THE NOTION OF VARIABLE IN SEMIOTIC CONTEXTS DIFFERENT

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INTRODUCTION

There are a lot of studies on the obstacles that the pupils meet in the passage from the arithmetic thought to the algebraic thought. Some of them reveal that the introduction of the concept of variable represents the point of critical transition (Matz,1982;Wagner,1981, 1983).

This concept is complex because it is used with diverse meanings in different situations. His management depends precisely from the particular way to use it in the problem-solving. The notion of variable could take on a plurality of conceptions: *generalized number* (it appears in the generalizations and in the general methods); *unknown* (it could be calculated considering the restrictions of the problem); *"in functional relation"* (relation of variation with other variables); *sign totally arbitrary* (it appears in the study of the structures); *register of memory* (in informatics) (Usiskin, 1988).

The high school privileges chiefly the first three conceptions: general number, unknown and functional relation, but it uses also the notion of register of memory in informatics.

The pupils meet many difficulties in the study of the algebra. It is possible that they derive from the inadequate construction of the concept of variable. This construction should include its principal conceptions and the possibility to pass from one to the other with flexibility, in relation to the exigencies of the problem to solve.

Kücheman (1981) has shown that most of the pupils between 13 and 15 years treat the letters in expressions or in equations like specific unknowns before as generalized numbers or variables in a functional relation. Trigueros, M. *et alii* (1996) have demonstrated that the beginners university students have a fairly poor conception of variable in its aspect of generalized number and a functional relation. They have difficulty, chiefly, in understanding the variation in a dynamic form, that is the relation of variation with other variables. The obstacles are greater when the resolution of the questions doesn't take place by the manipulation, but through the interpretation and the symbolization.

Panizza *et alii* (1999) have shown that the linear equation in two variables is not recognized by the pupils like an object that defines a set of infinite couples of numbers. The notion of unknown would not be effective to interpret the role of the letters in this type of equations. Instead, if the pupil uses the concept of function, he is in better conditions to calculate different solutions.

The present article intends to study the relational-functional aspect of the variable in the problem-solving, considering the semiotic contexts of the algebra and of the analytical geometry. We want to analyse if the unknown notion interferes with the interpretation of the functional aspect, and if the natural language and/or the arithmetic language prevail as the symbolic systems in absence of an adequate mastery of the algebraic language.

We have chosen to effect this research the linear equation in two variable for two motives: firstly, because it represents a nodal point to cause the conceptions of the pupils on the letters as unknowns or "things that vary". We anticipate that the students will find some difficulty to treat the equations with a plurality of solutions, in the ambit of problematic concrete situations (1). In the second place, this type of equation results an object well known from the pupils, treated under different viewpoints: linear function, equation of a straight line and component of the linear systems.

Even if from the mathematical point of view these three terms (linear function, equation of a straight line and component of the linear systems) represent the same object, for the pupil means to evoke different mental models (external) (2). According to Bagni (2001), the expression $ax + by + c = 0$ could be situated in a geometric context (to evoke, for instance, models of the concept of straight line in the ambit of the analytical geometry), or in a context purely algebraic (to speak that is of equation of first degree or, improperly, of polynomial). But this choice reflects an attitude quite different, that it has interesting motivations (they are also tied to the didactic contract) and remarkable didactic consequences.

HYPOTHESIS

1. The conception of variable as unknown interferes with the interpretation of the functional aspect in the algebraic context.

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2. The natural language and/or the arithmetic language prevail as symbolic systems in absence of a mastery of the algebraic language.
3. The translation of a functional relation from the algebraic language to the natural one doesn't happen spontaneously.

METHODOLOGY OF THE RESEARCH

One hundred eleven students of 16-18 years of the Experimental High School of the city of Ribera (AG)- Italy has participated to this study. They were thus distributed: 23 pupils of Fifth of the Classic High School and 88 of the Scientific High School, 37 of Third, 20 of Fourth and 31 of Fifth.

We want to explain that all the pupils knew the relative matters to: equations and inequations of first and of second degree, systems of equations, analytical geometry and functions. Particularly, the Fifth's students had already effected the graphic study of functions in the ambit of the mathematical analysis.

The questionnaire presents four questions (Appendix 1). In the first of them, the variable takes on the relational- functional aspect in the context of a problematic concrete situation. We also ask to think over the quantity of solutions. With this question we want to analyze the resolution strategy used and if the unknown notion interferes with the interpretation of the functional point of view.

The second question asks the formulation of a problem. This must be resolved by means of a given equation, namely, the student must carry out the translation from the algebraic language to the natural language. We consider that this activity represents a fundamental point. It allows to reveal the difficulties that the pupils meet to interpret the variable under the relational-functional aspect.

