research both in relationships between energy, sound and matter, and in those areas where resonance, beat, spin and other phenomena related to waves are studied.

We see here the helix effect and the torsion of the set (section 4.9) determined by the natural interweaving of the differences from harmonic partials 2, 3 and 5.

It is the partials, in this proportional scale (where s = 1), that combine to determine:
the difference on partial 2:

0.0005312769273…

the ±Δ difference on partials 3 and 4:

0.0021253899646…

and the incremental ratio in the chas semitonal scale:

1.0594865443501…

In this thirty-year-long research, the numbers dispel all doubt concerning the simplicity and power of this long-awaited entity. It is still believed that opening up a harmonic scale with just one ratio is impossible, like flattening a hemisphere on a plane. Yet we see how the *superparticulares* n/n+1 values give rise to a dynamic, coherent, balanced and perfectly synchronic set.

The number PHI describes a proportion on a plane; a fractal, starting from discretional values, describes auto-similarity.

The chas model "chorale" expresses a continuously evolving set, proportionate and auto-similar, with the power of its fundamental sounds and partial differences interacting in the dimension of time.

**NOTES, REFERENCES AND LINKOGRAPHY**:

[1] - The links selected relate to Italian universities or international institutes.
[2] - Joseph Sauveur - Principes d'acoustique et de musique - Paris, 1701.
[3] - Link: Università di Modena e Reggio Emilia - Dipartimento di Fisica - Battimenti
http://fisicaondemusica.unimore.it/Battimenti.html
[4] - Link: Articolo di Gianni Zanarini - Docente di Fisica e Acustica Musicale Unibo
Il divenire dei suoni: http://www.memex.it/SONUS/art8.htm
[5] - Link: Università di Modena e Reggio Emilia - Dipartimento di Fisica – Scala naturale
http://fisicaondemusica.unimore.it/Scala_naturale.html
[6] - Gioseffo Zarlino (1517-1590) – Istitutioni Harmoniche – Venezia, 1588.
[7] - Link: Carmine Emanuele Cella – Ricercatore IRCAM - Sulla struttura logica della musica
http://www.cryptosound.org/writings/music/files/StrutturaLogica.pdf
[8] - Link: David J. Benson – Music: A Mathematical Offering - University of Aberdeen – 2006
http://www.maths.abdn.ac.uk/~bensondj/html/music.pdf
[9] - Bernoulli J(1654–1705) – Specimen alterum Calculi differentialis… - 1691
[10] - Railsback O.L. (1938) : Scale Temperament as Applied to Piano Tuning – Link: The Journal of the Acoustical Society of America:
http://scitation.aip.org/getabs/servlet/GetabsServlet?prog=normal&id=JASMAN000009000003000274000005&idtype=cvips&gifs=yes –
Link: Robert W. Young (1952): Inharmonicity of Plain Wire Piano Strings - The Journal of the Acoustical Society of America:
http://www.afn.org/~afn49304/youngnew.htm#ftnote8

[11] – Referred to the traditional scale.
[12] - In the case of a planet's orbit calculated on a plane, the ratio between the distances from the sun of two planets which have orbit periods one double the other is 1.5874 …, a value that, together with the Delian Constant 2^(1/3) we find in the logarithmic scale in ratio 2:1 ratio. In the chas scale the corresponding scale value is obtained from the delta effect on partials 3 and 4, is the result of ((4 + **Δ**)^(1/2)*(4 + **Δ**)^3)/(3 – **Δ**)^4 = 1.5876… and has a difference of 0.0002.

**For more reading**:
- ALESSANDRO BERTIROTTI - Dispense del seminario tenuto all'interno del Corso di Etnologia
a.a. 2001-2002
http://eprints.unifi.it/archive/00000168/00/Bertirotti-Etnomusicologia_e_antropologia_della_musica.pdf
- FABIO BELLISSIMA - UNISI - SIMMETRIA DI SCALA
http://digilander.libero.it/initlabor/musica-simmetria/scala-pitagorica.htm
– CARLO SERRA – UNIMI - COMPONENTI PSICOLOGICHE– SAGGIO SU JACQUES CHAILLEY – PAR.1
http://users.unimi.it/~gpiana/dm10/serra/chailley.pdf
- GIANNI ZANARINI – CONSAPEVOLEZZE SCIENTIFICHE E PRATICHE COMPOSITIVE
http://www.ulisse.sissa.it/biblioteca/saggio/2004/Ubib041001s002/at_download/file/Ubib041001s002.pdf
- ALVISE VIDOLIN – CONSERVATORIO DI VENEZIA – MUSICA INFORMATICA E TEORIA MUSICALE – NUOVI SCENARI
http://www.dei.unipd.it/~musica/Dispense/cap1.pdf

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Rene' Descartes: Compendium Musicae – Stilo Editrice – Bari, 2008.
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Florindo Gazzola: L'accordatura degli antichi strumenti da tasto – Armelin Musica - Padova, 2003.
Pietro Righini: Accordature e accordatori – Berben - Ancona, 1978.