THE USE OF HAND-HELD TECHNOLOGY IN THE LEARNING AND TEACHING OF SECONDARY SCHOOL MATHEMATICS
The functionality of CABRI and DERIVE in a graphic calculator

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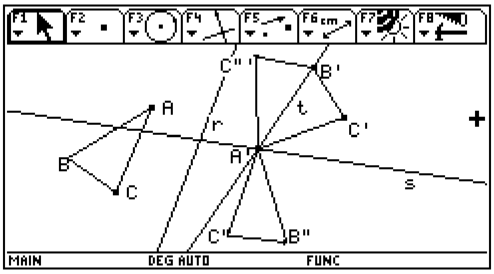
ABSTRACT
The workshop will focus on the use of hand-held technology to improve teaching and learning in mathematics in secondary school, with an emphasis on 14-16 education. The talk will be illustrated with practical examples using a new graphic calculator, which contains a version of CABRI and DERIVE. In particular it will be shown how to use the graphic calculator to teach *geometric plane isometries*, both from a synthetic point of view and from an analytical one.

1. Introduction
In this workshop we want to show how a teacher in a secondary school can illustrate "*geometric plane isometries*" using a new graphic calculator, which contains the functionality of CABRI and DERIVE. The didactic path will be divided in several units.

2. Explaining, by using CABRI, the following Theorem: *Every isometry of the plane is a product of at most three reflections in lines.*
The geometric construction (given two congruent triangles △*ABC* and △*A'B'C'*) is composed by the following steps:

(*i*) Using the "*Perpendicular bisector*" tool we draw the perpendicular bisector *r* of the line segment *AA'*. Then, using the "*Reflection*" tool, we take the triangle △*ABC* to △*A"B"C"* by the reflection in *r* (*A"* = *A'*). [Look at Figure]

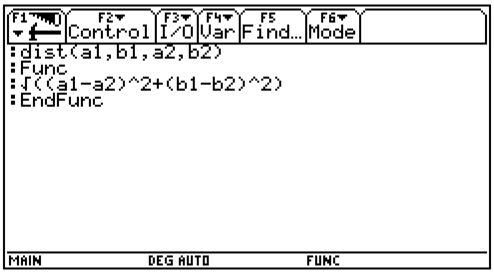


ii) If *B"* ≠ *B'*, using the "*Perpendicular bisector*" tool we draw the perpendicular bisector *s* of the line segment *B'B"*; *s* passes through point *A'* (= *A"*) because $\overline{A'B'} = \overline{A'B''}$. Then, using the "*Reflection*" tool, we take △*A"B"C"* to △*A'"B'"C'"* by the reflection in *s* (considering that *A'* = *A"* = *A'"* and *B'"* = *B'*).

(*iii*) If *C"'* ≠ *C'*, using the "*Perpendicular bisector*" tool we draw the perpendicular bisector *t* of the line segment *C"'C'*. Then, using the "*Reflection*" tool, we take (necessarily) △*A'B'C"'* to △*A'B'C'* by reflection in *t*. Consequently the composition of three reflections in lines *r*, *s*, *t* is the isometry which take △*ABC* to △*A'B'C'*.

3) Verifying the congruence of two triangles whose vertices are given and drawing its graph
We are given △*ABC* and △*A'B'C'*, whose vertices are *A*(−2,1), *B*(−7,9), *C*(−10,5) and *A'*(2,4), *B'*(10, −1), *C'*(6, −4). Using the function *dist* that we have included into the graphic calculator we can do the following check

[1] ADT is the Italian version of T-cubed or T^3



$\overline{AB} = \sqrt{89}$ $\overline{AC} = 4\sqrt{5}$ $\overline{BC} = 5$ $\overline{A'B'} = \sqrt{89}$ $\overline{A'C'} = 4\sqrt{5}$ $\overline{B'C'} = 5$

The congruence of triangles follows.

