School Mathematics Geometric Problems: Past, Present and future

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0. Why problem solving?

Through history, solving problems has been an essential part of dealing with mathematics. Through solving problems mathematics itself has been discovered and mathematical structures have been built. The previous words explain why solving problems has been an essential part of learning and teaching mathematics in formal and informal education.

1. Geometric problems in the past

Till the end of the 50s of the twentieth century, Euclidean Geometry had a special place to teach students solving mathematical problems and grasp understanding of mathematics nature. The biographies of distinguished mathematicians show that their love to mathematics started in solving Euclidean Geometry problems, among such mathematicians is Albert Einstein (French 1979, 53).

Before the 60s of the twentieth century, some mathematics educators were interested in using the structure of traditional Euclidean geometric problem solution 'Given, To be proved and Proof' to teach children how to solve algebraic problems and word problems in arithmetic. They saw in the structure of geometric problem solving a model for the process of thinking to solve a mathematical problem.

In the 60s and 70s of the twentieth century the interest in bringing Bourbakis' linear algebra to senior secondary school was the reason of the disappearance of Euclidean Geometry from secondary schools and the decline of geometry teaching in general (Malaty 1998). The interest of mathematics educators in developing children problem solving abilities was interrupted in the 60s and 70s by their involvement in other activities related to implementing the 'New Math' to schools. The formal structure of the 'New Math' was not the appropriate environment to develop children's problem solving abilities. Transformation Geometry, which some educators called Motion Geometry, is putting emphasis on motion more than in traditional Euclidean Geometry. This would be a good addition to geometric education, but not a replacement of Euclidean Geometry. In solving geometric problem, it is more natural to try first to find the relations of figure's properties in a static way before going to search for relevant motion to solve the problem. Finding appropriate transformation is in need of acquiring knowledge in Euclidean Geometry.

At the time of Transformation Geometry teaching, some solutions presented by teachers looked beautiful, but as well students were of a feeling that such solutions are coming from some place beyond their abilities as tricks. At that time the next fact was not emphasised: Transformation Geometry is not a Non-Euclidean Geometry, it has been invented upon knowledge on Euclidean Geometry.

2. Geometric problems in the present

In the previous chapter we demonstrated in brief the situation of geometric problems in the past. The present situation we shall discuss in relatively more details.

2.1. From 'Back-to-Basics' to 'Problem-Solving'

The beginning of the 80s saw the spread of the 'Back-to-Basics' movement. Back here did not mean back to teaching Euclidean Geometry, to writing textbooks for geometry or to providing weekly lessons in geometry. The word basics did not mean mathematical structures like Euclidean Geometric structure. The 'Back-to-Basics movement' put emphasis on mastering arithmetical skills. Therefore, in the modest part of geometry, calculating perimeters, areas and volumes became the main aim of teaching geometry. Rules for calculation were given mainly in a ready form and not as a result of solving a problem and giving proofs.

Mathematics educators were not happy with the 'Back-to-Basics' mechanical teaching. They know that solving problems is an essential task of learning, teaching and building mathematics. Towards the year 1985 'Problem-Solving' became a new slogan to reflect a new tendency for improving the movement 'Back-to-Basics'.

2.2. From 'restaurants puzzles' to textbooks problems

In the first half of the 80s it was not rare to find mathematics educator, who liked to present to other colleague or colleagues an interesting puzzle in a restaurant meeting, among others at the

time of conferences. The next was for me the first of such ones I received at that time in a conference dinner. The sticks (matchsticks) represent a shovel, where inside is a stone. Move two sticks to get the stone out.

(Figure 1)

This was not a bad puzzle for spending time in restaurants, but after a while such puzzles started to have a place in classroom teaching and even in mathematics textbooks.

To solve such type of puzzles there is a need to analyse the given spatial situation and use some logical judgement, but it is difficult to pronounce such puzzles as geometric problems. Such puzzles are not related to any systematic learning of mathematics for certain level. It could be even given in oral way to illiterates.

2.3. From tricks to simple Euclidean Geometry

Some of the puzzles offered to children are in need for special tricks, which are difficult to find, even by good mathematicians. As an example I can mention two matchsticks puzzles. In one puzzle, 6 sticks were given to construct 4 triangles.

