Effectively using new paradigms in the teaching and learning of mathematics: Action research in a multicultural South African classroom

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Abstract In this paper the author identifies some common conceptual errors that learners tend to make when working with exponents and reflects on the informal action research she undertook as a teacher in order to assist her learners in overcoming these. The error patterns, learning to adapt to teaching multicultural classes and the implementation of a new curriculum in South Africa were the key factors that initiated and formulated her new approach to teaching exponents to Grade 9(14 year old) learners. The results were interesting, especially those of the generally "weaker" learners and those who do not have English as their home language. The author gained far more enjoyment from approaching the teaching of exponents in this new manner and the majority of learners demonstrated growth in their ability to simplify and calculate expressions involving exponents. This paper aims to underline the importance of identifying and addressing conceptual errors and suggests an alternative approach to the teaching of exponents.

A. Introduction

Learners have always made mistakes in mathematics, but unless we are able to identify why they make these mistakes, we are unable to do something about it. As teachers, all our interventions in the classroom are guided by some theory – conscious or subconscious – of how children learn mathematics. These theories vary from teacher to teacher, as does our approach to addressing learners' mathematical mistakes. One such theory (although an escapist route) is to view many learners as rather dim, as not capable of understanding and label them as "weak". Attributing learners' mistakes to low intelligence, low mathematical aptitude, perceptual difficulties or learning disabilities is not useful in rectifying the mistakes. These factors naturally play a role but if it is our intention to help the individual learner, we need to examine available detail and determine the specific roots of the mistakes. (Olivier, 1989).

In this paper, I reflect on my experience as a teacher of becoming more aware of recurring mistakes made by my learners in the section on exponents, what valuable information these error patterns provided me with regarding the thinking and conceptual understanding of my learners and the paradigm shift that subsequently occurred in my teaching. Although I am no longer teaching in a school but lecturing mathematics methodology at a university, the same conceptual errors were appearing in the work of some of my undergraduate student mathematics teachers. This then prompted me to address the issue of teaching and understanding exponents with them and to subsequently write up the paper.

i) Contextualising the research

The school I was teaching in was a middle-class single sex girls' school consisting of a combination of learners from various backgrounds and cultures. The average class size was about 30 learners and one could expect at least one third of the learners to not speak the language of instruction (English) as their home language. These learners often struggled to access my bulky teacher-centered explanations. One such learner, in my Grade 9 class was Mpho, a learner labelled "weak" and not capable of continuing with mathematics beyond the Grade 9 level. Mpho admitted to struggling with mathematics and made no attempt to hide her obvious dislike of the subject and of anyone who had chosen to teach it! She subsequently rarely did her homework and we had frequent personal encounters regarding our obvious disagreement regarding this matter. Mpho had failed her mid-year examinations and was sharing my anxiety of the next five months we had to tackle. And I knew that the section I dreaded most was yet to come - exponents.

ii) The problem

I simply never enjoyed teaching exponents to Grade 9 learners. Looking back, I attribute this to the fact that I was originally textbook and "law" bound in my approach to this section. Normally we were allocated two weeks in our annual planning to complete exponents and my thinking then was that if I taught the laws quickly, the learners would still have ample time to practise the processes. It took me a few years to acknowledge the fact that although the teaching of exponents may be happening, this does not necessarily mean that the learning of exponents has taken place. It also does not mean that all learners have understood the notation and concept underlying exponents, even those that might be achieving high marks in their tests or exams. What led me to this conclusion (or acknowledgement) was my interest and subsequent frustration with the kinds of mistakes learners make regarding calculations and algebraic manipulations involving exponents. I began noticing a recurring trend in the types of mistakes being made by both learners who generally achieved low marks in mathematics

as well as those regularly achieving high marks. Some learners were achieving around 80%, but the 20% of mistakes that they did make were cause for concern.
I have summarised some of the more frequent types of mistakes below:

$2^3 \times 2^4 = 4^7$ $\quad$ $5^2 = 10$ $\quad$ $3^{-2} = -9$

$3^{-1} + 2^{-1} = 5^{-1}$ $\quad$ $4x^{-2} = \frac{1}{4x^2}$

These mistakes really concerned me. It is my opinion that they more often than not indicate a lack of understanding of the meaning of the notation used as well as the basic concept of what an exponent represents. Research has also shown that few errors are random or careless and that many errors are in fact conceptual and learned and have become habitual and consistent with advancing years in school (Baxter and Dole, 1990).
I subsequently began asking myself some questions:

- How are the learners benefiting from learning a set of laws to apply according to various algorithms, without necessarily gaining any understanding of the concept?
- Other than for the purpose of marks, do the learners benefit from practising these processes over and over without internalising an understanding of the concept?
- Has mathematics become a strict discipline governed by laws, rules and algorithms for the learners?