To draw the graph we can begin from the vector equation of line r_{AB}:

$$\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

where **p** is the position vector of a variable point $P \in r_{AB}$ and **a** and **b** are the position vectors of $A(x_1,y_1)$ and $B(x_2,y_2)$ and $t \in \mathbb{R}$ is the parameter. The previous equality can also be written in the following form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

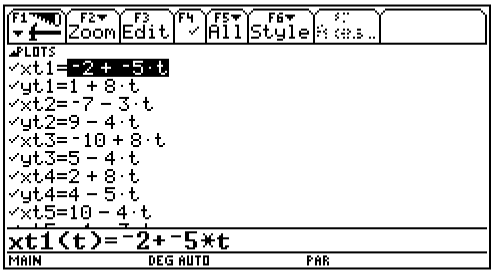
from which we get the parametric equations of the line

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \end{cases}.$$

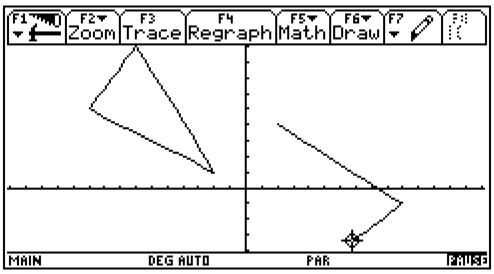
In particular, to represent the line segment AB it is enough to restrict the parameter t to the interval [0,1]. For the line segments that form our two triangles we have the following parametric equations

$$AB: \begin{cases} x = -2 - 5t \\ y = 1 + 8t \end{cases} \quad BC: \begin{cases} x = -7 - 3t \\ y = 9 - 4t \end{cases} \quad CA: \begin{cases} x = -10 + 8t \\ y = 5 - 4t \end{cases}$$

$$A'B': \begin{cases} x = 2 + 8t \\ y = 4 - 5t \end{cases} \quad B'C': \begin{cases} x = 10 - 4t \\ y = -1 - 3t \end{cases} \quad C'A': \begin{cases} x = 6 - 4t \\ y = -4 + 8t \end{cases}$$



REMARK. When we graph our two triangles on the graphic calculator we can observe that the first triangle $\triangle ABC$ is “traversed” in anticlockwise sense and the second one $\triangle A'B'C'$ in clockwise sense. Thus the two triangles are related by an isometry of second kind (or indirect or odd).



It is possible to show in an analytic way that two triangles are related by an isometry of the second kind or, more generally, if they have the same orientation, using the following theorem.

THEOREM. Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $A'(x_1',y_1')$, $B'(x_2',y_2')$, $C'(x_3',y_3')$ be the coordinates of the vertices of the oriented triangles $\triangle ABC$ and $\triangle A'B'C'$, respectively. In order that the triangles have the same orientation, it is necessary and sufficient that the determinants

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}$$

have the same sign.

For our two triangles we get

$$\begin{vmatrix} -2 & 1 & 1 \\ -7 & 9 & 1 \\ -10 & 5 & 1 \end{vmatrix} = 44 \qquad \begin{vmatrix} 2 & 4 & 1 \\ 10 & -1 & 1 \\ 6 & -4 & 1 \end{vmatrix} = -44 .$$

So the two triangles have opposite orientation.

REMARK. We note that the absolute value of the above determinant is twice the area of the triangle.

REMARK. Let us consider a transformation of the plane given by the following equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

We note that we can represent such an equation using only one object: a 3×3 matrix. In fact it is equivalent to the equation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

The last row of the matrix that represents the transformation is always formed by the vector [0,0,1]. A point of the plane is represented by a vector with three components, having the third component always equal to 1.

This representation is particularly convenient from an algorithmic point of view, because the transformation is completely described by only an object that is easy to implement.