In the other puzzle, 4 sticks were used to represent a cross and the trick needed was to move only one stick to get a square (Figure 2).

The 4 sticks were much more closed to each other than illustrates figure 2.

(Figure 2)

Figure 3 is related to other kind of matchsticks puzzles, which spread on the second half of the 80s. This type of puzzles is as well not related solely to a particular educational level (Pehkonen 1992), but in some way related to geometry and indeed Euclidean Geometry.

In this puzzle, student has to take away 4 sticks to get a figure of only 5 squares and nothing more.
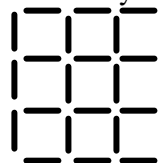
(Figure 3)

The sticks here do not construct squares in the geometric meaning of the word, but the word square could be accepted as the sticks are representing figures. The use of sticks brings some kind of motion different than that of transformation geometry, but such sticks' puzzles can also be solved in a more static way using paper and pencil, and drawing segments instead of sticks. The static version gives opportunity to analyse the figure better than the case of using sticks. The reason here is that the whole figure is available all the time. This helps the student in reflecting on his attempts. This type of puzzles could help in developing geometric thinking, but it is not enough to construct a curriculum for school geometry.

2.4. Computers and geometric problems

In the early 60s, mathematics educators started to be interested in using computers in teaching mathematics. They saw computers as the greatest promise for programmed instruction (Varga 1971, 31). The development of computer technology and the spread of computers in the late 80s and the beginning of the 90s have motivated mathematicians and mathematics educators to go in another direction in using computers in teaching mathematics, especially in teaching geometry.

At the moment mathematics educators are interested in using software packages like Cabri-Geometrie (Cabri-géomètre), Geometer's SketchPad (GSP) and Cinderella in teaching geometry.

In using any of these packages children can for instance draw a triangle, draw triangle's altitudes, and find that the altitudes are intersecting each other in one and only one point. These packages as well give chance to students to check as much as they like different cases of the investigating property. In the mentioned example, students can move any of triangle's vertices to twist the triangle up and down, left and right and to any direction else to explore that in all the cases the altitudes of a triangle intersect each other in one and only one point. That is why these packages are described as packages for Dynamic Geometry.

2.4.1. Some reflection The main problem we have to notice is that this kind of investigation does not prove anything. The software enables only to investigate as many as we like of different special cases in a short time. This can not be achieved manually by using paper and pencil. The

software dynamic property offers powerful tool for *inductive investigation*, but does not prove anything. This is well known to mathematicians and mathematics educators who co-operated with computer scientists in inventing such software to correspond Euclidean world. At the beginning, when Cabri-géomètre was invented in 1988, the aim of using the package was to give motivation to students to search for proofs. With the time, the experience showed that the opposite has happened; computers have convinced children on the result of investigation and therefore they have lost the motivation for finding the proofs (Malaty 1998).

Mathematicians who take part in inventing such mentioned above software are themselves able to give proofs to Euclidean Geometric theorems, that is why they are able to take part in inventing the packages to correspond Euclidean world. The question now is *how to get in the future such mathematicians, if the new generations of children are learning to be only users of such software?*

Despite the declaration of some educators that computer-based instruction is an appropriate way to Constructivistic Learning, in many cases the teacher has to provide children with the relevant way of investigation, i. e. he has to provide children with the trick needed. This is the case for instance, when the teacher asks the children to enlarge figures to see that a figure is a quadrilateral and not a triangle, as it seems in the screen. The main task in such case is to answer the question "Why": Why the figure could not be a triangle? In mathematics we can not accept answers based on using senses, but we need to see reasoning.

There is as well some minor problems related to the complexity of practice with geometric software. This could be noticed even in conferences presentations.

Dealing with computers makes us look modern in the eyes of children's parents and using computers brings to us fun. This has not to be our professional goal, we need to look forward for improving and developing the use of software and computers. One important goal has not to be changed in providing geometric problems; this is the developing of children's ability to use geometric structures to solve problems by giving proofs.

2.4.2. Some detailed analyses

In the mentioned above example of triangle's altitudes student is only giving orders using the provided menu to draw a triangle and to draw the altitudes. Student has only to remember the task of each order in the menu otherwise he has to ask the teacher. Asking the teacher to give help by different students makes the lesson proceed slowly. For this reason, teachers give to children details of how to proceed at the beginning of the lesson, after each step, or in a written recipe. The one thing left to student is the using of the mouse appropriately.

