General Aims of Mathematics Education Explained with Examples in Geometry Teaching

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Abstract: The connection between general aims and the subjects of mathematics teaching is worked out not very often. But for better understanding general aims it is good to give examples about such connections. In this article there will be given ideas about aspects of general aims in special subjects of geometry teaching like geometry in everyday life (grade 8 or 9), genetic approach to trigonometry (grade 10 or 11), shapes of triangles (grade 5 or 6), symmetry and its generalisations (grade 2 or 3) and polyominos (grade 3 to 11).

The meaning of mathematics teaching at regular school can not be just the mediation of mathematics. The main task rather is the development and education of young men. Thus general aims have to be in the fore. Normally you find such general aims in the guidelines or in the preface of the curriculum-guide. In the concrete plan of the lessons under the heading "objectives" you mostly find only subject orientated aims. Thus the general aims are often out of the mind of the teacher. In addition the assessment of general aims is difficult.

Therefore I plead for discussing and reflecting general aims before planning a concrete lesson. This has to start already during the study of pre-service-teacher and completed during in-service-teacher trainings. Also the group of mathematics teacher at one school should discuss the problems of general education in mathematics lessons from time to time. But with all this it is important to look out for the connections between general aims and special subjects too.

As initiation for that I will present some ***special topics*** (with short descriptions) ***together with their connections with general aims*** in the following[1].

1. Building out a loft – Geometry in everyday life

With a role play of a family which needs more room for the children we introduce the problem of building out a loft. Because of financial problems the people of the family want to do the work by themselves and have to make a financial plan which involves the costs of isolation (including computing areas), the size of the heating (including computing the volume) and the estimation of the costs of several small materials. The costs of some building materials have to be asked for in specialists shops. Finally the entire cost has to be determined.

The main intention of this unit concerns the *relevance of mathematics in everyday world* as well as the *readiness and ability of using mathematical tools with problems of real life*. This also has to do with the *ability of modelling* and *problem solving*. Moreover it can be learnt that in respect to a concrete problem of real life *limitations of mathematical modelling have to be reflected*. To get to know this best to the pupils it is good to work on situations[2] which are realistic (like here) or at least could be realistic. Besides in this unit the pupils deepen their knowledge of geometric figures, the computation of areas and volumes and train general computation skills. Also the handling with nets of solids and the *imagination of space* is deepened in this unit. Aims of the social dimension like *teamwork* and *good communication* will be reached if the method of teaching includes a lot of independent group work.

2. Genetic approach to trigonometry.

Beginning in grade ten with the problem of constructing a triangle (e.g. like Thales did to estimate the distance of a ship at the coast) we repeat the uniqueness of constructing a triangle with given three measures without the set of three sides and three angles. From geometry lessons in the last time we know that we can compute the length of the third side with the theorem of Pythagoras if there are given the length of two sides and a right angle. So we come to the looking out for a theorem which can help us to compute the measures of the two other angles. The teacher then can help with hints to history (the definition of slope in Egypt in ancient times) and to the gradient of streets in the mountains. Looking out for all right angled triangles with the same slope and thinking about similarity we find that the ratios of the sides are constant. This brings us to the definition of trigonometric functions where the six possible ratios give us six different trigonometric functions (called sin, cos,

[1] For more details in respect to general aims I refer to my paper "General Education in Mathematics Lessons" in this book.

[2] See e.g. Graumann 1976 or 1987a.

tan, ctg, sec, cosec). Now first it is easy to find a lot of connections between these six functions. Then we look out for special values of these functions by looking out for right angled triangles from which we know the ratios of its sides. We find such triangles within regular polygons. To get more values we again look out in history and discuss the reflections of Ptolemaus about 150 B.C.[3]
With this unit we first have the aim to see that the extending of a problem (computing unknown measures of a triangle from which we know that it is laid down clear) can lead us to new definitions which built an own theory. This means we learn a little bit about the *process of mathematics development*. Also we learn that to solve special problems (like determining values of trigonometric functions without using a modern calculator) we first have to find some theorems. This is underlined by the discussion of *history of mathematics* which can enriched with the way the *names of the trigonometric functions and the angles measure* came to us (which also has to be seen as general aim in respect to cultural knowledge) and the *importance of trigonometry in antiquity for astronomy*.
Another general aim we aspire to is the *training of combinatorical thinking* while defining the trigonometric functions the *discovery of connections* between the six functions as well as between special values and regular polygons. Moreover the *ability of problem solving and making investigations as well as creativity and independence* can be promoted in this unit. And with an appropriate method we also can support *social aims*.