B. Exploring some new options

Meanwhile Outcomes-Based Education (explained in B.i) was starting to be enforced at schools in South Africa and the emphasis was being shifted from the teaching onto the learning. I found this paradigm shift really liberating. It led me to realise that my focus needed to change from what I was teaching to what sense my learners were making of the content I was offering them.

i) Outcomes Based Education (OBE)

Outcomes-Based Education was introduced in South Africa in the late 1990's to address the imbalance in education and the changing demands in the market place arising from the need for a more skills-based workforce (Kramer, 1999).
According to Killen (2000), the most detailed articulation of the theory underpinning OBE is given in Spady (1994, 1998). In Spady's words: "Outcomes-Based Education means clearly focusing and organizing everything in an educational system around what is essential for all students to be able to do successfully at the end of their learning experiences. This means starting with a clear picture of what is important for students to be able to do, then organizing the curriculum, instruction and assessment to make sure this learning ultimately happens" (Spady, 1994:1). Killen expands on this definition in his article and goes on to say that three basic premises underpin OBE:

- All students can learn and succeed, but not all in the same time or in the same way.
- Successful learning promotes even more successful learning.
- Schools (and teachers) control the conditions that determine whether or not students are successful at school learning.

He also highlights an advantage of OBE being that it provides educators with a large degree of freedom to select the content and methods through which they will help their learners achieve the set outcomes. For me as a mathematics teacher, it meant an awareness of the need to shift the focus of my teaching away from the content to be covered towards the learners' development (Murray, 2000).

ii) A different approach

My frustration with the conceptual errors, a desire to make mathematics more accessible to learners such as Mpho and my newfound focus motivated me to change my whole approach to teaching the section on exponents. Not only had I never found great fulfilment in teaching it, it also seemed to present a real stumbling block for many learners, even beyond the Grade 9 level.
I first set about creating some outcomes to enable me to get "a clear picture" of what my learners should be able to do on completion of the section. I then designed my teaching and assessment around these outcomes.

- The learner is able recognise, read, expand and understand exponential notation.
- The learner is able to compute calculations involving powers and exponents.
- The learner is able to manipulate expressions involving exponential notation, in order to simplify them.
- The learner is able to critically query, think about and verify mathematical statements involving exponential notation.
- The learner is able to state and communicate their understanding of the concept of exponents to others.

None of these emphasised the need for learners to memorise and apply laws. Obviously the laws enable one to simplify an expression far quicker, but I decided not to teach any of them to the learners

this time but to allow them to be discovered and verified by the learner instead. It can be argued that some of the "weaker" learners (such as Mpho in this particular class) may never discover these laws on their own. This is in fact true, but then why should they use them? What I finally decided to embark on was to focus my two weeks of teaching exponents on ensuring that the learners know and understand the notation and concept of exponents and that they are able to convey this to others.

Many of the learners had already learnt from Grade 8 that "if you are multiplying bases that are the same, you may add the exponents" and that "if you are dividing bases that are the same, you may subtract the exponents." They did not however always apply these "discoveries" (or laws as they later became) correctly and I tried to encourage and challenge them to ensure that they understood the notation first rather than trying to apply laws that did not make sense to them.

We started off with the basic notation and terminology used in the section on exponents such as:

$2^3 = 2 \times 2 \times 2$ whereas

$3 \times 2 = 2 + 2 + 2$ and $2 \times 3 = 3 + 3$ $y^4 = y \times y \times y \times y$

I emphasised the terminology of base, exponent and power as well as pointing out the difference between 2^3 and 2×3. In the beginning I encouraged learners to make use of expansion and numbers (rather than variables) as much as possible as a means to verifying their instinctive short cuts and to check up on their initial thinking. All the time we kept returning to the notation and what it meant. My motivation for this was that at least they would have basics to go back to if they found themselves struggling at a later stage when things got more complicated.

