COMPUTER SIMULATIONS AND MODELLING IN MATHEMATICS EDUCATION

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The use of computer technology for simulations and modelling in mathematics education is now well established. Students of mathematics are directly involved in a process that is at the core of what mathematicians do: build mathematical models. There is little doubt that modelling and simulation programs are very powerful learning tools in mathematics education but what is it that makes a simulation program so educationally significant? As teachers, we all are familiar with simulations that seem to bear no relation to reality and are too far removed from concrete experience or are 'boring' and 'user unfriendly'. One of the important considerations when looking at simulation programs is whether the results the students get by manipulating the simulation are ones they could have anticipated but which also, once observed, can be seen to fit neatly into an evolving logical scheme. Only, if this is the case, will the exercise of discovery be likely to engage and maintain the interests of students. The visual representation of the phenomenon and the way the changing data is organized and presented on a computer screen are very important in a simulation program because unless the information that the students need to know is presented skillfully, the underlying processes may be obscured rather than made clearer. I also maintain that it is important that the mathematical data that students use has significance to the real world and can be manipulated in a meaningful way that has social, physical or human implications rather than purely mathematical ones.

An example of a mathematical learning activity that incorporates these features comes from the work of Lovitt and Lowe (1994) from the Victorian Curriculum Corporation. One investigation they designed explores the concept of 'half-life' and radioactive decay in the mathematical study of exponential functions. By carrying out a simulation and modelling program in a unique way that enhances student's cognition, the concept of exponential decay and half-life is powerfully realized. The context for this investigation might be a mathematics classroom where the nuclear debate becomes the topic for a classroom discussion. By way of introduction, the teacher talks about the issues and outlines the arguments 'for' and 'against' mining of uranium. The concepts such as 'radioactivity' and 'half life' are then briefly described. There are inevitably some blank faces as the teacher attempts to describe these scientific concepts. The students are then taken through a real life simulation activity where they are divided into small groups and each small group receives one die. They are then asked to pretend that they are atoms. Each student throws the die to give them selves a number from 1 to 6. The whole class then gathers together and then very dramatically, the teacher rolls the die and says that all those students with the number that appears uppermost on the die must sit down because they have just decayed. The activity continues with the teacher rolling a new number each year and then some of them (students) dropping out (decaying) until the last student has finally sat down. The small groups of four are then reformed and the students simulate the decay of ten atoms in ten smaller groups. At this stage a worksheet is given to students with ten squares in a row. Each square represented an atom and for each square in the line of ten the die is rolled once. If it comes up with a six that particular atom has decayed in that year and can be crossed out. The die is rolled again for each of the remaining atoms for the next year. The students continue rolling until all the atoms have decayed. The teacher, to show what happens to one hundred atoms then combines the ten groups of results. At this stage there is surprise at the range of results in the groups particularly as some took twice as many years as others to decay their atoms. The same process of each group watching the decay of ten atoms is repeated and combined and graphed on the board. It soon becomes clear that the graph takes on the classic shape of an exponential curve.

At this stage a computer program is introduced, initially on a data projector which I have shown here today that quickly selects a number for each atom left, removing the atoms with a six and counting those remaining. It even graphs results and allows you to carry out a fast run simulation where you can change the probability of decay occurring. This immediately became a very empowering and highly motivating learning tool because students can, not only see the model working, having only just finished the class simulation, but they can also understand quite clearly and powerfully the underlying statistical analysis. The social implications of the investigation also become apparent in a way that could not have been realized so acutely had this type of learning technology not been incorporated into the lesson in this unique way. Using the model of radioactive decay students can then manipulate the data by, for example, using a ten-sided die to demonstrate slower decay. For the social implications to become fully realized, teachers at this stage can explore the emotive issues surrounding the nuclear issue. Plutonium 239 is a waste product with a half-life of 24,390 years. Because it lasts so long it decays slowly but it's radioactivity is dangerous if there is any *of it to* decay. Each year a nuclear reactor will produce about 200 kg of it. Is Plutonium safe after 24,390 years? After 24,390 years there will be 100 kg left! How much is there after another 24,390 years? And, after yet another 24,390 years? Another side of the danger is the fact that Plutonium is the basic ingredient of the atom bomb. If there is any of it lying around it may be used by terrorists or others to create atom bombs! When students become aware of how many years it takes to get rid of one hundred atoms (between 20 and 40 years on average by simulation) they quickly appreciate the dangers. A 200kg sample of Plutonium represents a lot of atoms and 24,390 years is a long time to have it lingering around in the environment. This realization comes as quite a shock to students and powerfully illustrates the concept of radioactive decay and half-life in a way no standard textbook explanation could ever do.