In particular, for the isometries we have

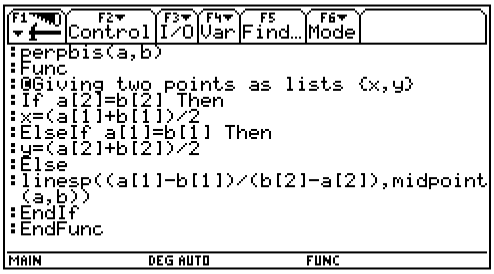
Isometries of the first kind: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & -b & c_1 \\ b & a & c_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Isometries of the second kind: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c_1 \\ b & -a & c_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

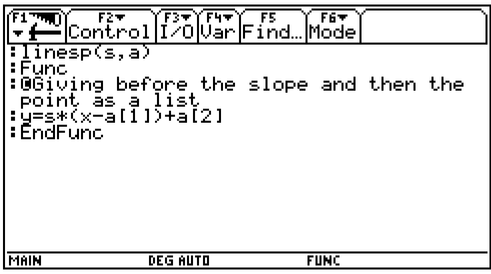
with $a^2 + b^2 = 1$.

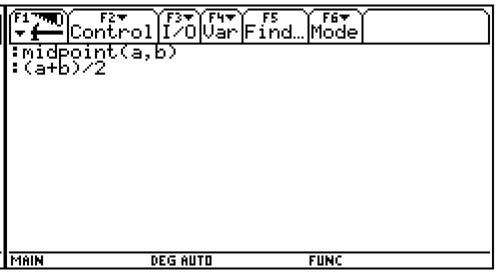
4) Determining analytically, using Theorem in 2), the isometry (of second kind) that take the triangle $\triangle ABC$ to $\triangle A'B'C'$.

Firstly we have to find the perpendicular bisector of the line segment *AA'* using the function *perpbis* that we have included into the graphic calculator. Such a function takes as input two points given as lists of two elements.



where *linesp* is the function that takes as input the slope and a point of line and *midpoint* is the function that calculates the midpoint of a line segment given as input its end points.

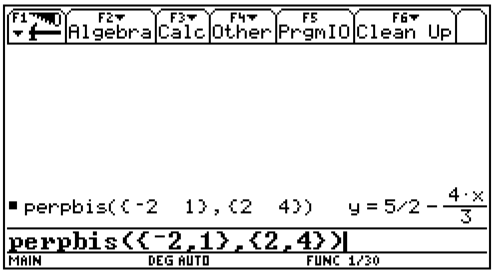




REMARK. As we can see, a student can build a “library” of functions to use in the resolution of particular types of problems. Such functions can be used to build other functions or programs in the same way we use the functions built in the graphic calculator.

In our case we have

$$\text{Perpendicular bisector}_{AA'}\colon\ y = -\frac{4}{3}x + \frac{5}{2}$$



Thus the slope is $m = -\dfrac{4}{3}$ and the intercept $q = \dfrac{5}{2}$.

In general the reflection in a line r whose equation is $y = mx + q$ can be obtained by the method of *double translation*: by means of a translation we take the line r to r' which is parallel to r and passes through the origin O, than we do a reflection in r' and lastly we do a translation by a vector that is opposite to the first one. We can choose the point on r whose coordinates are $(0,q)$ and therefore we can use the translation by vector $[0,-q]$. Consequently we have

$$\text{rifr}(m,q) = \text{trasl}([0,q])\cdot\text{rifor}(m)\cdot\text{trasl}([0,-q]),$$

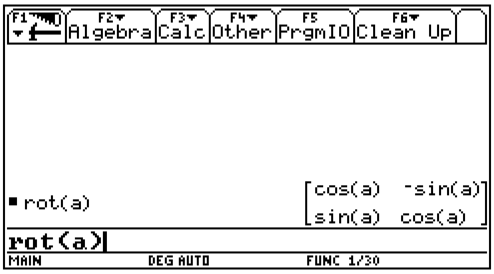
where rifor(m) denotes the reflection in a line that passes through the origin, whose equation is $y = mx$. We remember on this subject that the reflection in a line that passes through the origin and makes an angle α with the positive half of the x-axis can be obtained by means of a rotation (about the origin) through an angle $-\alpha$ (that takes the line r to the x-axis), then a reflection in the x-axis whose 2×2 matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