By understanding the notation, they were able to check up on themselves when simplifying expressions such as $(2y^3)^2$ by expanding if necessary:

They started reasoning that if y^2 meant $y \times y$, then $(\ )^2$ meant $(\ ) \times (\ )$

Therefore $(2y^3)^2 = (2y^3) \times (2y^3) = 2 \times y \times y \times y \times 2 \times y \times y \times y = 4y^6$

This also helped them avoid (or at least think about) assumptions such as:

$4y^2 = 4y \times 4y$, rather than the correct expansion of $4 \times y \times y$

I made use of the following types of tables to introduce and encourage a discussion on the use of negative exponents, as well as the concept of zero as an exponent always resulting in a value of one.

A lesson dedicated to group work and later a full class discussion on trends in the tables enabled us to agree that a negative exponent indicates division and that for example:

$3^{-2} = \frac{1}{9} = \frac{1}{3^2}$ And that $a^0 = 1$

With this basic knowledge of exponents in place, I jumped straight into giving the learners algebraic expressions to simplify as well as calculations involving the use of exponential notation. At first I gave them exercises such as the ones below, where the exponents were not

TABLE 1

Base	**Exponent**	**Power**	**Value**
10	2	10^2	100
10	1	10^1	10
10	0	10^0	1
10	-1	10^{-1}	$\frac{1}{10}$
10	-2	10^{-2}	$\frac{1}{100}$

TABLE 2

Base	**Exponent**	**Power**	**Value**
3	2	3^2	9
3	1	3^1	3
3	0	3^0	1
3	-1	3^{-1}	$\frac{1}{3}$

3	-2	3^{-2}	$\frac{1}{9}$

too large. The purpose of this was to encourage them to still use expansion where necessary to guide and verify their thinking.

$(ab^2)^2(a^2b)$ $\quad \frac{1}{4}(4a^3)^2 \quad$ $\frac{(2a^2b)^3.(2^3ab^{-2})}{2^5a^4b^3}$

$[3(2a^2b^3)]^3 \quad -2(4p)^2(-p)^2$ (Laridon et al, 1995)

I later also gave them more complex exercises (see examples below) to hopefully challenge some of them to search for ways of refining and/or improving their strategies and techniques. These also contained larger exponents which made the task of expanding more laborious.

$\frac{6x^7y^3}{(2x^2y^{-1})^4} \quad \frac{3^5.2^6}{2^8.3^{11}} \quad \frac{(2a^2.b^3)^4\times(8ab^{-2})^{-2}}{4ab^5\times12a^9b^7}$

$\frac{2\times10^5\times3\times10^8}{4\times10^9\times3\times10^6} \quad \frac{48^3\times27^2\times64}{32^4\times18^2}$ (Laridon et al, 1995)

C. Results

Throughout the two weeks, the growth that learners demonstrated in their thinking and reasoning was very encouraging. When asked if:

$$4y^{-3} = \frac{1}{4y^3}$$

many learners (even those labelled as "weak") were able to point out that the statement is incorrect and verify this by using their ability to expand exponential notation and their understanding of negative exponents in the following way:

$$4y^{-3} = 4\times y^{-3} = 4^1\times y^{-3} = 4\times\frac{1}{y^3} = \frac{4}{y^3}$$

They also managed to eliminate mistakes such as $2^3 \times 2^2 = 4^5$. Once they knew and understood the notation and were comfortable expanding it, they quickly realised that:

$2^3 \times 2^2 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$ which could not possibly be equal to $4^5 = 4 \times 4 \times 4 \times 4 \times 4$

The most interesting results, however, were evident when the learners wrote a traditional summative test at the end of the two weeks. One of the questions to calculate in the test was:

$$\frac{2^{-1}+3^{-1}}{2^{-1}\times3^{-1}}$$

Below is the solution of one of the "higher achieving" learners (Nichola) who had discovered and adopted the laws early on in this section:

$$\frac{2^{-1}+3^{-1}}{2^{-1}\times3^{-1}} = \frac{5^{-1}}{6^{-1}} = \frac{6}{5}$$

As you will notice, this learner has still made one of the mistakes I identified earlier. I attributed this to the tendency of this learner to find and use as many shortcuts as possible, without necessarily verifying that she understood the concept behind them.

