# It's not surprising that Euclid got excited about Geometry

**Thomas Hagspihl**
Director of Mathematics, St Stithians College, Johannesburg, South Africa

*As human beings we have the ability to appreciate - statement of the obvious. But why only poetry, art, a sunset, or a good movie and not Mathematics? Why is it that my pupils think I'm crazy when I tell them that Geometry is the most amazing subject on this planet? Is it perhaps because they haven't seen Morley's triangle or the connection between the Fibonacci sequence and Pythagoras? Is it perhaps because they haven't realised the link between the Sonic Boom of the Concorde and conic sections, or that cars can have square wheels and still drive smoothly, or is it perhaps just that their Mathematics teacher is a total maniac after all? In addressing this question we will look at some of the truly astounding discoveries made in Geometry over the last 2500 years.*

There are two things deeply rooted in every human being, viz. the ability to appreciate things (even if we don't necessarily like them) and an in-built inquisitiveness - wanting to know why things are the way they are and why things work in the way that they do. These abilities are often a bit suppressed or hidden and we need to work on them at times but nevertheless they are there. It is part of what makes us human.

Fortunately different people like different things - we aren't all enchanted by poetry or art or Physics for that matter, but I believe that we do ourselves a great disservice and miss out 'big-time' if we write off something because it doesn't happen to be our particular field of interest. Our experience on this planet is just so much more limited. Some years back I was asked to go on an Art tour for three days to help the lady art teacher look after the pupils. The initial apprehension and predictions of absolute boredom were absolutely shattered - it was one of the highlights of my year. I had made a beginning to understanding Art!

Have you ever wondered what actually happens when you put some salt in a cup of water and it just disappears? Where has it gone? I'm always amazed how that little bit of "magic" leaves some people ice cold. The bombs we made as children using brake fluid and granular pool chlorine continue to tickle me - why does it blow up so violently? Some people have become chemists because of a curiosity about why such things happen - and they have changed the world!

Have you ever wondered how an aeroplane weighing many hundreds of tons stays in the air? A jumbo 747 takes 170 000 litres of fuel (more than the average swimming pool) and has a take off speed of just over 310 km/h! It just doesn't seem to make sense that it all has to do with the shape of the wing. A curiosity about such things has made some people into physicists - and they have changed the world! Curiosity might have killed the cat, but without it we would still be in caves!

On the next page are two contributions from the arts that awakened my sense of appreciation when I saw them for the first time.

What has this all got to do with Mathematics? Well, how often do our pupils ask the question: "Where am I ever going to see this again?". One must put this to rest once and for all, early on with a class. When I do the rectangle of maximum area with them practically, they soon see a very good reason for Mathematics as it has to do with money. In addition however and probably more importantly - it seldom occurs to people that there is also a wonderful intrinsic beauty in Mathematics. A certain 'elegance' that you would have to be a brick wall if it left you stone cold!

Take the very simple example of "the ratio of the diameter of a circle to its diameter". Where the number pi hasn't already reared it's head - people have been fascinated by it since the beginning of man and it hasn't let them down.

Here are some beauties

$$\frac{p}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\ldots\ldots\ldots \qquad \text{(Gregory 1671)}$$

$$\frac{2}{p}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots\cdots\cdots \qquad \text{(Viete 1592)}$$

$$\frac{1}{p}+\frac{\sqrt{8}}{9.801}\sum_{n=0}^{\infty}\frac{(4n)!\left[1103+26390\ n\right]}{(n!)^4\,396^{4n}} \qquad \text{(Ramanujan 1914)}$$

**Pigeons** Richard Kell

***They paddle with staccato feet
in powder-pools of sunlight,
small blue busybodies.
Strutting like fat gentlemen
with hands clasped
under their swallowtail coats:
And, as they stump about,
their heads like tiny hammers
tap at imaginary nails
in non-existent walls.
Elusive ghosts of sunshine
slither down the green gloss
of their necks an instant,
and are gone.***