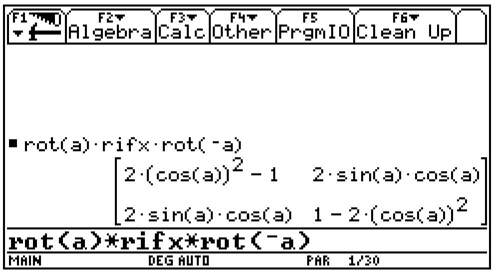
and again a rotation through an angle α. Therefore the matrix we are looking for is

$$\text{rot}(\alpha)\cdot\text{rif}x\cdot\text{rot}(-\alpha),$$

where rot(α) denotes the matrix that represents a rotation through an angle α.



The result of this calculation by a graphic calculator is



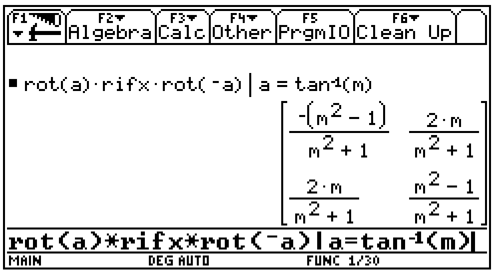
that, if we keep in mind the double-angle identities, coincides with the 2 × 2 matrix

$$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix},$$

that, as we know [see Impedovo] represents the reflection in a line that passes through the origin and makes an angle α with the positive half of the *x*-axis. Since $m = \tan\alpha$, we can use the double-angle identities to express cos2α e sin2α by means of tanα, so the above matrix becomes

$$\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2} \end{bmatrix},$$

or we can calculate the previous product with the condition $a = \tan^{-1}(m)$ and we obtain

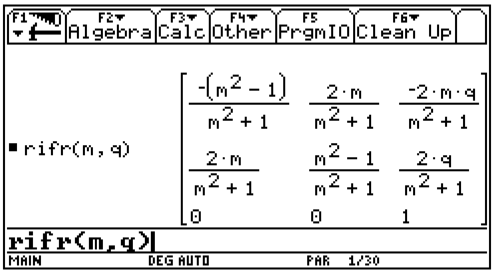


that is the same matrix written above.

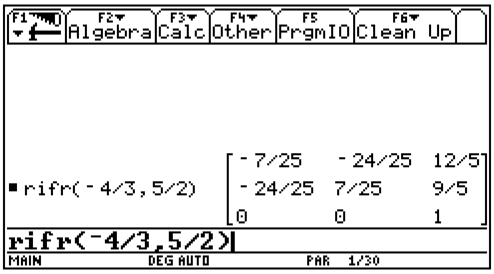
Thus, using again a 3 × 3 matrix, the reflection rifor(*m*) has the following form

$$\text{rifor}(m) = \begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} & 0 \\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Summing up, we obtain for the matrix rifr(*m*,*q*) the following form

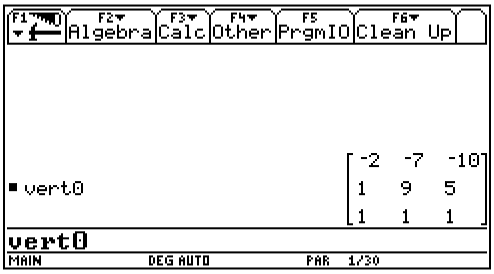


In particular, the reflection in the perpendicular bisector of *AA'* is

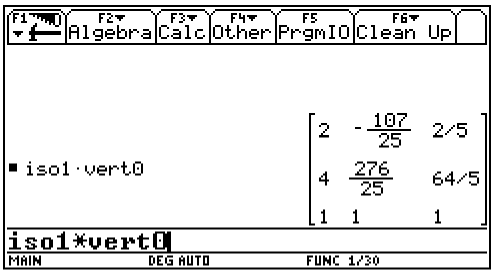


Such a matrix will be denoted with iso1.