In contrast to this, I have also provided the solution offered by one of the generally "weaker" learners (remember Mpho?):

$$\frac{2^{-1}+3^{-1}}{2^{-1}\times3^{-1}} = \frac{\frac{1}{2^1}+\frac{1}{3^1}}{\frac{1}{2^1}\times\frac{1}{3^1}} = \frac{\frac{2}{5}}{\frac{1}{6}} = \frac{2}{5}\times\frac{6}{1} = \frac{12}{5}$$

Although Mpho demonstrates another level of conceptual difficulty concerning addition of fractions, she does apply her understanding of negative exponents correctly.

In another part of the test Mpho offers a further interesting solution (see Appendix A on page 8) whereby she demonstrates her ability to simplify complex algebraic expressions involving the manipulation of exponents.

The most rewarding result, however, was evident in Mpho's increased self-confidence in the mathematics class and her newly acquired positive attitude to both the subject and myself. She passed the summative test, which caused much celebration from both parties, and also started doing her homework on a more regular basis.

D) Conclusion

Although this informal research was carried out a few years ago, it still plays a major role in my approach to teaching. Error analysis and diagnostic assessment have become a priority for me as a mathematics education lecturer and as a new researcher.

Research and literature, relating to mathematics education, over the years have also identified and discussed error patterns and misconceptions in mathematics (Radatz, 1980; Farrell, 1992; Clarkson, 1992; Gagatsis & Kyriakides, 2000) and suggested that errors can be overcome (Mestre, 1989) and foster a deeper and more complete understanding of mathematical content and the nature of mathematics itself (Borasi,1987). Some results suggest a greater need for emphasis on conceptual understanding in mathematics instruction (Woodward & Howard, 1994).

This paper has highlighted some of the recurring mathematical mistakes made by both "low achieving" and "high achieving" Grade 9 learners in the section on exponents. Regardless of the marks obtained by the learner for this section, some of these mistakes indicated a lack of conceptual understanding of the content. This in turn resulted in frustration and therefore necessary reflection of teaching methods for the author. Simultaneously the onset of OBE in South Africa initiated a paradigm shift from what was being taught to what was being learnt. The combination of circumstances created a framework for a new approach to be tried out in the teaching of exponents to a multicultural Grade 9 class. The results reflected a growth in most learners' ability to think about the expressions and calculations they were confronted with and to verify their conclusions. A learner labelled as "weak" demonstrated the required level of understanding necessary to manipulate and simplify complex algebraic expressions while still however showing conceptual gaps in her numerical skills.The new approach also resulted in learners in general experiencing mathematics as more enjoyable and accessible.

While this new approach in teaching exponents had positive results for the author with a particular Grade 9 class, it is acknowledged that this is not the only approach to ensure that learning has taken place. As mathematics educators, however, it should be our goal to ensure that learning and understanding are continually driving our teaching.

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APPENDIX A

$$\frac{\left(5x^{-3}\times y\right)^2}{4xy\times(x^2y)^2}\div\left(\frac{5xy}{2}\right)^2$$

$$=\frac{(5x^{-3}\times y)(5x^{-3}\times y)}{4xy\times(x^2y)(x^2y)}\div\frac{5xy}{2}\times\frac{5xy}{2}$$

$$=\frac{25x^{-6}y^2}{4x^5y^3}\div\frac{25x^2y^2}{4}$$

$$=\frac{25\times\frac{1}{x^6}\times y^2}{4x^5y^3}\div\frac{25x^2y^2}{4}$$

$$=\frac{\frac{25y^2}{x^6}}{\frac{4x^5y^3}{1}}\div\frac{25x^2y^2}{4}$$

$$=\frac{25y^2}{x^6}\times\frac{1}{4x^5y^3}\div\frac{25x^2y^2}{4}$$

$$=\frac{25y^2}{4x^{11}y^3}\times\frac{4}{25x^2y^2}=\frac{100y^2}{100x^{13}y^5}$$

$$=\frac{1y^{-3}}{1x^{13}}=\frac{\frac{1}{y^3}}{\frac{x^{13}}{1}}=\frac{1}{y^3}\times\frac{1}{x^3}=\frac{1}{x^3y^3}$$