***Summer hangs drugged
from sky to earth
in limpid fathoms of silence:
Only warm dark dimples of sound
slide like slow bubbles
from the contented throats.***

***Raise a casual hand -
with one quick gust
they fountain into air.***

$$\frac{p-3}{4} = \frac{1}{2\times3\times4} - \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} - \cdots\cdots\cdots$$

Continued fractions:

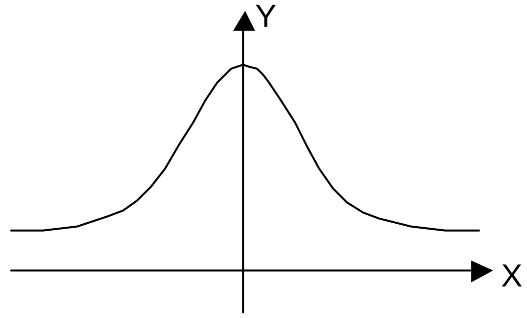
$$\frac{4}{p} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cdots}}}}}$$

Golden mean: $f = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}}$

$$\frac{1}{\left(\sqrt{f\sqrt{5}} - f\right)e^{\frac{2p}{5}}} = 1 + \cfrac{e^{-2p}}{1 + \cfrac{e^{-4p}}{1 + \cfrac{e^{-6p}}{1 + \cfrac{e^{-8p}}{1 + \cfrac{e^{-10p}}{1 + \ldots}}}}}$$

$e^{ip} = -1$ Leonhard Euler

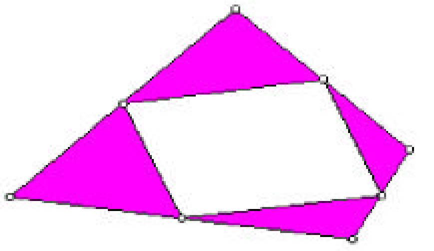
$$\sqrt{p} \;=\; \int_{-\infty}^{\infty} e^{-x^2}\, dx$$




- The probability of two positive integers, chosen at random, being relatively prime is $\frac{6}{p^2}$. I actually did this with a class as I didn't believe it and got $\frac{220}{515}$ which makes the value of pi 3.74. Not too bad with a small sample.
- If a needle of length L is thrown at random into a horizontal plane with parallel lines spaced a distance L from each other, the probability that the needle will intersect one of these lines is $\frac{2}{p}$.

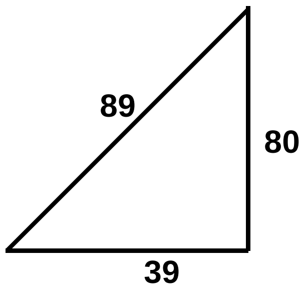
**Now if that leaves you cold you must be a brick wall!!**

Varignon's Theorem is wonderfully simple yet so profound: "If the mid-points of the sides of any quadrilateral are joined in turn, a parallelogram is formed". On sketchpad one can beautifully demonstrate this theorem. Furthermore the area of the 4 shaded parts is equal to that of the parallelogram.



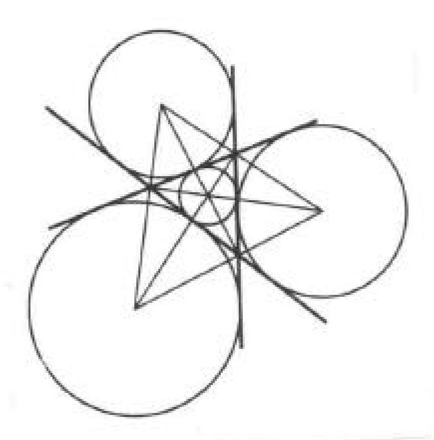
Even Napoleon produced a mathematical theorem - so the legend goes. *"If you construct equilateral triangles onto the sides of any triangles and join the three centres, they will form another equilateral triangle".*

Here is a beautiful and rather unexpected link between the Fibonacci Sequence and the Theorem of Pythagoras. Take any 4 consecutive Fibonacci numbers, say 3;5;8;13. Then $3 \times 13 =$ **39** and $2 \times (5 \times 8) =$ **80**. Both 39 and 80 Are then the shorter sides of a right-angled triangle and the Hypotenuse, which in this case is **89** is also a Fibonacci number. What's more the area of the triangle is $3 \times 5 \times 8 \times 13$ !! Now that is stunning.