Let us denote with vert0 the 3 × 3 matrix whose columns are the vectors $[x_i,y_i,1]$, $1 \le i \le 3$, where x_i,y_i are the coordinates of vertices *A*, *B*, *C* of our first triangle. That is



The product iso1·vert0 give us a matrix, that we call vert1, whose columns contain the coordinates of the transformed points *A''*, *B''*, *C''*.

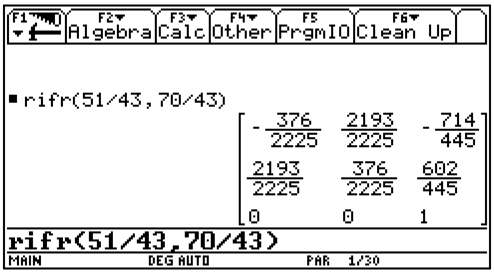


We can note that *A''* = *A'*.

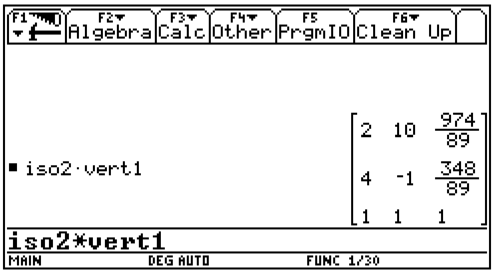
Now we find the perpendicular bisector of the line segment *B''B'*, with $B''\left(-\frac{107}{25},\frac{276}{25}\right)$ and *B'*(10,–1). We get

$$\text{perpendicular bisector}_{B''B'}\text{:}\ \ y=\frac{51}{43}x+\frac{70}{43}$$

and we can easily verify that such a line passes through *A'*. The reflection in the line *s* (*s*: $y=\frac{51}{43}x+\frac{70}{43}$) is expressed by the matrix



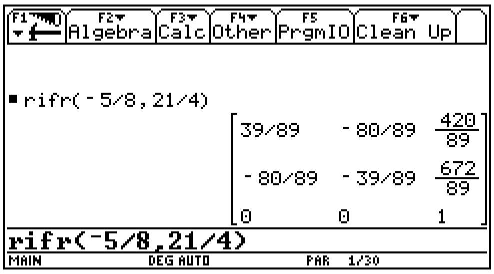
that we call iso2. The product iso2·vert1 give us a matrix, that we call vert2, whose columns contain the coordinates of the transformed points *A'''* = *A'*, *B'''* = *B'*, *C'''*.



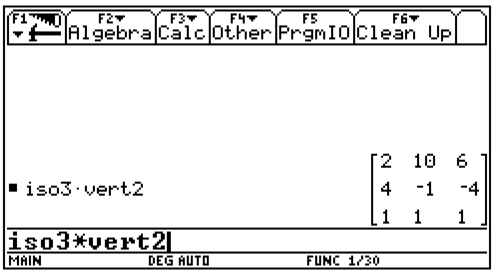
Lastly we find the perpendicular bisector of the line segment *C'''C'*, with $C'''\left(\frac{974}{89},\frac{348}{89}\right)$ and *C'*(6,–4). We get

$$\text{perpendicular bisector}_{C'''C'}:\ y=-\frac{5}{8}x+\frac{21}{4}$$

and we can easily verify that such a line passes though *A'* and *B'*. The reflection in the line *t* (t: $y=-\frac{5}{8}x+\frac{21}{4}$) is expressed by the matrix



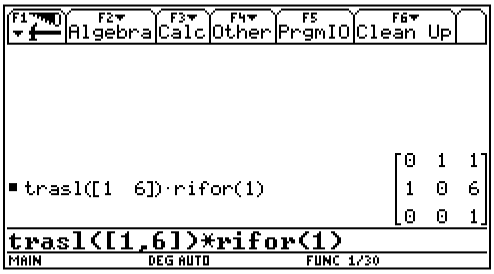
that we call iso3. We can verify that the product iso3·vert2 gives us a matrix whose columns contain the coordinates of vertices of △*A'B'C'*.