3. Shapes of triangles – Combinatoric in geometry teaching.

In grade five or six normally the children have to get more familiar with different possible shapes of triangles and especially have to learn different types of triangles like isosceles, right-angled, equilateral and isosceles-right-angled triangles but also acute angled and obtuse triangles. In this context the pupils also should make experiences with very acute angled and very obtuse triangles to get an idea of the big range of possible shapes of triangles.
A nice point of start for this is the problem field of the analysis of triangles with integers as length[4]. Such triangles are natural for the pupils because at this time they mostly know only natural numbers. By treating such triangles they are lead to construct such triangles and discuss their shapes. For example it is a good problem to find all triangles whose sides have a length of 1, 2, 3 or 4 units. For this you have to find a systematic to get all of them and by constructing them you find out the theorem of the inequality of triangles. Also you find different types of triangles. Very acute or very obtuse triangles then you can construct by having one side equal to 1 and the others equal to n (for n= 5,6,7,8,...) respectively one side equal to 2n+1 and the others equal to n . For concrete experiences it is good to use strips of paper or plastic which have holes with constant differences (you can produce them with a punch). You then can connect such strips with clips in different ways[5].
An intention of this unit is (as already said) to cause experiences for the children in respect to different shapes of triangles. But some general aims also can be developed during this investigation. First of all the pupils must use their *combinatorial thinking* and *find a systematic* to be sure that all possibilities have been discussed. This is a special aspect of mathematical thinking which is helpful not only by working on mathematical problems. In addition a general aim is the experience that on one hand at least with 4 or more units for the length a systematic procedure is necessary and on the other hand different systematic procedures are possible. Moreover by looking out for a formula by given 1, 2, 3, 4 or 5 different units of length for the sides the *creativity* and *ability of discovering connections* can be developed. Also the pupils can learn that *theoretical knowledge (like mathematical theorems) can be tools to solve problems*.

4. Symmetry and its generalisations in primary school

Normally in grade two or three the children are confronted with symmetry especially axial symmetry in plane. A possible introduction to this theme is to produce blot-pictures on a folded paper and cutting out figures from a folded paper. With this the children can have a lot of associations but also find out the congruence of the two halves. This then is deepened by completing figures into symmetric figures, looking out for axis respectively planes of symmetry and making experiences with balance and the symmetry of our own body as well as animals or machines like aeroplanes. Finally we look out for

[3] For more details see e.g. Graumann 1987, Graumann 2001 or van der Waerden 1974.
[4] Cf. also Graumann 2001.
[5] For more details see e. g. Graumann 2001.

generalized symmetry in language (words whose meaning do not change if we read them forward or backwards) and mathematics (palindroms and sums of them)[6].

Symmetry in its geometric sense but also the fundamental idea of symmetry (in the sense of balance, optimality and regularity[7]) is an aspect of mathematics and science which should become acquainted with all people. Thus the *general phenomena symmetry* is to be seen as one general aim of mathematics teaching whereat the *aesthetical aspect as well as the functional view* is important. But also with this theme the children can see the *relevance of mathematics in everyday world* as well as the way mathematics concepts can help to *understand the world around* and to make *communication about special aspects* better. Furthermore in this unit the children can *train their perception of details and their psychomotoric abilities* especially in respect to using drawing equipments and to make drawings by hand. By working together in groups and presenting their new knowledge or products they also can *develop their abilities of argumentation and communication as well as cooperation*.

5. Polyominos – a glass bead game.

In primary school it is normal to discuss parquets with special shapes as basis or to solve problems with puzzles like Tangram. In this context the topic of polyominos also gives a lot of possibilities to work with. A polyomino is a figure which consists of congruent squares at which each two neighbouring squares have on side in common.

At school you may start with two congruent squares and demonstrate that with all possibilities to put these two squares together (with one side in common) there comes out only one figure if we don't look for different positions (i. e. we look out for the shapes which are independent in respect to congruence). If we then take a third square and move it along the sides of the "double-square" we find that there exist exact two different shapes of the so-called triominos.

The children then by themselves have to find out all shapes of quadrominos and quintominos. For help they can get a set of congruent squares made of plastic or cut out of cardboard.

For differentiating or deepening we can ask for sixtominos which build a net of the cube or we ask for all numbers at which we can get a big square or a rectangle. Or we can change the basic form from a square to an equilateral triangle or an isosceles triangle with one right angle or even to a cube (cf. the Soma-cube) or a tetrahedron.

A main intention to confront the children with this topic is to show that *mathematics is not only computation but also can be playful* (as matter for leisure time too) and then can lead us to theoretical questions. Another main intention is to train the *ability of systematisation and handling of problems as wells as training the imagination of space* (in respect to congruent figures in different positions). Also the *knowledge about geometric concepts and shapes* will be deepened. Moreover the *ability of creativity, imagination, discovery, problem solving strategy and building analogies* will be trained with such a topic. Finally *social aims* can be reached if several problems and problem posings within this topic is worked on in groups and the results are presented to other pupils.

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[6] For more details see Graumann & Graumann 2001.

[7] See Graumann & Graumann 2001, p. 82.