The third question asks to interpret, by a "short answer", the following relations of equality: $ax + by + c = 0$ e $y = mx + q$. We try to understand to which model and in which context, these equations are associated by the students. The purpose of this question is to compare the evoked models from these expressions with those activated from the problems 1 and 4.

In the fourth question, the variable take on its relational-functional aspect in the context of a problematic concrete situation. We also ask to think over the quantity of solutions. While in the first problem the pupil was frees himself to choose the resolutive context, in this one, instead, we force him to operate in the ambit of the analytical geometry.

We have effected a priority analysis for each query of the questionnaire. The objective was to determine all the possible strategies that the pupils could use. Because the students could make some mistakes in the application of these strategies, we have also individualized the possible mistakes.

We have administered of the questionnaire during the last week of April 2002. The pupils have resolved the proof individually, we have not allowed the consultation of books or of notes. The granted time has been of sixty minutes.

We have completed a table with a double input "pupils/strategy". For every pupil we have indicated the strategies that he used with the value 1 and those ones, he didn't applied with the value 0.

The data have been analyzed in a quantitative way, using the implicative analysis of the variable of Regis Gras (1997, 2000) and with the help of the CHIC software 2000 available under Windows 95.

RESULTS AND DISCUSSION

We observe in the table of frequencies that the highest percentages of answers are obtained in the first problem with 95% and in the two questions of the third query with the 98% and the 99%. We find the 75% of the answers in the fourth problem, while only the 59% in the second question.

The 71% of the students interpret the expression $ax + by + c= 0$ of the third query in the ambit of the analytical geometry (equation of: straight line 44%, circumference or parabola 26% and sheaf of straight lines 1%), while the 27% recall the algebraic context (equation of first degree in two variable 22% and polynomial 5%). For the expression $y = mx + q$, instead, the totality of the pupils reference to the ambit of the analytical geometry (equation of: straight line 68%, sheaf of straight lines 30% and parabola 1%). These expressions coincide with the explicit and implicit equations of the straight line (presented generally in the books of text and from the teacher). This coincidence directs the interpretation toward the context of the analytical geometry. These results find comparison with those of Bagni (2001).

The graphics of the CHIC (Appendix 2) have underlined the fundamental knots of the questionnaire and the relations between them.

- In the first problem we observe basically the use of three types of strategy:
    - * *Algebraic*: the pupil translates the text of the problem to an equation of first degree in two unknowns and he applies the method of "substitution into one same" (he writes a variable in function of the other and he then substitutes in the original equation) (3). When the student arrives to the expression of an identity, he abandons generally the resolution or he tries another method. Only the 4% of the students

arrive to the correct solution. This procedure is also used to solve the equation of the second query. This result is got by the links of similarity, implicative and hierarchical implicative between the variables AL5, AL7 and IAL3.

* *For attempts and mistakes in natural language or in half-formalized language*: the student that applies this method shows only some solutions that verify the equation. This result is confirmed by the links of similarity, implicative and hierarchical implicative between the variables AL3 and ALb3.

* *Procedure in natural language*: the pupil adds a datum (he divides the win in half or he considers that the played sums are equal) and he finds only a particular solution that verifies the equation. This result is got by the links of similarity, implicative and hierarchical implicative between the variables AL2, AL4, Alb2 and ALb1.

We want to underline that solely the 29% of the pupils consider that the first problem has a plurality of solutions. However, they have tried before to solve this problem applying the preceding procedures.

Many pupils answer the question on the possible solutions and they consider the single solution. This result is got by the links of similarity, implicative and hierarchical implicative between the variables ALb2 and ALb1.

- The second problem (translation from the algebraic language to the natural one) is resulted difficult for the pupils. The 59% answers and only the 6% writes a text correctly. The only meaningful result that emerges from the graphic is the following: the pupil that shows a particular solution that verifies the equation, produces a text that considers only constant (links of similarity, implicative and hierarchical implicative between the variables IAL4 and IAL7).

  The results of the table of frequencies show that some pupils produce a text that considers an only variable (5%) and others regard the two variables, but the formulation is not correct (13%), because they omit the question or because the relationship doesn't answer exactly to the equation.
- If the pupil translates the fourth problem to an equation of first degree in two unknowns and he answers on the possible solutions then he considers that a plurality of solutions exists. This emerges from the links of similarity and implicative between the variables GAa4, GAbc1 and GAbc6.