This example clearly shows how the computer program is used to enhance the processes of cognition and encourage higher order thinking rather than being used as a tool to replace the teacher. The program used here has the features I have already described because the data obtained by the students can be seen to fit into an evolving logical scheme and the mathematical model itself is clearly discernable. Above and beyond this, it is highly motivating because it brings mathematical concepts out of the classroom and into a social, scientific and political context (a 'real' world setting) through the use of computer technology. Not only does it take mathematical concepts out of the textbook setting but it also crosses cultural and language barriers. People of all nations are divided on the nuclear debate; this universal appreciation of the mathematical model illustrated heightens young people's sensitivities to environmental issues.

I would like to demonstrate the power of this simulation in this workshop today and also show other examples of modelling and simulation programs that engage students in the way already described. With the help of the graphic calculator we can extend the above example where we used a computer program that quickly selected a number for each atom left, removed the atoms with a six, and counted those remaining. The computer simulation made a table and a graph of the results and even allowed us to change the probability of decay occurring. One weakness with this computer simulation is that atoms do not wait until the start of the year to decide to decay. The step-wise decay rate of one in six per year is a rough approximation of the actual continuous decay rate. The usefulness of this classroom activity depends on being able to do it in discrete stages. In reality, it is impossible to predict exactly which nuclei in a sample will disintegrate, however it is possible to predict on average the percentage of nuclei that will decay during a given time period. This percentage, expressed as a decimal, is called the *decay constant*.

Mathematically, the decay process is modelled exponentially: $N=N_o e^{-8t}$

Where N_o is the original number of nuclei present and N is the number present at time t

The *half-life*, $t_{1/2}$ of a radioactive sample is the time required for one-half of the nuclei present to decay. If the above exponential equation is solved for t when $N = N_o/2$, the result is: $t_{1/2} = \ln 2/8 = 0.693/8$

The graphic calculator program can be used to simulate the process of radioactive decay. Based on the data you collect, you can calculate the decay constant and half-life of the sample.

<u>Instructions and analysis:</u>

1. Start the **HALFLIFE** program on the TI-83 plus calculator as shown on the viewscreen. . Set the number of trials for 10 and press **ENTER** to start collecting data. The nuclei are represented by a 4 X 4 grid of squares . When a nuclei decays, its square disappears from the screen. Record elapsed time and nuclei count in a table .

Elapsed time	Nuclei Count
1	
………………………………………………………..	
10	

2. When all the trials are complete press **ENTER** to display a plot of nuclei count verses time. Notice that the count is 150 when $t = 0$. Note that $N_o = 150$.
3. Press **2^nd^ [DRAW]** 3 to display a horizontal line on the screen. Use the arrow keys to move the line up and down until the nuclei count reads very close to $y = 75 = N_o/2$
   Use the left and right arrow keys to position the cursor so that it lies on the imaginary curve connecting all the data points. The displayed value of x corresponds to the half-life of the sample. Record this value as $t_{1/2}$ in your lab notebook.
4. Calculate and record the decay constant , 8 using the formula given.
5. Enter the nuclei decay equation , $y = N_o\, e^{-8t}$ in the Y= list with the appropriate values for N_o and 8. Press **GRAPH** after the equation has been entered. How does the exponential model compare with the data collected ?
6. The actual decay constant used in the calculator program has been stored as ***L***
   .Press **ALPHA L ENTER** to retrieve this value. How does it compare to the decay constant found experimentally?

Using the exponential modelling equation students can actually fit the mathematical model to their actual results obtained from their own experimental simulation. The data obtained by the students can be seen to fit into an evolving logical scheme where the mathematical model itself is clearly discernable, consequently this becomes a highly motivating and enriching learning activity. Through manipulation of the data and exponential regression analysis the initial computer modelling simulation can be extended with the wider mathematical applications becoming more acutely realized and more highly appreciated. Another learning activity that incorporates the important features of modelling and simulation programs I have described and also involves fitting a mathematical model to an exponential curve relates to Newton's Law of Cooling.