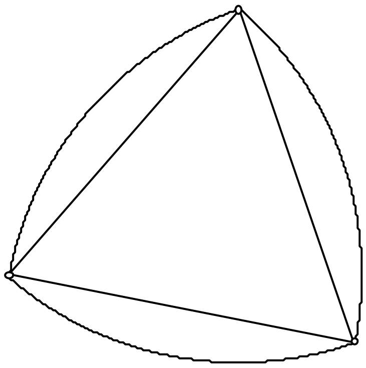



We know that the bisectors of the angles of a triangle are concurrent. If one bisects the exterior angles of a triangle they are also concurrent and intersect at what are called the "Excentres" - these are the centres of the "Escribed Circles" of the triangle. If r is the radius of the inscribed circle and $r_a$, $r_b$, and $r_c$ are the radii of the escribed circles then:

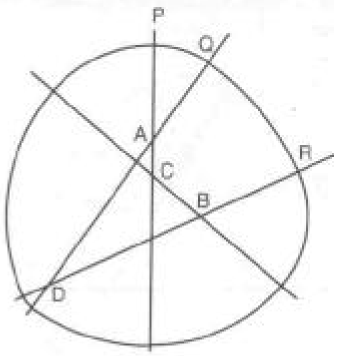
$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \quad \textit{and Area of Triangle} = \sqrt{r \times r_a \times r_b \times r_c}$$



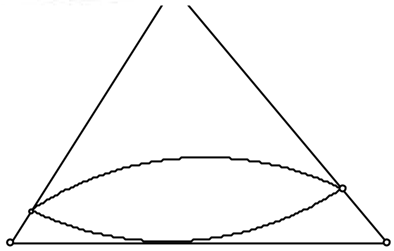
Lets briefly talk about rollers and rotors. It's is obvious that a circle is a shape of constant diameter, i.e. if you put two circular discs between two rulers, they will move parallel to one another without any bumps. The surprising thing is however that the circle is not the only shape for which this is true. If you take an equilateral triangle, use the three vertices as centres to draw three arcs with a radius the length of the side of the triangle, you will get another shape of constant diameter.



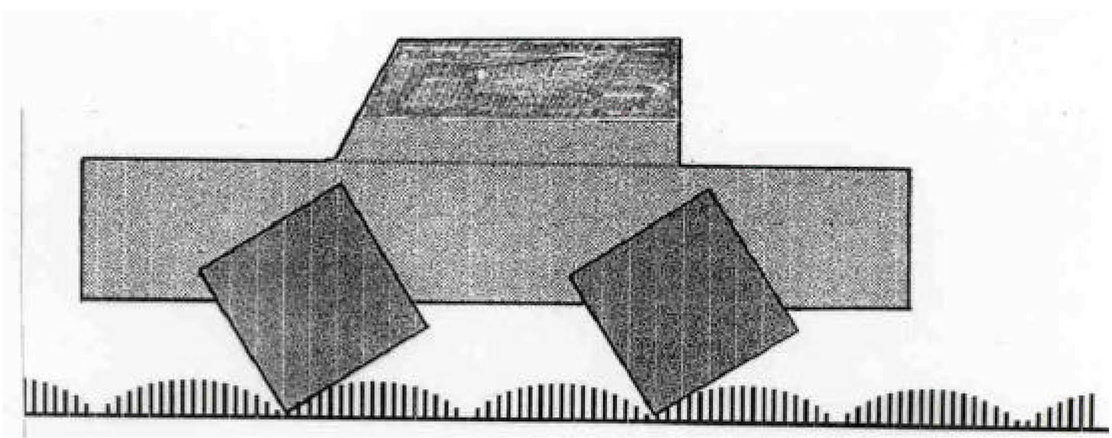
In fact the shape on which this kind of roller is based does not have to be a triangle, let alone an equilateral one. Given the 4 lines in the figure, place your compass point on A and draw arc PQ. Then move to point B and draw the next arc and continue until you get back to A. All shapes with constant diameter ***d*** have the same perimeter as a circle of the same diameter viz. ***pd.***