Therefore the matrix that represents the isometry we are looking for is

$$\text{iso3}\cdot\text{iso2}\cdot\text{iso1}=\begin{bmatrix}0&1&1\\1&0&6\\0&0&1\end{bmatrix}.$$

Looking at the matrix we can easily infer that we can obtain our isometry by means of a reflection in the line $y = x$ and a translation by vector [1,6] (*first* the reflection and *then* the translation).

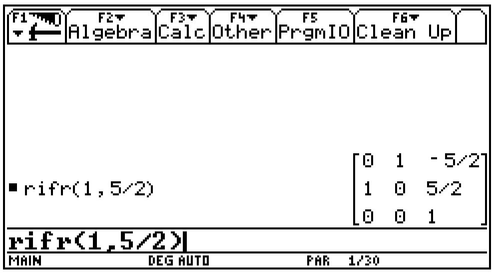


5) Determining the glide reflection that generates our isometry

To find such glide reflection we first observe that its axis must pass through the midpoints of line segments *AA'*, *BB'* e *CC'*. If we denote such midpoints with $M_{AA'}$, $M_{BB'}$ e $M_{CC'}$, we have

$$M_{AA'}\left(0,\frac{5}{2}\right) \quad M_{BB'}\left(\frac{3}{2},4\right) \quad M_{CC'}\left(-2,\frac{1}{2}\right).$$

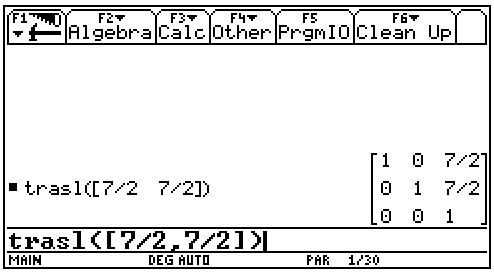
The equation of line r that passes through $M_{AA'}$ and $M_{BB'}$ is $y = x + \frac{5}{2}$ and r is parallel to the line $y = x$. We can verify that $M_{CC'} \in r$. The reflection in the line r is expressed by the matrix



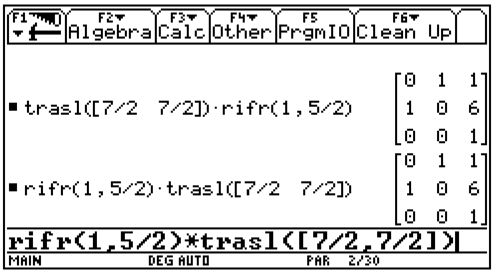
The vector **v** of the translation is parallel to the line r and thus must be of the form **v** = $[k,k]$. But the reflection takes the point $A(-2,1)$ to the point $A^*\left(-\frac{3}{2},\frac{1}{2}\right)$ and the translation by vector **v** has to take A^* to $A'(2,4)$. Consequently

$$\begin{cases} -\frac{3}{2}+k=2 \\ \frac{1}{2}+k=4 \end{cases} \quad \Rightarrow \quad k=\frac{7}{2}.$$

Therefore the vector **v** is $\left[\frac{7}{2},\frac{7}{2}\right]$. The translation by vector **v** = $\left[\frac{7}{2},\frac{7}{2}\right]$ is



The product of such a matrix with the matrix of the reflection gives us the same result we have obtained before and we can verify that this product is commutative.



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