As soon as a hot liquid is poured, it begins to cool. The cooling process is rapid at first, and then levels off. After a long period of time, the temperature of the liquid eventually reaches room temperature.Temperature variations for such cooling objects were summarized by Newton. He stated that the rate at which a warm body cools is approximately proportional to the temperature difference between the temperature of the warm object and the temperature of its surroundings. Stated mathematically:

$$)T/)t = -k(T - C)$$

where $)T$ represents the object's temperature change during a very small time interval, $)t$. T is the body's temperature at some instant, C is the surrounding temperature, and k is a proportionality constant. This equation can be solved for T using advanced techniques: $T - C = (T - T_o)e^{-kt}$

Where T_o is the body's temperature when $t = 0$.

We can use data obtained experimentally to attempt to verify the mathematical model developed by Newton. This experiment makes use of the **DATAMATE** program in the CBL 2. The calculator simply needs to be put in **RECEIVE** mode and the program can be transferred from the CBL 2. The CBL 2 detects the calculator to which it is connected and sends the appropriate version of the built-in **DATAMATE** software. The Stainless Steel Temperature sensor is connected to Channel 1 of the CBL 2 and then **DATAMATE** can be run.**DATAMATE** automatically identifies the stainless Steel Temperature sensor, loads its calibration factors, and displays the name of the sensor, as well as the temperature in degrees C. The calculator can be put in time graph mode to show you a graph of the temperature verses time as the experiment is in progress. This scale on this graph can be set to record the temperature over specified intervals of time for the duration of the experiment. To perform the experiment a beaker is simply filled with boiling water, the probe is immersed in the water and the program is instructed to start. Immediately, the graph appears on the screen, the probe is taken out of the water to allow cooling to occur. Graphing tools allow you to perform an exponential regression analysis (line of best fit) for the results and thus an exponential equation can be formulated and compared to Newton's Law of Cooling equation to verify its validity.

According to Newton's law of cooling , the quantity $y = T - C$ varies exponentially with time. To model this relationship we must first subtract room temperature from the collected temperature values which are stored in the **LISTS** of the calculator.To do this, press **2 nd [L4] – ALPHA C STO 2 nd [L4] ENTER** at the homescreen, where C is the room temperature value that was recorded earlier. By performing an exponential regression on the collected data from the **STAT CALC** menu on the TI 83 plus we can compare the cooling curve with our exponential curve obtained mathematically. Because the times have been stored in L2 and the temperature data has been stored in list L4, the appropriate regression command is **ExpReg L2,L4**. The regression equation and the correlation coefficient can be recorded and compared with other class results. The power of this mathematical modelling is clearly self evident. Students quickly grasp the concept and because the data is real and can be analysed mathematically at almost the same time that the experiment is performed the power of this modelling program is unquestionable.

The data can be seen to fit into an evolving logical scheme and again can be quickly related back to a real world setting.

The final modelling program I wish to describe involves the behaviour of a bouncing ball. Unlike the two modelling programs described above that involve analysis of exponential curves (log functions), this program involves parabolic motion and the fitting of a quadratic equation or mathematical model to a parabolic curve. Despite this apparent difference similar mathematical principles are involved. Of all the modelling programs described, perhaps this is the most motivating of all. This mathematical investigation can be quickly related back to ball games and physical sport a sure winner in any classroom because it has such universal appeal across all continents and in all cultures.

Real-world concepts such as free-falling objects, gravity and constant acceleration are examples of parabolic functions. This investigation explores the values of height (y), time (X) and the coefficient A in the quadratic equation:

$$Y = A(X - H)^2 + K$$

Physics theory suggests that the height of a bouncing ball is a second degree function of time where y = height in metres and x = time in seconds.

In this investigation data is collected to find an equation for this relationship.

An introductory activity that simply involves finding the equation of a parabola and sketching it if the vertex V is (2,5) and another point is A (3,6) begins this simulation. If A is (3,7) instead repeat and if A is (4,-3) repeat then the equation for a family of parabolas with vertex (2,5) is found.