  The answer on the existence of a plurality of solutions is strengthened by the hierarchical strong implications: GAbc6 $\rightarrow$ GAbc1 to the first level, but also of di GAa8 $\rightarrow$ [GAa7 $\rightarrow$ (GAbc6 $\rightarrow$ GAbc1)], namely the pupil that considers (in a mistaken way) that between x and y exists a proportionality relation then he represents the relation graphically in a correct way and therefore he considers the plurality of solutions. The implicative link (GAa2, GAa7 e GAa8) $\rightarrow$ Gab6 is also important, namely if the pupil has adopted different strategy of resolution (procedure in natural language, graphic representation correct or proportionality relation) then he answers that a plurality of solutions exists.

CONCLUSIONS

From the analysis of the data we observe the used strategies for to solve the first problem. They are the followings:

- *procedure in natural language*: it results the more used, it conducts to a unique solution, the conception of predominant variable it is that of unknown.
- *for attemps and mistakes in natural language and/or in half-formalized language*: generally arithmetic, it conducts to several solutions, the dependence of the variable is evoked, but a strong conception of the relational-functional aspect doesn't appear still.
- *algebraic method*: it is little used and generally abandoned.

Therefore, the hypothesis 2 is confirmed, that is in absence of adequate mastery of the algebraic language prevail the natural language and/or the arithmetic language as symbolic systems.

It is interesting to observe that no pupil uses the context of the analytical geometry to solve the first problem, and that many students consider that the problematic situation has a unique solution. For the fourth problem (with a concrete similar situation to the preceding), instead, the students that answer on the possible solutions consider directly the plurality of solutions.

This analysis would seem to point out that: the pupils could consider the plurality of solutions more easily, in the context of the analytical geometry evoking the mental model of the equation of the straight line.

Almost all the pupils have interpreted the expressions $ax + by + c = 0$ and $y = mx + q$ in the ambit of the analytical geometry, but the model of straight line has not been evoked with the equation of the first problem, possibly for a matter of the didactic contract. Usually, the problems with equations of the school are solved in an algebraic context and the variable engages the unknown aspect. The problematic concrete situations generally are never solved in the ambit of the analytical geometry, recalling representative visual registers. The problems of analytical geometry of the school are different. In the fourth problem, when the pupil is forced to use the model of straight line with its Cartesian representation, he "visualizes" more easily the plurality of solutions.

Independently from the motive of the choice of the context, if we consider the preceding analysis: we observe that an algebraic context with the variable under the unknown aspect, it could conduct more easily to the oneness of the solution of the equation in two variables. Accordingly, we could affirm that there is a certain interference of the conception of unknown on the functional aspect. However, we believe that the matter must be still deepened, for example, analyzing in every particular the risolutive used strategy. We should investigate how these conceptions are activated in the resolution of a problematic situation in an algebraic context and how could the passage occur from one to the other without interference.

The translation from the algebraic language to the natural language results a difficult exercise for the pupils, therefore the third hypothesis finds confirmation. Some interesting particularities are emerged, for example, the students who generate a text with constants, they produce the question referred to the second member of the equation, that is at 18. Therefore, for these pupils, the expression is an unidirectional relation and with the answer to the right side. Thus, this shows a return to the primitive perceptions that the students of 12-13 years have on the equations of first degree in an unknown (Kieran, 1981). Some pupils have needed to know the values of the unknown before to involve it in the elaboration of the problem. This shows an obstacle of the language at level purely syntactic that it should be analyzed more thoroughly.

Other salient questions that emerge from this research are: the importance of the visualization in problem-solving and of the coordination of different representative registers (Duval, 1999).

NOTES:

(1) On purpose we don't want to use the term "infinite solutions". The possible connotations of the word "infinite" have not been examine in the present study.

(2) We think opportune to explain the used terminology in this research; to this purpose we will follow D'Amore & Frabboni (1996). We call mental image what it is elaborate from the pupil, also unintentionally, before of an any request (interior or external): it is an interior image, therefore not express, at least initially. All the mental images of a concept constitute the mental relative model to this concept (Johnson-laird, 1988). Thus, the built conceptions must often be express, communicated by means of a specific translation, therefore, an external model is created and is expressible frequently in a well determined language. Every form of communication of a content or an any mathematical message occurs therefore with the use of external models (Shepard, 1980).

(3) We have called "procedure of substitution in one self": the incorrect method that consists of writing a variable in function of the other, then, to replace it in the original equation and in this way to obtain an identity. That is, the pupil applies the method of substitution used to solve the systems of equations to a single equation.

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APPENDIX 1: QUESTIONNAIRE

1- Charles and Lucy win the lottery the total sum of e 300. We know that Charles wins the triple of the money bet, while Lucy wins the quadruple of her own. (a)Determine the sums of money that Charles and Lucy have bet. Comment the procedure that you have followed. (b)How many are the possible solutions? Motivate your answer.