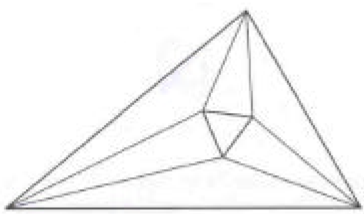

A square or hexagon will rotate inside a circle - we know that. What about other shapes? And of course there are! The German engineer Wankel used this fact to manufacture the rotary engine which you might remember in Mazda cars in the seventies. The RPMs of these engines far exceeded those of normal combustion engines but problems with wear and tear took them off the market.



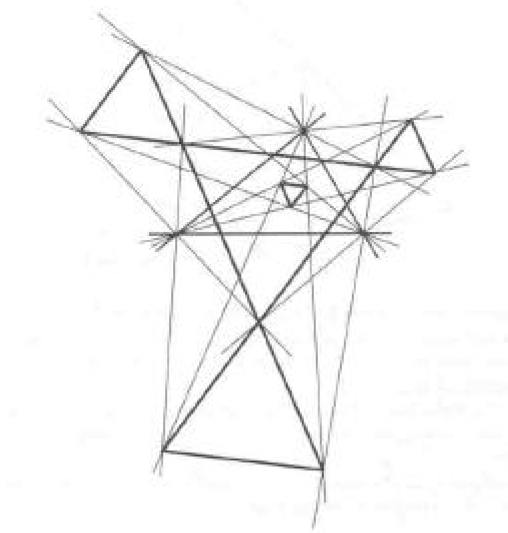
Talk about cars. Have you ever wondered if wheels need to be round - the answer is of course no - it's just the road that needs to be modified! It's quite a nice problem to work out the equation of the curves of the road given a particular size of square.



To my mind one of the most beautiful



theorems in geometry
is Morley's Triangle.
***"Trisect any three angles of any triangle. These lines intersect to form an equilateral triangle. If the three exterior angles are trisected, three more equilateral triangles are formed. Finally if the sides of these triangles are produced, another equilateral triangle is formed."***
And remember that this works for any triangle!



To end off with I want to share a little experience I had, which just goes to show that there are still many fascinating things in Geometry out there that still need to be discovered. Some years ago I gave a grade 8 class an investigation on circles. One of their tasks was to develop a method to find the centre of a given circle. I of course expected them to draw two chords, perpendicularly bisect them and where they meet would be the centre of the circle. One boy however did something else. He drew a number of rectangles into the circle and noticed that all the diagonals passed through the centre. He thus stated his method of finding the centre of a circle:

***"Inscribe any rectangle into a circle and the diagonals will intersect at the centre."***

Six years later (he was now in Post-Matric) he reminded me of all those projects we had done and asked if I still had some of them. I went to the cupboard and sure enough there was his project on circles - very dusty and a bit faded. Next to his method of finding the centre of a given circle, written in my scrawl, were the words "Can you prove this?" He looked at it and said; "let's see if we can prove it now". It did not take him long to find the proof that he had been unable to do in grade 8, and what had remained a conjecture for six years, now became Gain's theorem:

***"The diagonals of any cyclic rectangle intersect at the centre of the circle."***

D.R. Gain (1995)

References: *The Penguin Dictionary of Curious and Interesting Geometry*. David Wells. (Penguin Books)
*A History of Pi.* Petr Beckmann. (Barnes and Noble Books)