Laurentian
UNIVERSITY•UNIVERSITÉ
Laurentienne

Ramsey Lake Road
Sudbury, Ontario
Canada P3E 2C6
www.laurentian.ca

Chemin du lac Ramsey
Sudbury (Ontario)
Canada P3E 2C6
www.laurentienne.ca

Report concerning the Ph.D. Thesis "The concept of variable in the passage from the arithmetical language to the algebraic language in different semiotic contexts", submitted by Elsa del Pilar Malisani.

The main goal of the thesis is to provide new insights into the difficulties that students encounter in the transition from arithmetic to algebra. This is a topic that has been widely discussed in mathematics education since the early 1980s. The novelty of the research conducted by Mrs Malisani resides in the fact that it tackles the problem of the students' difficulties by focusing on one key conceptual distinction involved in the aforementioned transition: the distinction between the concept of unknown and the concept of variable.

From a didactic viewpoint, this choice is extremely interesting. Yet, to be successful, such a choice has to be supported by a strong theoretical framework and a robust methodology. Mrs Malisani is aware of the risks and, to cope with them, makes recourse to well designed research. Her theoretical framework is, indeed, based on Brousseau's well-known *théorie des situations* and a fine-grained historical-epistemological reflection of the concepts of unknown and variable.

École des sciences de l'éducation
B. Ed. Alternatif
(705) 675-1151
ext./poste 5006
(705) 671-3855

The methodology is based on quantitative and qualitative data. The statistical analyses –carried out with the help of the implicative method CHIC (Classification Hiérarchique Implicative et Cohésisve) and classical factorial analysis (SPSS)– provide us with a general map of the situation. The qualitative analyses –based on transcripts of various audio-recordings of experimental situations– offer particular details that complete the statistical analyses. In the end, the research design allows Mrs Malisani to reach several interesting results and to raise some conjectures.

The thesis consists of five chapters. The first one deals with the historical evolution of algebraic language. After a short discussion of the evolution of algebraic symbolism, a survey of pre-algebraic and algebraic methods to solve equations is presented. The survey covers topics such as the method of false position, the geometric oriented procedures of Babylonian scribes, the more symbolically and numerically oriented methods of Diophantus as found in his *Arithmetica*, Abu Kamil's and al-Khwarizmi's methods, and so on. The historical survey puts into evidence the variety of problems that have been historically tackled through a diversity of methods that we now put together under the label of "algebra". The historical study leads Mrs Malisani to revisit the question of the difference between unknowns and variables. This point is essential for the success of the whole thesis, for it rests on the distinction between these two concepts. While many contemporary mathematics educators still maintain that both concepts are the same, mainly because –on the level of symbols– they can be manipulated in the same way, the historical-epistemological analysis makes it possible to show, in a decisive way, that they belong to two different conceptual formations.
The historical survey also allows Mrs Malisani to show a subtle interplay of different semiotic systems and to give us an idea of the level of semantic reduction that the contemporary algebraic symbolism imposes on its users.

The second chapter presents the results of an experiment carried out with Middle and High School students. The purpose is to start exploring the difficulties in the transition from arithmetic to algebra. A problem concerning a magic square was proposed to the

students. The sum of each row, column and diagonal should be equal to "a+26". The instructions required the students to make sense of this algebraic expression. The results show how difficult it was for the young students to achieve an algebraic understanding of such an expression. The analysis suggests a variety of interpretations of the letter "a".

The goal of the third chapter is to investigate the relation-functional aspect of the concept of variable in two problem-solving contexts: an algebraic and a geometric one. More than one hundred students from a High School participated in this experiment in which they had to solve 4 problems. Several hypotheses were presented. The first one, in my opinion the more interesting, tried to elucidate to what extent the students were making recourse to a concept of unknown or to a relational-functional aspect of algebraic variables. Two problems among the four proposed problems shed an interesting light on the research question. They present a contextual situation which can be translated into one equation with two variables, say "x" and "y". More specifically, the first problem leads to the equation $3x+4y=300$. The question asked to the students is: How many are the possible solutions?

Many students add an extra condition (e.g. $x=y$) in order to obtain one solution. Others interpret the problem as having multiple solutions (even if they do not reach a symbolic expression for them). In the first case the students are seen as resorting to the concept of unknown, while in the second case, the students are seen as resorting to a relational-functional aspect of variables. There are several interesting issues here. One of them, which relates to those students who add an extra condition in order to have a single solution, may be explained by a didactic contract. But it may also be the case that the problem is related to a deep cognitive problem. This problem may be stated as follows: The students transform the statement of the problem into a model that is, cognitively speaking, within their reach. In other words, they cannot "interpret" the problem in relational-functional terms because this aspect of the concept of variable is beyond their grasp. So they need to add extra data in order to make the problem interpretable within their cognitive scope. The transcripts suggest that the transformation of the statement of the problem into another problem (the one that results from the original one plus the added information, e.g. $x=y$) may be the cause. We are led to another problem that

requires further research: what would be the role of symbolism, natural language, drawings, etc. in the creation of a deeper understanding in which the students become aware of the relation-functional aspect of the variables?

Chapter four presents a follow-up investigation of the previous chapter. An analysis of two pairs of students (different from the students that participated in the experiment reported in Chapter 3) is carried out when the students worked on the four problems considered in Chapter 3. The protocol analysis deepens the results previously obtained.

Chapter five presents a well written summary of the thesis, some conclusions and a list of questions for further research. One interesting point concerning the role of symbols in cognition is the fact that natural language (as reported in the students' protocols) opens a certain range of semantic control during the problem solving procedures. As Mrs Malisani suggests, the students follow the pace of the spoken thought. When algebraic symbolism is used, the pace of thought is disrupted. A highly encoded syntax and semantics replace verbal reasoning and a new configuration now guides the students' pace of thought. The qualities and particularities of this new cognitive configuration are still unknown and certainly further research is required to study it. This is a valuable research path that Mrs Malisani's Ph.D. thesis opens for us. By way of conclusion, I want to say that, for the reasons presented previously, I value her work in a very positive way.

September 25, 2005

Luis Radford
École des sciences de l'éducation
Université Laurentienne
Sudbury, Ontario
Canada, P3E 2C6