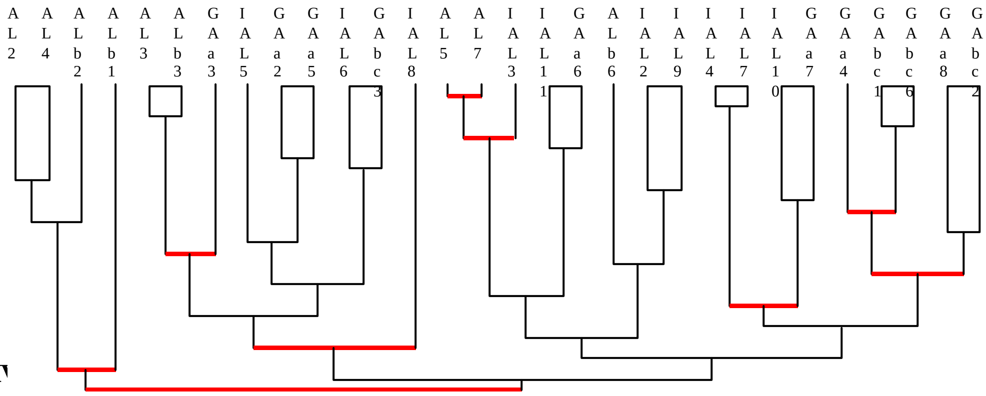
2- Invent a possible situation-problem that could solve using the following relation of equality: $6x - 3y = 18$. Comment the procedure that you have followed.

3- What is it? Interpret by a "short answer" the following expressions: a) $ax + by + c = 0$ b) $y = mx +$

4- A person make use of a fixed telephone installed beside of an another one, she arranges with him of pay monthly e 5 more e 2 for hour phone calls effect. Let: *x* the number of monthly hours of phone calls effects and *y* the total sum that she has paid monthly (a)Establish which type of relation intervenes between x and y and represent it graphically in the plain Cartesian.(b)Determine the total sum that she has paid monthly and the number of monthly hours of phone calls effects. Explain your answer. (c)How many are the possible solutions? Motivate your answer.

APPENDIX 2: GRAPHICS

SIMILARITY TREE



IMPLICATI

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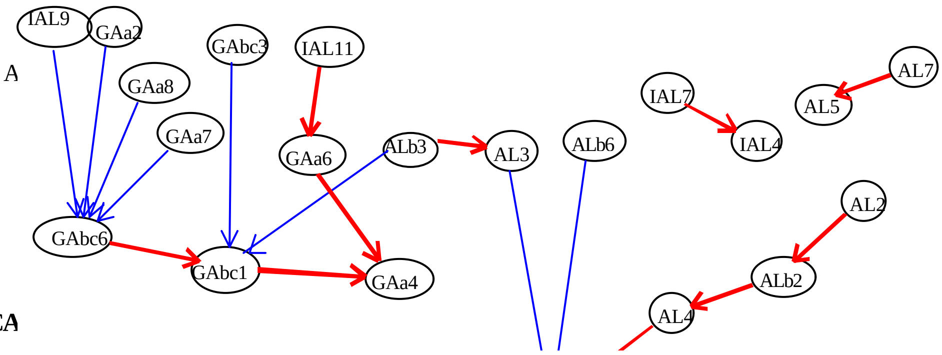

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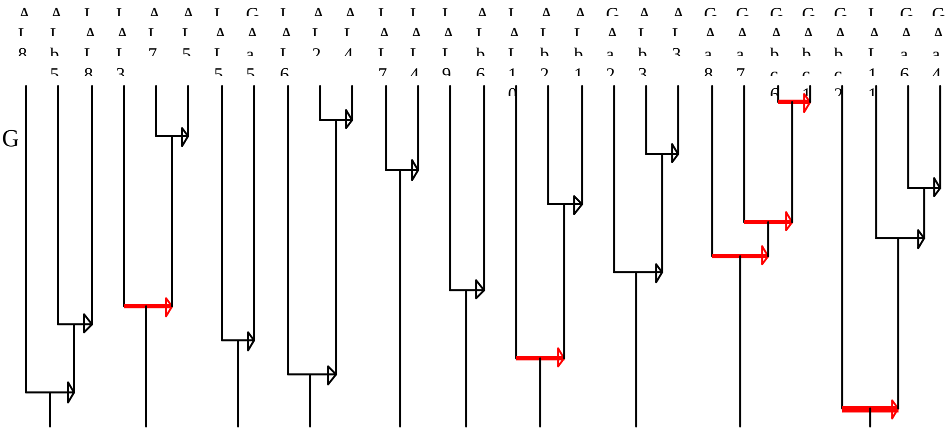

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