The software itself makes the drawing thanks to its inventors' knowledge. The student finds ready that the altitudes intersect each other in one and only one point. This could be an exciting result to student to the level of seeing miracle, but this does not mean that he understands why this is happening.

When the triangle is obtuse; two altitudes are outside the triangle and as well the point of intersection. Student may notice this case, but he does not understand 'why'. He as well does not understand why in the case of acute triangle all the altitudes are inside the triangle and as well the point of intersection. Without being able to answer the question why, we can not say that teaching mathematics is in a right way.

3. Some other kind of present attempts

Different changes have taken place in geometry teaching since the beginning of the 60s. Today in different countries, there are attempts to bring traditional Euclidean Geometry back to schools. In some of these countries there are as well attempts to write again textbooks for geometry and indeed Euclidean Geometry. In some other countries like Hungary, Russia and Czech republic Euclidean Geometry has always been the main part of geometric education. Here we have to notice the diversity happened in mathematics education since the 60s between what we still call 'Eastern Countries' and those we call 'Western Countries' (Malaty 1998). Finland, for instance, had been one of the Nordic Countries, which started radical changes in mathematics education in the 60s and now is trying to bring back Euclidean Geometry.

3.1. An example of the attempts

In Finland, in 1993 and for the first time textbooks were published in Geometry. These textbooks are textbooks for senior secondary school.

Figure 4 shows one of the problems, which appears at the beginning of one of these textbooks (Jäppinen, Kupiainen, Räsänen 2001, 14).

I have made some changes to the original figure in the textbook by adding names to 7 points in the figure and changing the naming of the angles from α, β, γ to be coded by numerals 2, 1,3.
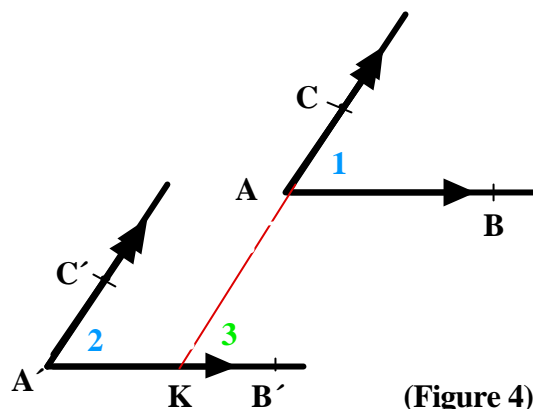
(Figure 4)

The textbook mentions that angles α, γ have the same measure and gives in brief the reason and then added that in analogy β and γ, thus α and β have the same measure.

This problem was introduced before in the 60s before the "New Math" in a geometry textbook for much younger students of Junior Secondary school (Väisälä 1961, 20). In this older textbook the proof was written in a similar way, but the reasoning of corresponding angles' measures equality was justified by a theorem discussed above.

3.2. Some reflection

The example of last chapter is now coming back to school after disappearance of 40 years, but for older children and for only some elite. The secondary textbook mentioned above is for those students who choose to study more mathematics than others do.

This problem itself is a simple one, which should be presented to all students of not more than 13 years. It is also relevant to make some modifications, as we shall propose below.

When we use numerals in naming angles for young children, they can concentrate on the number of angles we are concerning about (3) and the naming 1, 2, 3 is playing a role of ordinal numbers. It would be better to write the proof in a similar way of the next form.

By such writing we can use geometry to understand the transitivity of the relation of equality. This transitivity we tried to teach in learning relations at the time of the "New Math" in a formal way.

$$\widehat{1} = \widehat{3} \; \text{———————} \; \textbf{I} \quad \textbf{(Theorem)}$$

$$\widehat{3} = \widehat{2} \; \text{——————} \; \textbf{II} \quad \textbf{(Theorem)}$$

I & II ⊢

$$\widehat{1} = \widehat{2} \qquad\qquad\qquad \textit{Q.E.D.}$$

This writing shows what we do lose when we do not teach Euclidean Geometry. Euclidean Geometry is not only needed for the study of Cartesian Geometry, Trigonometry and Analyses, but as well it offers in a visual way insight into algebraic relations. Deductive thinking was discovered in dealing with Euclidean Geometry since Thales (ca. 624-548 B.C.), that is why Pythagoreans were able to deal with algebraic relations in elegant geometric way. Such part of the history of mathematics is needed for today's children to see this elegant thought, which can bring understanding and appreciation of mathematics and its algebraic thinking.