To collect the data, the CBR is connected to the TI83 plus calculator and the **RANGER** program is transferred with the calculator in **RECEIVE** mode. The **RANGER** probe is an ultrasonic motion detector that allows motion to be analysed by sending out ultrasonic sound waves at moving objects. These objects reflect the waves. By calculating the time intervals between sound waves leaving the sensor and returning to the sensor, motion can be analysed mathematically. In principle it works in a similar way to a policemans speed gun. To conduct the experiment, begin with a test bounce and drop a large bouncy ball. Position the CBR from a position 0.5 metres above the height of the highest bounce. Hold the sensor directly over the ball and make sure that there is nothing in the clear zone. Run the **RANGER** program and from the **MAIN MENU** choose **APPLICATIONS**. Choose **METRES**. From the **APPLICATIONS** menu choose **BALL BOUNCE**. General instructions are displayed. **BALL BOUNCE** automatically takes care of the settings. Hold the ball with arms extended. Press **ENTER**. The **RANGER** program is now in **TRIGGER** mode. At this point, you may detach the CBR from the calculator.

Press **TRIGGER**. When the green light begins flashing, release the ball, and then step back (If the ball bounces to the side , move to keep the CBR directly above the ball, but be careful not to change the height of the CBR. You will hear a clicking sound as the data is collected . Data is collected for time and distance , and calculated for velocity and acceleration. If you have detached the CBR, reattach it when the data collection is finished Press **ENTER** (If the plot doesn't look good, repeat the sample) Study the plot and observe that **BALL BOUNCE** automatically flipped the distance data.

Looking at the display on the calculator you will see that the distance-time plot of the bounce forms a parabola. Press **ENTER**. From the **PLOT MENU**, chose **PLOT TOOLS**, and then **SELECT DOMAIN**. We want to select the first bounce. Move the cursor to the base at the beginning of the bounce , and then press **ENTER**. Move the cursor to the base at the end of that bounce , and then press **ENTER**. The plot is redrawn, focusing on a single bounce. With the plot in **TRACE** mode determine the vertex of the bounce. Press **ENTER** to return to the **PLOT MENU**. Choose **MAIN MENU** Choose **QUIT**.

Analysis

The vertex form of the quadratic equation , $Y = A(X - H)^2 + K$, is appropriate for this analysis. Press *Y* =

In the *Y* = editor, turn off any functions that are selected. Enter the vertex form of the quadratic equation : $Y_n = A*(X - H)^2 + K$

On the Homescreen, store the value you recorded in question 5 for the height in variable *K*; store the corresponding time in variable *H*; store 1 in variable *A*.

Press **GRAPH** to display the graph. Try different values of *A*. Start with *A* = 2,0,-1

Choose values of your own for A until you have a good match for the plot. Record your choices . Which choice is closest to the data (best fit). Leave this as Y_1 in your function list. The data for your experiment is in L_1(time) and L_2(height).

Using the data for height and time . Press **STAT** and select 1:**EDIT**. The time values are listed as L_1, and the height values as L_2. Press **STAT** and choose **CALC** and **6: Quadreg,** and **ENTER** L_1,L_2, Y_2. To enter Y_2, press **VARS**, select **Y-VARS**. And !:Function and press **ENTER**. Set up a **STAT PLOT**. Press **GRAPH** to view the data and regression curve. Compare the graphs of Y_1 and Y_2 . Make a comment about accuracy and any outlier data points. Which data point is furthest away from the curve Y_2 and by how much. Finally, complete the square for your Y_2 expression so that you can compare coefficients with those obtained for Y_1. In conclusion, compare results with other students and give to an appropriate level of accuracy , the values of the coefficients for the quadratic equation.

These four simulation and modelling programs I have described have all the qualities that I have alluded to at the beginning of this paper. The graphical analysis undertaken by the students allows them to manipulate the data and the results the students get by manipulating the model are ones that could have been anticipated but which also, once observed, can be seen to fit neatly into an evolving logical scheme. This exercise of discovery is likely to engage and maintain the interest of students. The visual representation of the phenomenon and the way the changing data is organized and presented on the screen of the computer or the graphic calculator are very important in these simulation programs. Unless the information that the students need to know is presented skillfully, the underlying processes may be obscured rather than made clearer. All these examples I have shown here today make the processes of learning and the concepts clearer. Above and beyond this, they have significance to the real world and can be manipulated in a meaningful way that has implications relevant to the students own field of experience that goes beyond the purely mathematical principles involved.

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