From the geometric point of view, it is better to not draw the segment AK ready as was given in the problem of figure 4. This idea of drawing has to be found by the children. We have also to ask them to find other versions and other solutions. When the children find that it is possible to draw the segment AK as extension of the ray AB to meet the ray A´C´ in a point K, we have to appreciate this other version which could be of help in dealing with other problems in the future. We have also to encourage them to see in the new version the similarity of thinking. Others solutions could be found. The next solution brings other property of equalities.
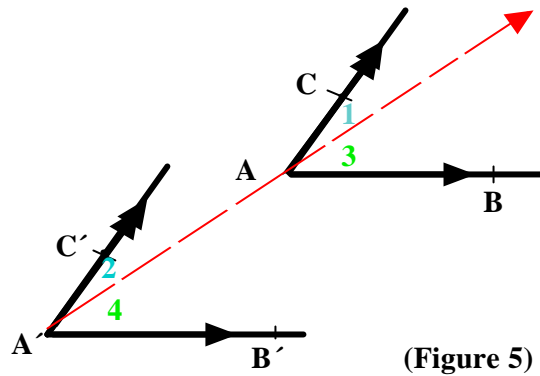
$$\hat{1} = \hat{2} \quad \text{———— I} \quad \textbf{(Theorem)}$$

$$\hat{3} = \hat{4} \quad \text{———— II} \quad \textbf{(Theorem)}$$

I & II ⊨

$$\hat{1} + \hat{3} = \hat{2} + \hat{4}$$

⊨ $C\hat{A}B = C'\hat{A}'B'$

Q.E.D.

(Figure 5)

3.3. Variation of solutions The geometric problem discussed above is an easy one. Proving that $C\hat{A}B = C'\hat{A}'B'$ in figures 4 and 5 can be done in Transformation Geometry through simple translation. Also a simple motion of Traditional Euclidean Geometry is enough. By motion here I mean, the idea of superposing one figure upon another. In teaching solving problems, we have to ask children to search for different solutions. The variation of solutions enriches children experience and makes them more able to face new problems.

4. Geometric problems in the future In a country like Finland, where the schools are well equipped with computers, the interest of teaching traditional Euclidean Geometry is increasing all the time. This can mean that the changes of the last 40 years in teaching geometry were only accidental waves. For geometric education world wide, the role of mathematics educators in 'Eastern Countries' could be a prominent one. From one hand they have continuous experience in teaching systematic Euclidean Geometry, and from the other hand some of them have shown interest in inventing and using software in teaching geometry. Despite the fact that this interest has been stimulated by the acquaintance of 'Western Countries' activities in this field, the use of software in 'Eastern Countries' is quite different. This is due to the culture of mathematics education there. In these countries solving problems by giving proofs is the main thing to learn in secondary schools. This is the reason why the interest of using software and computers in 'Eastern Countrie s' could bring the needed improvements in using software.

4.1. Changes and future Where Euclidean Geometry was a main part of school mathematics for hundreds of years, geometric education has seen different changes in the last 40 years. Today after different coups in mathematics education, Euclidean Geometry is coming back. One of the factors, which brought back Euclidean Geometry, is the interest in computer-based instruction. School geometry software is of Euclidean Geometry. The common factor in the changes in geometry teaching is the increasing of motion. Motion was in Transformation Geometry, in some puzzles like matchstick puzzles and indeed in the use of software. We need to have some of Transformation Geometry in secondary schools, some type of puzzles we can provide from time to time and the using of software has to be developed. We have not to make mixture between the using of computers to make investigations and solving a problem. Solving a problem can not be accepted in secondary schools, unless there is a proof. The dynamic property of software can be of help to explore new geometric problems, but not solving them. Our work as mathematics educators is not to avoid teaching proving, but it is our challenge to develop our strategies and our methods to make proving possible and enjoyable task for present and coming generations of students.

Finally I am leaving here a problem to the reader to find what we need to know from Geometry to solve it. This problem is, *'How to divide a given segment into 3 parts of equal measures with the use of only a non-scaled ruler and compasses'.*

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